The Ramsey numbers for disjoint union of trees versus W_4

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Abstract. The Ramsey number for a graph G versus a graph H, denoted by $R(G,H)$, is the smallest positive integer n such that for any graph F of order n, either F contains G as a subgraph or $\overline{F}$ contains H as a subgraph. In this paper, we investigate the Ramsey numbers for union of stars versus small cycle and small wheel. We show that if n_i is odd and $2n_{i+1} \geq n_i$ for every i, then $R(\bigcup_{i=1}^{k} T_{n_i}, W_4) = R(T_{n_k}, W_4) + \sum_{i=1}^{k-1} n_i$ for $k \geq 1$.. Furthermore, we show that

1. If n_i is even and $2n_{i+1} \geq n_i + 1$ for every i, then $R(\bigcup_{i=1}^{k} S_{n_i}, W_4) = 2n_k + \sum_{i=1}^{k-1} n_i$ for $k \geq 2$,
2. If n_i is odd and $2n_{i+1} \geq n_i$ for every i, then $R(\bigcup_{i=1}^{k} S_{n_i}, W_4) = R(S_{n_k}, W_4) + \sum_{i=1}^{k-1} n_i$ for $k \geq 1$.

Keywords : Ramsey number, Cycle, Wheel

1 Introduction

For given graphs G and H, the *Ramsey number* $R(G,H)$ is defined as the smallest positive integer n such that for any graph F of order n, either F contains G or $\overline{F}$ contains H, where $\overline{F}$ is the complement of F. Chvátal and Harary [6] established a useful lower bound for finding the exact Ramsey numbers $R(G,H)$, namely $R(G,H) \geq (\chi(G)-1)(C(H)-1)+1$, where $\chi(G)$ is the chromatic number of G and $C(H)$ is the number of vertices of the largest component of H. Since then the Ramsey numbers $R(G,H)$ for many combinations of graphs G and H have been extensively studied by various authours, see a nice survey paper [9]. In particular, the Ramsey numbers for combinations involving union of stars have also been investigated. Let S_n be a star of n vertices and W_m a wheel with m spokes.

For a combination of stars with wheels, Surahmat et al. [10] determined the Ramsey numbers for large stars versus small wheels. Their result is as follows.

Theorem A.(Surahmat and E. T. Baskoro, [10]) *For* $n \geq 3$,

$$R(S_n, W_4) = \begin{cases} 2n+1, & \text{if } n \text{ is even,} \\ 2n-1, & \text{if } n \text{ is odd.} \end{cases}$$

Parsons in [?] considered about the Ramsey numbers for stars versus cycles as presented in Theorem .

Theorem B. (Parsons's upper bound, [?]) *For* $p \geq 2$, $R(S_{1+p}, C_4) \leq p + \sqrt{p} + 1$.

Hasmawati et al. in [?] and [?] proved that $R(S_6, C_4) = 8$, and $R(S_6, K_{2,m}) = 13$ for $m = 5$ or 6 respectively.
Let G be a graph. The number of vertices in a maximum independent set of G denoted by $\alpha_0(G)$, and the union of s *vertices-disjoint* copies of G denoted sG . S. A. Burr et al. in [3], showed that if the graph G has n_1 vertices and the graph H has n_2 vertices, then

$$n_1 s + n_2 t - D \leq R(sG, tH) \leq n_1 s + n_2 t - D + k,$$

where $D = \min\{s\alpha_0(G), t\alpha_0(H)\}$ and k is a constant depending only on G and H. Recently, Baskoro et al. in [2] determined the Ramsey numbers for multiple copies of a star versus a wheel. Their results are given in the next theorem.

Theorem C. [2] *For* $n \geq 3$,

$$R(kS_n, W_4) = \begin{cases} (k+1)n & \textit{if } n \textit{ is even and } k \geq 2, \\ (k+1)n - 1 & \textit{if } n \textit{ is odd and } k \geq 1. \end{cases}$$

In this paper, we study the Ramsey numbers for disjoint union of stars versus small cycle and small wheel. The results are presented in the next two theorems.

Theorem 1. *Let* n_i *is natural number for* $i = 1, 2, \ldots, k$ *and* $n_i \geq n_{i+1} \geq 3$ *for every* i. *If* n_i *is odd and* $2n_{i+1} \geq n_i$ *for every* i, *then* $R(\bigcup_{i=1}^{k} T_{n_i}, W_4) = R(T_{n_k}, W_4) + \sum_{i=1}^{k-1} n_i$ *for* $k \geq 1$.

Theorem 2. *Let* n_i *is natural number for* $i = 1, 2, \ldots, k$ *and* $n_i \geq n_{i+1} \geq 3$ *for every* i.

1. *If* n_i *is even and* $2n_{i+1} \geq n_i + 1$ *for every* i, *then* $R(\bigcup_{i=1}^{k} S_{n_i}, W_4) = 2n_k + \sum_{i=1}^{k-1} n_i$ *for* $k \geq 2$,
2. *If* n_i *is odd and* $2n_{i+1} \geq n_i$ *for every* i, *then* $R(\bigcup_{i=1}^{k} S_{n_i}, W_4) = R(S_{n_k}, W_4) + \sum_{i=1}^{k-1} n_i$ *for* $k \geq 1$.

Before proving the theorems let us present some notations used in this note. Let G be any graph with the vertex set $V(G)$ and the edge set $E(G)$. The *order* of G, denoted by $|G|$, is the number of its vertices. The graph $\overline{G}$, the *complement* of G, is obtained from the complete graph on $|V(G)|$ vertices by deleting the edges of G. A graph $F=(V',E')$ is a *subgraph* of G if $V' \subseteq V(G)$ and $E' \subseteq E(G)$. For $S \subseteq V(G)$, $G[S]$ represents the *subgraph induced* by S in G. If G is a graph and H is a subgraph of G, then denote $G[V(G)\backslash V(H)]$ by $G\backslash H$. For $v \in V(G)$ and $S \subset V(G)$, the *neighborhood* $N_S(v)$ is the set of vertices in S which are adjacent to v. Furthermore, we define $N_S[v] = N_S(v) \cup \{v\}$. If $S = V(G)$, then we use $N(v)$ and $N[v]$ instead of $N_{V(G)}(v)$ and $N_{V(G)}[v]$, respectively. The *degree* of a vertex v in G is denoted by $d_G(v)$. Let S_n be a *star* on n vertices and C_m be a *cycle* on m vertices. We denote the *complete bipartite* whose partite sets are of order n and p by $K_{n,p}$.

Proof of Theorem 1

Let n_i be odd and $2n_{i+1} \geq n_i$ for every i. Consider $F = K_{-1+\sum_{i=1}^{k} n_i} \cup K_{n_k-1}$. Clearly, the graph F has order $-2+2n_k+\sum_{i=1}^{k-1} n_i$, without containing $\sum_{i=1}^{k} T_{n_i}$ and $\overline{F}$ contains no W_4. Hence,

$$R(\bigcup_{i=1}^{k} T_{n_i}, W_4) \geq -1 + 2n_k + \sum_{i=1}^{k-1} n_i = R(T_{n_k}, W_4) + \sum_{i=1}^{k-1} n_i. \quad (1)$$

To obtain the Ramsey number we use an induction on k. For $k=1$, we have $R(T_{n_1}, W_4) = 2n_1 - 1$ (by Theorem 1). For $k=2$, we show that $R(T_{n_1} \cup T_{n_2}, W_4) = 2n_2 - 1 + n_1 = R(T_{n_2}, W_4) + n_1$.

Let F' be a graph with $|F'| = 2n_2 - 1 + n_1 = 2n_1 - 1 + 2n_2 - n_1$. Assume that $\overline{F}'$ contains no W_4. We show that F' contains $T_{n_1} \cup S_{n_2}$. Since $2n_2 \geq n_1$, then $|F'| \geq 2n_1 - 1$. By Theorem **??** and Theorem 1, F' contains T_{n_1}. Write $L = F'\backslash T_{n_1}$. Thus $|L| = 2n_2 - 1$, such that L contains T_{n_2}. Hence, F' contains $T_{n_1} \cup T_{n_2}$. Therefore, $R(T_{n_1} \cup T_{n_2}, W_4) \leq 2n_2 - 1 + n_1$.

Suppose the theorem holds for every $r < k$. Let F_1 be a graph of order $-1+2n_k+\sum_{i=1}^{k-1} n_i$. Suppose $\overline{F_1}$ contains no W_4. By the assumption, F_1 contains $\sum_{i=1}^{k-1} T_{n_i}$. Let $L' = F_1 \backslash \sum_{i=1}^{k-1} T_{n_i}$. Thus $|L'| = 2n_k - 1$. Since $\overline{L'}$ contains no W_4, then by Theorem **??** and 1, $L' \supset T_{n_k}$.

Hence, F_1 contains $\sum_{i=1}^{k} T_{n_i}$. Therefore, we have

$$R(\bigcup_{i=1}^{k} T_{n_i}, W_4) \leq -1 + 2n_k + \sum_{i=1}^{k-1} n_i = R(T_{n_k}, W_4) + \sum_{i=1}^{k-1} n_i. \quad (2)$$

Proof of Theorem 2
We only prove the Theorem 2.2.

Let n_i be odd and $2n_{i+1} \geq n_i$ for every i. Consider $F \simeq K_{-1+\sum_{i=1}^{k} n_i} \cup K_{n_k-1}$. Clearly, the graph F has order $-2 + 2n_k + \sum_{i=1}^{k-1} n_i$, without containing $\sum_{i=1}^{k} S_{n_i}$ and $\overline{F}$ contains no W_4. Hence,

$$R(\bigcup_{i=1}^{k} S_{n_i}, W_4) \geq -1 + 2n_k + \sum_{i=1}^{k-1} n_i. \quad (3)$$

To obtain the Ramsey number we use an induction on k. For $k = 1$, we have $R(S_{n_1}, W_4) = 2n_1 - 1$ (by Theorem 1). For $k = 2$, we show that $R(S_{n_1} \cup S_{n_2}, W_4) = 2n_2 - 1 + n_1 = R(S_{n_2}, W_4) + n_1$.
Let F_1 be a graph with $|F_1| = 2n_2 - 1 + n_1 = 2n_1 - 1 + 2n_2 - n_1$. Assume that $\overline{F}_1$ contains no W_4. We show that F_1 contains $S_{n_1} \cup S_{n_2}$. Since $2n_2 \geq n_1$, then $|F_1| \geq 2n_1 - 1$. By Theorem 1, F_1 contains S_{n_1}. Write $L = F_1 \backslash S_{n_1}$. Thus $|L| = 2n_2 - 1$, such that L contains S_{n_2}. Hence, F_1 contains $S_{n_1} \cup S_{n_2}$. Therefore, $R(S_{n_1} \cup S_{n_2}, W_4) \leq 2n_2 - 1 + n_1$.

Suppose the theorem holds for every $r < k$. Let F_2 be a graph of order $-1 + 2n_k + \sum_{i=1}^{k-1} n_i$. Suppose $\overline{F_2}$ contains no W_4. By the assumption, F_2 contains $\bigcup_{i=1}^{k-1} S_{n_i}$. Let $L' = F_2 \backslash \bigcup_{i=1}^{k-1} S_{n_i}$. Thus $|L'| = 2n_k - 1$. Since $\overline{L'}$ contains no W_4, then by Theorem 1, $L' \supset S_{n_k}$. Hence, F_2 contains $\bigcup_{i=1}^{k} S_{n_i}$. Therefore, we have

$$R(\bigcup_{i=1}^{k} S_{n_i}, W_4) = -1 + 2n_k + \sum_{i=1}^{k-1} n_i = R(S_{n_k}, W_4) + \sum_{i=1}^{k-1} n_i. \quad (4)$$

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