Applied Mathematical Sciences, Vol. 5, 2011, no. 71, 3531 - 3540

Quaternion Algebra-Valued Wavelet Transform

Mawardi Bahri

Department of Mathematics
Universitas Hasanuddin
KM 10 Tamalanrea 90245, Makassar, Indonesia
mawardibahri@gmail.com

Abstract

In the previous paper [7], we proposed the two-dimensional continuous quaternion wavelet transform (CQWT). In the present paper, using the orthogonality of harmonic exponential functions we give an alternative proof for inner product relation property of the CQWT.

Mathematics Subject Classification: 11R52, 42C40

Keywords: Quaternion Fourier transform, Admissible quaternion wavelets

1 Introduction

The quaternion Fourier transform (QFT), which is a nontrivial generalization of the real and complex Fourier transform (FT) using quaternion algebra has been of interest to researchers for some years (see e.g. [3, 5]). It was found that many FT properties still hold but others have to be modified. Based on the (right-sided) QFT, one can extend the classical wavelet transform (WT) to quaternion algebra while enjoying the same properties as in the classical case. In [10], using the (two-sided) QFT Traversoni proposed a discrete quaternion wavelet transform which was applied by Bayro-Corrochano [2] and Zhou et al. [11]. In [6, 8], we introduced an extension of the WT to Clifford algebra by means of the kernel of the Clifford Fourier transform [4].

The purpose of this paper is to provide an alternative proof for inner product relation property of the two-dimensional continuous quaternion wavelet transform (CQWT), which was recently studied in [7]. This property is very important to derive the inversion formula for the CQWT.

2 Basic Concept

In this section, we briefly review some basic ideas on quaternions, the (right-sided) QFT and the similitude group $SIM(2)$.

The quaternion algebra over $\mathbb{R}$, denoted by $\mathbb{H}$, is an associative non-commutative four-dimensional algebra,

$$\mathbb{H} = \{q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R}\}, \tag{1}$$

which obey Hamilton's multiplication rules

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}, \quad \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{2}$$

The quaternion conjugate of a quaternion q is given by

$$\bar{q} = q_0 - \boldsymbol{i}q_1 - \boldsymbol{j}q_2 - \boldsymbol{k}q_3, \qquad q_0, q_1, q_2, q_3 \in \mathbb{R}, \tag{3}$$

and it is an anti-involution, i.e.

$$\overline{qp} = \bar{p}\bar{q}. \tag{4}$$

From (3) we obtain the norm of $q \in \mathbb{H}$ defined as

$$|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \tag{5}$$

It is not difficult to see that

$$|qp| = |q||p|, \qquad \forall p, q \in \mathbb{H}. \tag{6}$$

It is convenient to introduce the inner product of two quaternion functions, $f, g : \mathbb{R}^2 \to \mathbb{H}$, as follows:

$$(f, g)_{L^2(\mathbb{R}^2;\mathbb{H})} = \int_{\mathbb{R}^2} f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\, d^2\boldsymbol{x}. \tag{7}$$

In particular, if $f = g$, then we obtain the associated norm

$$\|f\|_{L^2(\mathbb{R}^2;\mathbb{H})} = (f, f)^{1/2}_{L^2(\mathbb{R}^2;\mathbb{H})} = \left(\int_{\mathbb{R}^2} |f(\boldsymbol{x})|^2\, d^2\boldsymbol{x}\right)^{1/2}. \tag{8}$$

Based on quaternions we can define the (right-sided) QFT.

Definition 2.1 *The QFT of $f \in L^1(\mathbb{R}^2;\mathbb{H})$ is the function $\mathcal{F}_q\{f\}\colon \mathbb{R}^2 \to \mathbb{H}$ given by*

$$\mathcal{F}_q\{f\}(\boldsymbol{\omega}) = \widehat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2}\, d^2\boldsymbol{x}, \tag{9}$$

where $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$, $\boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2$, and the quaternion exponential product $e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2}$ is called the quaternion Fourier kernel.

Theorem 2.2 *Suppose that $f \in L^2(\mathbb{R}^2;\mathbb{H})$ and $\mathcal{F}_q\{f\} \in L^1(\mathbb{R}^2;\mathbb{H})$. Then the QFT of f is an invertible transform and its inverse is given by*

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \mathcal{F}_q\{f\}(\boldsymbol{\omega}) e^{\boldsymbol{j}\omega_2 x_2} e^{\boldsymbol{i}\omega_1 x_1}\, d^2\boldsymbol{\omega}. \tag{10}$$

As in the classical case, we obtain Plancherel's formula, specific to the (right-sided) QFT [3, 5],

$$(f,g)_{L^2(\mathbb{R}^2;\mathbb{H})} = \frac{1}{(2\pi)^2}(\mathcal{F}_q\{f\},\mathcal{F}_q\{g\})_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{11}$$

In particular, if $f = g$ we get Parseval's formula,

$$\|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})} = \frac{1}{(2\pi)^2}\|\mathcal{F}_q\{f\}\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{12}$$

Following Antoine et al. (see [1]), we consider the similitude group *SIM(2)* denoted by $\mathcal{G}$ on $\mathbb{R}^2$ associated with wavelets as follows.

$$\mathcal{G} = \mathbb{R}^+ \times SO(2) \times \mathbb{R}^2 = \{(a, r_\theta, \boldsymbol{b}) \,|\, a \in \mathbb{R}^+, r_\theta \in SO(2), \boldsymbol{b} \in \mathbb{R}^2\}, \tag{13}$$

where $SO(2)$ is the special orthogonal group of $\mathbb{R}^2$. The multiplication defined by (13) follows the group law

$$\{a, \boldsymbol{b}, r_\theta\}\{a', \boldsymbol{b}', r_{\theta'}\} = \{aa', \boldsymbol{b} + a r_\theta \boldsymbol{b}', r_{\theta+\theta'}\}. \tag{14}$$

The rotation $r_\theta \in SO(2)$ acts on $\boldsymbol{x} \in \mathbb{R}^2$ as usual,

$$r_\theta(\boldsymbol{x}) = (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta), \qquad 0 \le \theta < 2\pi. \tag{15}$$

The left Haar measure on $\mathcal{G}$ is given by

$$d\lambda(a,\theta,\boldsymbol{b}) = d\mu(a,\theta)d^2\boldsymbol{b},$$

where $d\mu(a,\theta) = \frac{da d\theta}{a^3}$ and $d\theta$ is the Haar measure on $SO(2)$. For the sake of simplicity, we write $d\mu = d\mu(a,\theta)$ and $d\lambda = d\lambda(a,\theta,\boldsymbol{b})$.

3 Construction of 2-D Quaternion Wavelets

Based on quaternions and the (right-sided) QFT, one can extend the real (or complex) wavelet transform to a quaternion wavelet transform. This section briefly introduces the basic facts of the two-dimensional continuous quaternion wavelet transform (CQWT).

3.1 Admissible Quaternion Wavelet

In the following we introduce the admissible quaternion wavelet.

Definition 3.1 (Admissible quaternion wavelet) *Let AQW denote the class of admissible quaternion wavelets $\psi \in L^2(\mathbb{R}^2;\mathbb{H})$ which satisfy the following admissibility condition, i.e.*

$$\int_{SO(2)}\int_{\mathbb{R}^+} |\hat{\psi}(ar_{-\theta}(\boldsymbol{\omega}))|^2 \frac{da d\theta}{a} \tag{16}$$

is a real positive constant independent of $\boldsymbol{\omega}$ satisfying $|\boldsymbol{\omega}| = 1$. Denote by C_ψ, the real positive constant.

Notice that according to (5) C_ψ is an invertible real constant. Using (1) we may decompose $\psi \in AQW$ into the following form

$$\psi(\boldsymbol{x}) = \psi_0(\boldsymbol{x}) + \boldsymbol{i}\psi_1(\boldsymbol{x}) + \boldsymbol{j}\psi_2(\boldsymbol{x}) + \boldsymbol{k}\psi_3(\boldsymbol{x}), \tag{17}$$

where $\psi_s \in L^2(\mathbb{R}^2;\mathbb{R})$ for $s = 0, 1, 2, 3$. Using (9) and the linearity of the (right-sided) QFT we may write (17) in the quaternionic frequency domain in the form

$$\begin{aligned}\mathcal{F}_q\{\psi\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} (\psi_0(\boldsymbol{x}) + \boldsymbol{i}\psi_1(\boldsymbol{x}) + \boldsymbol{j}\psi_2(\boldsymbol{x}) + \boldsymbol{k}\psi_3(\boldsymbol{x}))\, e^{-\boldsymbol{i}\omega_1 x_1} e^{-\boldsymbol{j}\omega_2 x_2}\, d^2\boldsymbol{x} \\ &= \mathcal{F}_q\{\psi_0\}(\boldsymbol{\omega}) + \boldsymbol{i}\mathcal{F}_q\{\psi_1\}(\boldsymbol{\omega}) + \boldsymbol{j}\mathcal{F}_q\{\psi_2\}(\boldsymbol{\omega}) + \boldsymbol{k}\mathcal{F}_q\{\psi_3\}(\boldsymbol{\omega}), \end{aligned} \tag{18}$$

where we assume that $\mathcal{F}_q\{\psi_s\} \in L^2(\mathbb{R}^2;\mathbb{R})$ for $s = 0, 1, 2, 3$.

Like for classical wavelets [9], the zero *th* moment of $\psi \in AQW$ vanishes,

$$\int_{\mathbb{R}^2} \psi(\boldsymbol{x})\, d^2\boldsymbol{x} = \int_{\mathbb{R}^2} (\psi_0(\boldsymbol{x}) + \boldsymbol{i}\psi_1(\boldsymbol{x}) + \boldsymbol{j}\psi_2(\boldsymbol{x}) + \boldsymbol{k}\psi_3(\boldsymbol{x}))\, d^2\boldsymbol{x} = 0. \tag{19}$$

It means that the integral of every component ψ_s of the quaternion mother wavelet is zero

$$\int_{\mathbb{R}^2} \psi_s\, d^2\boldsymbol{x} = 0, \quad s = 0, 1, 2, 3. \tag{20}$$

Definition 3.2 *For $\psi \in L^2(\mathbb{R}^2;\mathbb{H}), a \in \mathbb{R}^+, \boldsymbol{b} \in \mathbb{R}^2$, and $r_\theta \in SO(2)$, we define the unitary linear operator*

$$U_{a,\theta,\boldsymbol{b}} : L^2(\mathbb{R}^2;\mathbb{H}) \longrightarrow L^2(\mathcal{G};\mathbb{H}),$$

as

$$\left(U_{a,\theta,\boldsymbol{b}}(\psi)\right) = \psi_{a,\theta,\boldsymbol{b}}(\boldsymbol{x}) = \frac{1}{a}\,\psi\left(r_{-\theta}\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)\right). \tag{21}$$

The family of wavelets $\psi_{a,\theta,\boldsymbol{b}}$ are called *daughter quaternion wavelets* where a is a dilation parameter, $\boldsymbol{b}$ a translation vector parameter, and θ an $SO(2)$ rotation parameter.

By straightforward calculations we obtain the following lemma (see [7]).

Lemma 3.3 *Let ψ be an admissible quaternion function. Daughter quaternion wavelets (21) can be written in terms of the (right-sided) QFT as*

$$\mathcal{F}_q\{\psi_{a,\theta,\boldsymbol{b}}\}(\boldsymbol{\omega}) = a\, e^{-\boldsymbol{i}\omega_1 b_1} \left\{\widehat{\psi_0}(ar_{-\theta}(\boldsymbol{\omega})) + \boldsymbol{i}\widehat{\psi_1}(ar_{-\theta}(\boldsymbol{\omega}))\right\} e^{-\boldsymbol{j}\omega_2 b_2} + a\, e^{\boldsymbol{i}\omega_1 b_1} \left\{\boldsymbol{j}\widehat{\psi_2}(ar_{-\theta}(\boldsymbol{\omega})) + \boldsymbol{k}\widehat{\psi_3}(ar_{-\theta}(\boldsymbol{\omega}))\right\} e^{-\boldsymbol{j}\omega_2 b_2}. \tag{22}$$

3.2 2-D Continuous Quaternion Wavelet Transform

Definition 3.4 (CQWT) *The CQWT of a quaternion-valued function $f \in L^2(\mathbb{R}^2;\mathbb{H})$ with respect to $\psi \in AQW$ in two dimensions is defined by*

$$\begin{aligned} T_\psi : L^2(\mathbb{R}^2;\mathbb{H}) &\to L^2(\mathbb{R}^2;\mathbb{H}) \\ f &\mapsto T_\psi f(a,\theta,\boldsymbol{b}) = (f, \psi_{a,\theta,\boldsymbol{b}})_{L^2(\mathbb{R}^2;\mathbb{H})} \\ &= \int_{\mathbb{R}^2} f(\boldsymbol{x})\, \frac{1}{a}\, \overline{\psi\left(r_{-\theta}\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)\right)}\, d^2\boldsymbol{x}. \end{aligned} \tag{23}$$

The following results can be found in [7], which will be necessary to prove the main theorem.

Lemma 3.5 *Suppose that $\psi \in AQW$. If $\psi \in L^2(\mathbb{R}^2;\mathbb{H})$, then the CQWT (23) has a quaternion Fourier representation of the form*

$$T_\psi f(a,\theta,\boldsymbol{b}) = \frac{a}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega})\, e^{\boldsymbol{j} b_2\omega_2} \times \left[\overline{\widehat{\psi_{0l}}(ar_{-\theta}(\boldsymbol{\omega}))}\, e^{\boldsymbol{i} b_1\omega_1} + \overline{\widehat{\psi_{1l}}(ar_{-\theta}(\boldsymbol{\omega}))}\, e^{-\boldsymbol{i} b_1\omega_1}\right] d^2\boldsymbol{\omega}, \tag{24}$$

where $\widehat{\psi_{0l}}(ar_{-\theta}(\boldsymbol{\omega}))$ and $\widehat{\psi_{1l}}(ar_{-\theta}(\boldsymbol{\omega}))$ are defined by

$$\begin{aligned} \widehat{\psi_{0l}}(ar_{-\theta}(\boldsymbol{\omega})) &= \widehat{\psi_0}(ar_{-\theta}(\boldsymbol{\omega})) + \boldsymbol{i}\widehat{\psi_1}(ar_{-\theta}(\boldsymbol{\omega})), \\ \widehat{\psi_{1l}}(ar_{-\theta}(\boldsymbol{\omega})) &= \boldsymbol{j}\widehat{\psi_2}(ar_{-\theta}(\boldsymbol{\omega})) + \boldsymbol{k}\widehat{\psi_3}(ar_{-\theta}(\boldsymbol{\omega})). \end{aligned} \tag{25}$$

Lemma 3.6 *Let $\psi \in L^2(\mathbb{R}^2;\mathbb{H})$ be a quaternion valued wavelet. If $\mathcal{F}_q\{\psi\} = \mathcal{F}_q\{\psi_0\} + \boldsymbol{k}\mathcal{F}_q\{\psi_3\}$, then equation (24) can be expressed as*

$$T_\psi f(a,\theta,\boldsymbol{b}) = \mathcal{F}_q^{-1}\left(a\widehat{f}(\cdot)\widehat{\psi_0}(ar_{-\theta}(\cdot))\right)(\boldsymbol{b}) + \mathcal{F}_q^{-1}\left(a\widehat{f}(\cdot)\boldsymbol{k}\overline{\widehat{\psi_3}(ar_{-\theta}(\cdot))}\right)(-\boldsymbol{b}). \tag{26}$$

The following proposition is a particular case of the above lemma.

Proposition 3.7 *Let $\psi \in L^2(\mathbb{R}^2;\mathbb{H})$ be a quaternion valued wavelet.*

(i). If $\mathcal{F}_q\{\psi\} = \mathcal{F}_q\{\psi_0\} \in \mathbb{R}$, then equation (24) has the form

$$T_\psi f(a,\theta,\boldsymbol{b}) = \frac{a}{(2\pi)^2}\int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega})\,\widehat{\psi}(ar_{-\theta}(\boldsymbol{\omega}))\,e^{\boldsymbol{j}b_2\omega_2}e^{\boldsymbol{i}b_1\omega_1}\,d^2\boldsymbol{\omega}. \tag{27}$$

Or, equivalently,

$$\mathcal{F}_q(T_\psi f(a,\theta,\boldsymbol{.}))(\boldsymbol{\omega}) = a\widehat{f}(\boldsymbol{\omega})\,\widehat{\psi}(ar_{-\theta}(\boldsymbol{\omega})). \tag{28}$$

(ii). If $\mathcal{F}_q\{\psi\} = \boldsymbol{k}\mathcal{F}_q\{\psi_3\}$, then we may rewrite equation (24) in the form

$$T_\psi f(a,\theta,\boldsymbol{b}) = \frac{a}{(2\pi)^2}\int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega})\overline{\widehat{\psi}(ar_{-\theta}(\boldsymbol{\omega}))}\,e^{-\boldsymbol{j}b_2\omega_2}e^{-\boldsymbol{i}b_1\omega_1}\,d^2\boldsymbol{\omega}. \tag{29}$$

Or, equivalently,

$$T_\psi f(a,\theta,\boldsymbol{b}) = \mathcal{F}_q^{-1}\left(a\widehat{f}(\cdot)\overline{\widehat{\psi}(ar_{-\theta}(\cdot))}\right)(-\boldsymbol{b}). \tag{30}$$

4 Reproducing Formula

In an attempt to reconstruct a original signal f from its CQWT, we have the following result. Using the orthogonality of harmonic exponential functions we give an alternative proof of this fundamental property.

Theorem 4.1 (Inner product relation) *Suppose that $\psi = \psi_0 + \boldsymbol{i}\psi_1 + \boldsymbol{j}\psi_2 + \boldsymbol{k}\psi_3 \in L^2(\mathbb{R}^2;\mathbb{H}) \in AQW$ be a quaternion admissible wavelet which defines the CQWT (23).*

(i). Assume that $\mathcal{F}_q\{\psi\} = \mathcal{F}_q\{\psi_0\} \in \mathbb{R}$, then for every $f, g \in L^2(\mathbb{R}^2;\mathbb{H}) \cap L^1(\mathbb{R}^2;\mathbb{H})$ we have

$$(T_\psi f, T_\psi g)_{L^2(\mathcal{G};\mathbb{H})} = C_\psi (f,g)_{L^2(\mathbb{R}^2;\mathbb{H})}, \tag{31}$$

(ii). Assume that $\mathcal{F}_q\{\psi\} = \boldsymbol{k}\mathcal{F}_q\{\psi_3\}$, then for every $f, g \in L^2(\mathbb{R}^2;\mathbb{H}) \cap L^1(\mathbb{R}^2;\mathbb{H})$ we have

$$(T_\psi f, T_\psi g)_{L^2(\mathcal{G};\mathbb{H})} = C_\psi (f,g)_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{32}$$

Remark 4.1 *It is easy to see that the above theorem is not valid if $\mathcal{F}_q\{\psi\}$ is full quaternion. It is worth noting here that if $f = g$, then Theorem 4.1 takes the form*

$$\|T_\psi f\|^2_{L^2(\mathcal{G};\mathbb{H})} = C_\psi \|f\|^2_{L^2(\mathbb{R}^2;\mathbb{H})}. \tag{33}$$

Proof. Compare to the proof of Theorem 2 in [7].

(i). To prove this theorem, we note that since $\mathcal{F}_q\{\psi\} = \mathcal{F}_q\{\psi_0\} \in \mathbb{R}$, then (24) becomes (27). By inserting (27) into the left side of (31), we immediately obtain

$$\begin{aligned}
&(T_\psi f, T_\psi g)_{L^2(\mathcal{G};\mathbb{H})} \\
&= \int_{SO(2)} \int_{\mathbb{R}^+} \frac{a^2}{(2\pi)^4} \left(\int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega}) e^{\boldsymbol{j} b_2\omega_2} \hat{\psi}(a r_\theta^{-1}(\boldsymbol{\omega})) e^{\boldsymbol{i} b_1\omega_1} d^2\boldsymbol{\omega} \right.\right. \\
&\qquad \left.\left. \times \int_{\mathbb{R}^2} \overline{\left\{ \hat{g}(\boldsymbol{\omega}') e^{\boldsymbol{j} b_2\omega_2'} \hat{\psi}(a r_\theta^{-1}(\boldsymbol{\omega}')) e^{\boldsymbol{i} b_1\omega_1'} \right\}} d^2\boldsymbol{\omega}' \right] d^2\boldsymbol{b} \right) d\mu.
\end{aligned} \tag{34}$$

Since $\hat{\psi}(a r_\theta^{-1}(\boldsymbol{\omega}))$ is a real valued wavelet, then equation (34) reduces to

$$\begin{aligned}
&(T_\psi f, T_\psi g)_{L^2(\mathcal{G};\mathbb{H})} \\
&= \int_{SO(2)} \int_{\mathbb{R}^+} \frac{a^2}{(2\pi)^4} \left(\int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega}) e^{\boldsymbol{j} b_2\omega_2} \hat{\psi}(a r_\theta^{-1}(\boldsymbol{\omega})) e^{\boldsymbol{i} b_1\omega_1}\, d^2\boldsymbol{\omega} \right.\right. \\
&\qquad \left.\left. \times \int_{\mathbb{R}^2} e^{-\boldsymbol{i} b_1\omega_1'}\, \hat{\psi}(a r_\theta^{-1}(\boldsymbol{\omega}')) e^{-\boldsymbol{j} b_2\omega_2'}\, \overline{\hat{g}(\boldsymbol{\omega}')} d^2\boldsymbol{\omega}' \right] d^2\boldsymbol{b} \right) d\mu.
\end{aligned} \tag{35}$$

Notice that $\hat{\psi}(a r_\theta^{-1}(\boldsymbol{\omega}))\hat{\psi}(a r_\theta^{-1}(\boldsymbol{\omega}')) \in \mathbb{R}$. Hence,

$$\begin{aligned}
&(T_\psi f, T_\psi g)_{L^2(\mathcal{G};\mathbb{H})} \\
&= \int_{SO(2)} \int_{\mathbb{R}^+} \frac{a^2}{(2\pi)^4} \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega}) e^{\boldsymbol{j} b_2\omega_2} e^{\boldsymbol{i} b_1\omega_1} e^{-\boldsymbol{i} b_1\omega_1'} \right. \\
&\qquad \left. \times\, \hat{\psi}(a r_\theta^{-1}(\boldsymbol{\omega}))\hat{\psi}(a r_\theta^{-1}(\boldsymbol{\omega}')) e^{-\boldsymbol{j} b_2\omega_2'}\, \overline{\hat{g}(\boldsymbol{\omega}')} d^2\boldsymbol{\omega}' d^2\boldsymbol{\omega} d^2\boldsymbol{b} \right) d\mu.
\end{aligned} \tag{36}$$

Furthermore, we get

$$\begin{aligned}
&(T_\psi f, T_\psi g)_{L^2(\mathcal{G};\mathbb{H})} \\
&= \int_{SO(2)} \int_{\mathbb{R}^+} \frac{a^2}{(2\pi)^4} \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega}) \hat{\psi}(a r_\theta^{-1}(\boldsymbol{\omega})) e^{\boldsymbol{j} b_2\omega_2} e^{\boldsymbol{i} b_1(\omega_1 - \omega_1')} d^2\boldsymbol{b} \right. \\
&\qquad \left. \times\, e^{-\boldsymbol{j} b_2\omega_2'} \hat{\psi}(a r_\theta^{-1}(\boldsymbol{\omega}')) \overline{\hat{g}(\boldsymbol{\omega}')} d^2\boldsymbol{\omega}'\, d^2\boldsymbol{\omega} \right) d\mu.
\end{aligned} \tag{37}$$

It follows, therefore, from the orthogonality of harmonic exponential functions we easily obtain

$$
\begin{aligned}
&(T_\psi f, T_\psi g)_{L^2(\mathcal{G};\mathbb{H})} \\
&= \int_{SO(2)}\int_{\mathbb{R}^+}\frac{1}{(2\pi)^2}\left(\int_{\mathbb{R}^2}\hat{f}(\boldsymbol{\omega})\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}))\right. \\
&\qquad\qquad \left.\times\int_{\mathbb{R}^2}\delta(\boldsymbol{\omega}-\boldsymbol{\omega}')\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}'))\overline{\hat{g}(\boldsymbol{\omega}')}d^2\boldsymbol{\omega}'\,d^2\boldsymbol{\omega}\right)\frac{dad\theta}{a} \\
&= \frac{1}{(2\pi)^2}\int_{SO(2)}\left(\int_{\mathbb{R}^+}\hat{f}(\boldsymbol{\omega})\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}))\right. \\
&\qquad\qquad \left.\times\int_{\mathbb{R}^2}\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}))\overline{\hat{g}(\boldsymbol{\omega})}\,d^2\boldsymbol{\omega}\right)\frac{dad\theta}{a} \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\hat{f}(\boldsymbol{\omega})\underbrace{\left(\int_{SO(2)}\int_{\mathbb{R}^+}|\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}))|^2\frac{dad\theta}{a}\right)}_{C_\psi \text{ is a real constant}}\overline{\hat{g}(\boldsymbol{\omega})}\,d^2\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^2}C_\psi\int_{\mathbb{R}^2}\hat{f}(\boldsymbol{\omega})\overline{\hat{g}(\boldsymbol{\omega})}\,d^2\boldsymbol{\omega} \\
&\overset{(11)}{=} C_\psi\int_{\mathbb{R}^2}f(\boldsymbol{x})\overline{g(\boldsymbol{x})}\,d^2\boldsymbol{x} \\
&= C_\psi(f,g)_{L^2(\mathbb{R};\mathbb{H})}.
\end{aligned}
\tag{38}
$$

In the third equality we applied Fubini's theorem to reverse the integration order. □

(ii). From the assumption of $\mathcal{F}_q\{\psi\} = \boldsymbol{k}\mathcal{F}_q\{\psi_3\}$, then (24) becomes (29). By inserting (29) into the left side of (32), we immediately obtain

$$
\begin{aligned}
&(T_\psi f, T_\psi g)_{L^2(\mathcal{G};\mathbb{H})}\\
&= \int_{SO(2)} \int_{\mathbb{R}^+} \frac{a^2}{(2\pi)^4} \left(\int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega}) e^{\boldsymbol{j} b_2\omega_2} \overline{\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}))} e^{-\boldsymbol{i} b_1\omega_1} d^2\boldsymbol{\omega} \right.\right.\\
&\qquad \times \left.\left. \int_{\mathbb{R}^2} \overline{\left\{ \hat{g}(\boldsymbol{\omega}') e^{\boldsymbol{j} b_2\omega_2'} \overline{\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}'))} e^{-\boldsymbol{i} b_1\omega_1'} \right\}} d^2\boldsymbol{\omega}' \right] d^2\boldsymbol{b} \right) d\mu\\
&= \int_{SO(2)} \int_{\mathbb{R}^+} \frac{a^2}{(2\pi)^4} \left(\int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega}) \overline{\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}))} e^{-\boldsymbol{j} b_2\omega_2} e^{-\boldsymbol{i} b_1\omega_1} d^2\boldsymbol{\omega} \right.\right.\\
&\qquad \times \left.\left. \int_{\mathbb{R}^2} \overline{\left\{ \hat{g}(\boldsymbol{\omega}') \overline{\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}'))} e^{-\boldsymbol{j} b_2\omega_2'} e^{-\boldsymbol{i} b_1\omega_1'} \right\}} d^2\boldsymbol{\omega}' \right] d^2\boldsymbol{b} \right) d\mu\\
&= \int_{SO(2)} \int_{\mathbb{R}^+} \frac{a^2}{(2\pi)^4} \left(\int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega}) \overline{\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}))} e^{-\boldsymbol{j} b_2\omega_2} e^{-\boldsymbol{i} b_1\omega_1}\, d^2\boldsymbol{\omega} \right.\right.\\
&\qquad \times \left.\left. \int_{\mathbb{R}^2} e^{\boldsymbol{i} b_1\omega_1'}\, e^{\boldsymbol{j} b_2\omega_2'}\, \hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}')) \overline{\hat{g}(\boldsymbol{\omega}')} d^2\boldsymbol{\omega}' \right] d^2\boldsymbol{b} \right) d\mu\\
&= \int_{SO(2)} \int_{\mathbb{R}^+} \frac{a^2}{(2\pi)^4} \left(\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega}) \overline{\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}))} e^{-\boldsymbol{j} b_2\omega_2} e^{\boldsymbol{i} b_1(\omega_1'-\omega_1)} \right.\\
&\qquad \times \left. \int_{\mathbb{R}^2} e^{-\boldsymbol{j} b_2\omega_2'} \hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}')) \overline{\hat{g}(\boldsymbol{\omega}')} d^2\boldsymbol{\omega}'\, d^2\boldsymbol{\omega} \right) d^2\boldsymbol{b}\, d\mu\\
&= \int_{SO(2)} \int_{\mathbb{R}^+} \frac{1}{(2\pi)^2} \left(\int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega}) \overline{\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}))} \right.\\
&\qquad \times \left. \int_{\mathbb{R}^2} \delta(\boldsymbol{\omega}'-\boldsymbol{\omega}) \hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}')) \overline{\hat{g}(\boldsymbol{\omega}')} d^2\boldsymbol{\omega}'\, d^2\boldsymbol{\omega} \right) \frac{da d\theta}{a}\\
&= \int_{SO(2)} \int_{\mathbb{R}^+} \frac{1}{(2\pi)^2} \left(\int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega}) \overline{\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}))} \right.\\
&\qquad \times \left. \int_{\mathbb{R}^2} \hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega})) \overline{\hat{g}(\boldsymbol{\omega})}\, d^2\boldsymbol{\omega} \right) \frac{da d\theta}{a}\\
&= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega}) \underbrace{\left(\int_{SO(2)} \int_{\mathbb{R}^+} |\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}))|^2 \frac{da d\theta}{a} \right)}_{C_\psi \text{ is a real constant}} \overline{\hat{g}(\boldsymbol{\omega})}\, d^2\boldsymbol{\omega}\\
&= \frac{1}{(2\pi)^2} C_\psi \int_{\mathbb{R}^2} \hat{f}(\boldsymbol{\omega}) \overline{\hat{g}(\boldsymbol{\omega})}\, d^2\boldsymbol{\omega}\\
&\overset{(11)}{=} C_\psi \int_{\mathbb{R}^2} f(\boldsymbol{x}) \overline{g(\boldsymbol{x})}\, d^2\boldsymbol{x}\\
&= C_\psi (f,g)_{L^2(\mathbb{R};\mathbb{H})}, \qquad (39)
\end{aligned}
$$

where in the second equality we have used the fact that $e^{\boldsymbol{j} b_2\omega_2}\hat{\psi} = \hat{\psi}\, e^{-\boldsymbol{j} b_2\omega_2} (\hat{\psi} = \boldsymbol{k}\hat{\psi}_3)$. This proves the theorem. □

References

[1] J. P. Antoine and R. Murenzi, Two-dimensional directional wavelet and the scale-angle representation, *Signal Processing,* **52** (1996), 259–281.

[2] E. Bayro-Corrochano, The theory and use of the quaternion wavelet transform, *Journal of Mathematical Imaging and Vision,* **24** (2006), 19–35.

[3] E. Hitzer, Quaternion Fourier transform on quaternion fields and generalizations, *Advances in Applied Clifford Algebras,* **17** (2007), 497–517.

[4] E. Hitzer and B. Mawardi, Clifford Fourier transform on multivector fields and uncertainty principle for dimensions $n = 2$ (mod 4) and $n = 3$ (mod 4), *Advances in Applied Clifford Algebras,* **18** (2008), 715–736.

[5] B. Mawardi, E. Hitzer, A. Hayashi, and R. Ashino, An uncertainty principle for quaternion Fourier transform, *Computers and Mathematics with Applications,* **56** (2008), 2411–2417.

[6] B. Mawardi, S. Adji and J. Zhao, Clifford algebra-valued admissible wavelet transform on multivector fields, *Advances in Applied Clifford Algebras,* **21** (2011), 13–30.

[7] B. Mawardi, R. Ashino and R. Vaillancourt, Two-dimensional quaternion wavelet transform, *Applied Mathematics and Computation,* **218** (2011), 10–21.

[8] B. Mawardi and E. Hitzer, Clifford algebra $Cl_{3,0}$-valued wavelet transformation, Clifford wavelet uncertainty inequality and Clifford Gabor wavelets, *International Journal of Wavelets, Multiresolution and Information Processing,* **5** (2007), 997–1019.

[9] S. Mallat, *A Wavelet Tour of Signal Processing,* Academic Press, San Diego, CA, 1999.

[10] L. Traversoni, Imaging analysis using quaternion wavelet, *Geometric Algebra with Applications in Science and Engineering, E. B. Corrochano and G. Sobczyk (ed.),* Birkhäuser, Boston, 2001.

[11] J. Zhou, Y. Xu and X. Yang, Quaternion wavelet phase based stereo matching for uncalibrated images, *Pattern Recognition Letters,* **28** (2007), 1509–1522.

Received: May, 2011