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Note

The Ramsey numbers for disjoint unions of trees

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Abstract

For given graphs G and H, the *Ramsey number* $R(G, H)$ is the smallest natural number n such that for every graph F of order n: either F contains G or the complement of F contains H. In this paper, we investigate the Ramsey number $R(\cup G, H)$, where G is a tree and H is a wheel W_m or a complete graph K_m. We show that if $n \geqslant 3$, then $R(kS_n, W_4) = (k+1)n$ for $k \geqslant 2$, even n and $R(kS_n, W_4) = (k+1)n - 1$ for $k \geqslant 1$ and odd n. We also show that $R(\bigcup_{i=1}^{k} T_{n_i}, K_m) = R(T_{n_k}, K_m) + \sum_{i=1}^{k-1} n_i$.

Keywords: Ramsey number; Star; Wheel; Tree

1. Introduction

For given graphs G and H, the *Ramsey number* $R(G, H)$ is defined as the smallest positive integer n such that for any graph F of order n, either F contains G or $\overline{F}$ contains H, where $\overline{F}$ is the complement of F.

In 1972, Chvátal and Harary [6] established a useful lower bound for finding the exact Ramsey numbers $R(G, H)$, namely $R(G, H) \geqslant (\chi(G) - 1)(c(H) - 1) + 1$, where $\chi(G)$ is the chromatic number of G and $c(H)$ is the number of vertices of the largest component of H. Since then the Ramsey numbers $R(G, H)$ for many combinations of graphs G and H have been extensively studied by various authors, see a nice survey paper [9].

Let P_n be a *path* with n vertices and let W_m be a *wheel* of $m+1$ vertices that consists of a cycle C_m with one additional vertex being adjacent to all vertices of C_m. A *star* S_n is the graph on n vertices with one vertex of degree $n-1$, called the *center*, and $n-1$ vertices of degree 1. T_n is a *tree* with n vertices and a *cocktail-party graph* H_s is the graph which is obtained by removing s disjoint edges from K_{2s}.

Several results on Ramsey numbers have been obtained for wheels. For instance, Baskoro et al. [1] showed that for even $m \geqslant 4$ and $n \geqslant (m/2)(m-2)$, $R(P_n, W_m) = 2n - 1$. They also showed that $R(P_n, W_m) = 3n - 2$ for odd $m \geqslant 5$, and $n \geqslant ((m-1)/2)(m-3)$.

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For a combination of stars with wheels, Surahmat et al. [10] investigated the Ramsey numbers for large stars versus small wheels. Their result is as follows.

Theorem 1 (*Surahmat and Baskoro [10]*). *For* $n \geqslant 3$,

$$R(S_n, W_4) = \begin{cases} 2n+1 & \text{if } n \text{ is even,} \\ 2n-1 & \text{if } n \text{ is odd.} \end{cases}$$

For odd m, Chen et al. have shown in [4] that $R(S_n, W_m) = 3n-2$ for $m \geqslant 5$ and $n \geqslant m-1$. This result was strengthened by Hasmawati et al. in [8], by showing that this Ramsey number remains the same, as given in the following theorem.

Theorem 2 (*Hasmawati et al. [8]*). *If* m *is odd and* $n \geqslant ((m+1)/2) \geqslant 3$, *then* $R(S_n, W_m) = 3n-2$.

If $n \leqslant (m+2)/2$, Hasmawati [7] gave $R(S_n, W_m) = n+m-2$ for even m and odd n, or $R(S_n, W_m) = n+m-1$, otherwise.

Let G be a graph. The number of vertices in a maximum independent set of G denoted by $\alpha_0(G)$, and the union of s *vertices-disjoint* copies of G denoted sG.

Burr et al. in [3], showed that if the graph G has n_1 vertices and the graph H has n_2 vertices, then

$$n_1 s + n_2 t - D \leqslant R(sG, tH) \leqslant n_1 s + n_2 t - D + k,$$

where $D = \min\{s\alpha_0(G), t\alpha_0(H)\}$ and k is a constant depending only on G and H.

In the following theorem Chvátal gave the Ramsey number for a tree versus a complete graph.

Theorem 3 (*Chvátal [5]*). *For any natural number* n *and* m, $R(T_n, K_m) = (n-1)(m-1)+1$.

In this paper, we determine the Ramsey numbers $R(\cup G, H)$ of a disjoint union of a graph G versus a graph H, where G is either a star or a tree, and H is either a wheel or a complete graph.

The results are presented in the next three theorems.

Theorem 4. *If* m *is odd and* $n \geqslant (m+1)/2 \geqslant 3$, *then* $R(kS_n, W_m) = 3n-2+(k-1)n$.

Theorem 5. *For* $n \geqslant 3$,

$$R(kS_n, W_4) = \begin{cases} (k+1)n & \text{if } n \text{ is even and } k \geqslant 2, \\ (k+1)n-1 & \text{if } n \text{ is odd and } k \geqslant 1. \end{cases}$$

Theorem 6. *Let* $n_i \geqslant n_{i+1}$ *for* $i = 1, 2, \ldots, k-1$. *If* $n_i \geqslant (n_i - n_{i+1})(m-1)$ *for any* i, *then* $R(\bigcup_{i=1}^{k} T_{n_i}, K_m) = R(T_{n_k}, K_m) + \sum_{i=1}^{k-1} n_i$ *for an arbitrary* m.

Before proving the theorems, we present some notations used in this note. Let $G(V, E)$ be a graph. For any vertex $v \in V(G)$, the *neighborhood* $N(v)$ is the set of vertices adjacent to v in G. Furthermore, we define $N[v] = N(v) \cup \{v\}$. The degree of a vertex v in G is denoted by $d_G(v)$. The order of G, $|G|$ is the number of its vertices, and the minimum (maximum) degree of G is denoted by $\delta(G)(\Delta(G))$. For $S \subseteq V(G)$, $G[S]$ represents the subgraph induced by S in G. If G is a graph and H is a subgraph of G, then denote $V(G) \backslash V(H)$ by $G \backslash H$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. The *union* $G = G_1 \cup G_2$ has the vertex set $V = V_1 \cup V_2$ and the edge set $E = E_1 \cup E_2$. Their *join*, denoted $G_1 + G_2$, is the graph with the vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$.

2. The proofs of theorems

Proof of Theorem 4. Let m be odd and $n \geqslant (m+1)/2 \geqslant 3$. We shall use an induction on k. For $k = 1$, we have $R(S_n, W_m) = 3n-2$ (by Theorem 2). Assume the theorem holds for any $r < k$, namely $R(rS_n, W_m) = (3n-2)+(r-1)n$. We will show that $R(kS_n, W_m) = (3n-2)+(k-1)n$.

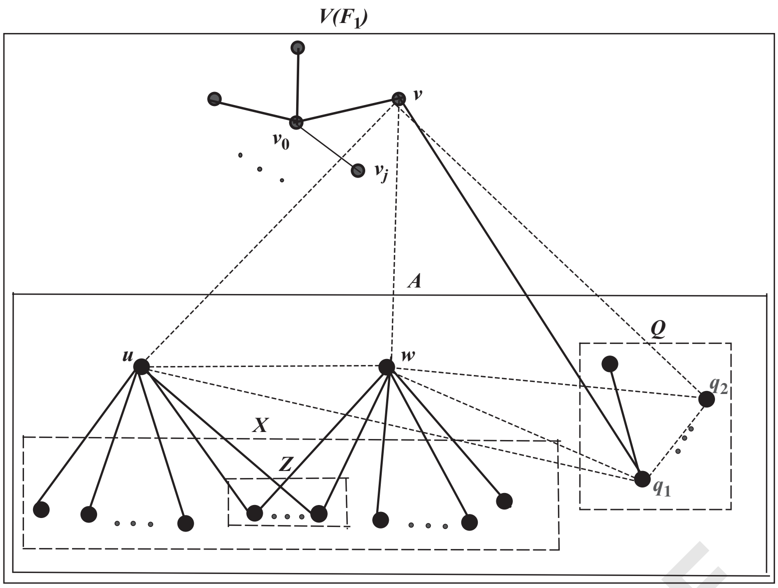

Fig. 1. The illustration of the proof of $R(2S_n, W_4) \leqslant 3n$.

Let F be a graph with $|F| = 3n - 2 + (k-1)n$. Suppose that $\overline{F}$ contains no W_m. Since $|F| \geqslant R(rS_n, W_m)$, then $F \supset (k-1)S_n$. Let $A = F \backslash (k-1)S_n$ and $T = F[A]$. Thus, $|T| = 3n - 2$. Since $\overline{T}$ contains no W_m, then by Theorem 2, $T \supseteq S_n$. Thus, F contains kS_n. Hence, we have $R(kS_n, W_m) \leqslant (3n-2) + (k-1)n$.

On the other hand, it is not difficult to see that $F_1 = K_{kn-1} \cup 2K_{n-1}$ contains no kS_n and its complement contains no W_m. Observe that F_1 has $3n - 3 + (k-1)n$ vertices. Therefore, we have $R(kS_n, W_m) \geqslant (3n-2) + (k-1)n$, and the assertion follows. □

Proof of Theorem 5. Let n be even, $n \geqslant 4$ and $k \geqslant 2$. Consider $F = (H_{(kn-2)/2} + K_1) \cup H_{n/2}$. Clearly, graph F has $(k+1)n - 1$ vertices and contains no kS_n. Its complement contains no W_4. Hence, $R(kS_n, W_4) \geqslant (k+1)n$. We will prove that $R(kS_n, W_4) = (k+1)n$ for $k \geqslant 2$. First we will show that $R(2S_n, W_4) = 3n$.

Let F_1 be a graph of order $3n$. Suppose $\overline{F}_1$ contains no W_4. By Theorem 1, we have $F_1 \supseteq S_n$. Let $V(S_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ with center v_0, $A = F_1 \backslash S_n$ and $T = F_1[A]$. Thus $|T| = 2n$.

If there exists $u \in T$ with $d_T(u) \geqslant (n-1)$, then T contains S_n. Hence F_1 contains $2S_n$. Therefore we assume that for every vertex $u \in T$, $d_T(u) \leqslant (n-2)$. Let $u, w \in T$ where $(u, w) \notin E(T)$. Consider $H = N[u] \cup N[w]$, $Q = T \backslash H$, $Z = N(u) \cap N(w)$, and $X = H \backslash \{u, w\}$ (see Fig. 1).

By contradiction suppose $d(u) \leqslant n - 3$. Then $0 \leqslant |Z| \leqslant n-3$, $2 \leqslant |H| \leqslant 2n-3$ and $2n - 2 \geqslant |Q| \geqslant 3 + |Z|$.

Observe that every $q \in Q$ is adjacent to at least $|Q| - 2$ other vertices of Q. (Otherwise, there exists $q \in Q$ which is not adjacent to at least two other vertices of Q, say q_1 and q_2. Then $\overline{T}$ will contain a $W_4 = \{q_1, u, q_2, q, w\}$ with w as a hub, a contradiction). Then, for all $q \in Q$, $d_Q(q) \geqslant |Q| - 2$.

Let $E(X \backslash Z, Q) = \{uv : u \in X \backslash Z, v \in Q\}$. If there exists $x \in X \backslash Z$ not adjacent to at least two vertices of Q, say q_1 and q_2, then $\overline{T}$ will contain $W_4 = \{q_1, x, q_2, u, w\}$ with w or u as a hub, a contradiction. Hence every $x \in X \backslash Z$ is adjacent to at least $|Q| - 1$ vertices in Q. Therefore, we have $|E(X \backslash Z, Q)| \geqslant |X \backslash Z| \cdot (|Q| - 1)$.

On the other hand, every vertex $q \in Q$ is incident with at most $(n-2) - d_Q(q) \leqslant (n-2) - (|Q| - 2) = n - |Q|$ edges from $X \backslash Z$. Thus $|E(X \backslash Z, Q)| \leqslant |Q| \cdot (n - |Q|)$.

Now, we will show that $|X \backslash Z| \cdot (|Q| - 1) > |Q| \cdot (n - |Q|)$, which leads to a contradiction.

Writing $|X \backslash Z| \cdot (|Q| - 1) = |X \backslash Z| \cdot |Q| - |X \backslash Z|$ and substituting $|X \backslash Z| = 2n - 2 - |Q| - |Z|$, we obtain $|X \backslash Z| \cdot (|Q| - 1) = |Q| \cdot (n - |Q|) + |Q| \cdot (n - 2 - |Z|) + |Q| - n - (n - 2 - |Z|)$. Noting $|Q| \geqslant 3 + |Z|$, it can be verified that $|Q| \cdot (n - 2 - |Z|) + |Q| - n - (n - 2 - |Z|) > 0$. Thus $|X \backslash Z| \cdot (|Q| - 1) > |Q| \cdot (n - |Q|)$. Hence there is no $u \in T$ such that $d(u) \leqslant n - 3$.

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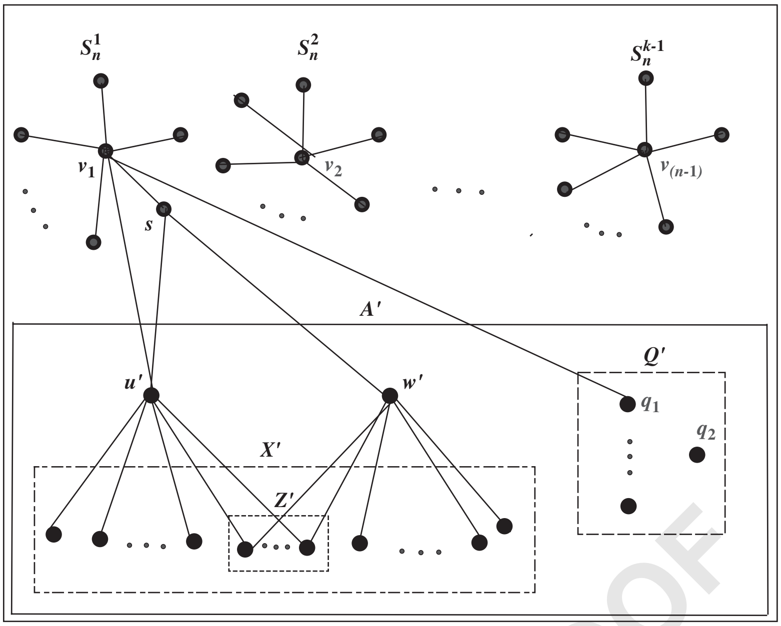

Fig. 2. The illustration of the proof of $R(kS_n, W_4) \leqslant (k+1)n$.

Therefore every vertex $u \in T$, $d_T(u) = n-2$. This implies $|Q| = 2 + |Z|$. Next, suppose $F_1[Q]$ is a complete graph. Then for every $q \in Q$, $d_Q(q) = |Q| - 1$. Consequently, every vertex in Q is incident with $(n-2) - d_Q(q) = n - 1 - |Q|$ edges from $X \backslash Z$. Similarly as in the previous argument, this implies that $|X \backslash Z| \cdot (|Q| - 1) > |Q| \cdot (n - 1 - |Q|)$, which is impossible. Hence $F_1[Q]$ is not a complete graph. Now, choose two vertices in Q which are not adjacent, call q_1 and q_2. Let $Y = \{q_1, q_2\} \cup \{u, w\}$, it is clear that Y is an independent set.

If there exists a vertex $v \in V(S_n)$ adjacent to at most one vertex in Y say q_1, then $\{v, u, q_1, q_2, w\}$ will induce a W_4 in $\overline{F}_1$, with a hub w, a contradiction. Therefore, every vertex $v \in V(S_n)$ is adjacent to at least two vertices in Y. Suppose v_0 and v_j in $V(S_n)$ are adjacent to y_1, y_2 and to y_3, y_4 in Y, respectively. Note that at least two $y_i's$ are distinct. Without loss of generality, assume $y_1 \neq y_3$. Since Y is independent, then we have two new stars, namely S_n' and S_n'', where $V(S_n') = S_n \backslash \{v_j\} \cup \{y_1\}$ with v_0 as the center and $V(S_n'') = N[y_3] \cup \{v_j\}$ with y_3 as the center (see Fig. 1). So, we have $F_1 \supseteq 2S_n$. Hence, $R(2S_n, W_4) = 3n$.

Now, assume the theorem holds for every $r < k$. We will show that $R(kS_n, W_4) = (k+1)n$. Let F_2 be a graph of order $(k+1)n$. Suppose $\overline{F}_2$ contains no W_4. We will show that $F_2 \supseteq kS_n$. By induction, $F_2 \supseteq (k-1)S_n$. Denote the $(k-1)S_n$ as $S_n^1, S_n^2, \ldots, S_n^{k-1}$ with the center $v_1, v_2, \ldots, v_{k-1}$, respectively. Writing $A' = F_2 \backslash (k-1)S_n$ and $T' = F_2[A']$. Thus $|T'| = 2n$.

Similarly, as in the case $k=2$, every vertex $u \in T'$, must have degree $n-2$. Next, let $u', w' \in T$ where $(u', w') \notin E(T')$, $H' = N[u'] \cup N[w']$, $Q' = T' \backslash H'$, and $Y' = \{q_1, q_2\} \cup \{u', w'\}$, where $q_1, q_2 \in Q'$ and $(q_1, q_2) \notin E(T')$ (see Fig. 2).

If the vertex $v \in V((k-1)S_n)$ is adjacent to at most one vertex in Y', say u', then $\overline{F}_2$ will contain $W_4 = \{u', q_1, v, q_2, w'\}$ with w' as a hub, a contradiction.

Therefore, every vertex $v \in V((k-1)S_n)$ is adjacent to at least two vertices in Y'. Suppose v_1 and s in $V(S_{n_1})$ are adjacent to u', q_1 and to u', w', respectively (see Fig. 2). Then, we will alter S_n^1 into $S_n^{1'}$ with $V(S_n^{1'}) = (S_n^1 \backslash \{s\}) \cup \{q_1\}$ and create a new star S_n^k where $V(S_n^k) = N[u'] \cup \{s\}$ with the center u'. Hence, we now have k disjoint stars, namely $S_n^{1'}, S_n^2, S_n^3, \ldots, S_n^{k-1}$ and S_n^k. Therefore, we have $R(kS_n, W_4) = (k+1)n$.

Let n be odd. Consider $F_3 = K_{kn-1} \cup K_{n-1}$. Clearly, the graph F_3 has order $(k+1)n - 2$, without containing kS_n and $\overline{F_3}$ contains no W_4. Hence, $R(kS_n, W_4) \geqslant (k+1)n - 1$. To obtain the Ramsey number we use an induction on k. For $k=1$, we have $R(S_n, W_4) = 2n-1$. Suppose the theorem holds for every $r < k$. We show that $R(kS_n, W_4) = (k+1)n - 1$. Let F_4 be a graph of order $(k+1)n - 1$. Suppose $\overline{F_4}$ contains no W_4. By the assumption, F_4 contains $(k-1)S_n$. Let

$B = F_4 \backslash (k-1)S_n$ and $L = F_4[B]$. Thus $|L| = 2n-1$. Since $\overline{L}$ contains no W_4, then by Theorem 1, $L \supset S_n$. Therefore, F_4 contains kS_n. The proof is now complete. □

Proof of Theorem 6. Let $n_i \geqslant n_{i+1}$ and $n_i \geqslant (n_i - n_{i+1})(m-1)$ for any i. Since $F = (m-2)K_{n_k - 1} \cup K_{\sum_{i=1}^{k} n_i - 1}$ has no $\bigcup_{i=1}^{k} T_{n_i}$ and its complement contains no K_m, then $R(\bigcup_{i=1}^{k} T_{n_i}, K_m) \geqslant (m-1)(n_k - 1) + \sum_{i=1}^{k-1} n_i + 1$. We fix m and apply an induction on k. For $k = 2$, we show that $R(T_{n_1} \cup T_{n_2}, K_m) = (m-1)(n_2 - 1) + n_1 + 1$.

Let F_1 be a graph with $|F_1| = (m-1)(n_2-1)+1+n_1$. Suppose $\overline{F_1}$ contains no K_m. Since $n_1 \geqslant n_2$, then we can write $n_1 - n_2 = q \geqslant 0$. Substitute $n_2 = n_1 - q$, then we obtain $|F_1| = (m-1)(n_1-q-1)+n_1+1 = (m-1)(n_1-1)-q(m-1)+n_1+1$ or $|F_1| = (m-1)(n_1-1)+1+[n_1-(n_1-n_2)(m-1)]$. Noting $n_1 - (n_1 - n_2)(m-1) \geqslant 0$, it can be verified that $|F_1| \geqslant (m-1)(n_1-1)+1$ i.e. $|F_1| \geqslant R(T_{n_1}, K_m)$. Hence, $F_1 \supseteq T_{n_1}$. Now, let $A = F_1 \backslash T_{n_1}$, and $H = F_1[A]$. Then $|H| = (m-1)(n_2-1)+1$. Since $\overline{H}$ contains no K_m, then by Theorem 3, $H \supseteq T_{n_2}$. Therefore, F_1 contains a subgraph $T_{n_1} \cup T_{n_2}$.

Next, assume the theorem holds for all $r < k$, namely $R(\bigcup_{i=1}^{r} T_{n_i}, K_m) = (m-1)(n_r - 1) + \sum_{i=1}^{r-1} n_i + 1$. We shall show that $R(\bigcup_{i=1}^{k} T_{n_i}, K_m) = (m-1)(n_k-1) + \sum_{i=1}^{k-1} n_i + 1$. Take an arbitrary graph F_2 with order $(m-1)(n_k-1)+\sum_{i=1}^{k-1} n_i + 1$. Suppose $\overline{F_2}$ contains no K_m. By induction, F_2 contains $\bigcup_{i=1}^{k-1} T_{n_i}$.

Writing $B = F_2 \backslash \bigcup_{i=1}^{k-1} T_{n_i}$, and $Q = F_2[B]$. Then $|Q| = (m-1)(n_k-1)+1$. Since $\overline{Q}$ contains no K_m, then Q contains T_{n_k}. Hence F_2 contains $\bigcup_{i=1}^{k} T_{n_i}$. Therefore, we have $R(\bigcup_{i=1}^{k} T_{n_i}, K_m) = (m-1)(n_k-1) + \sum_{i=1}^{k-1} n_i + 1$. The proof is now complete. □

3. Uncited reference

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