Acta Mathematica Sinica, English Series

DOI: 10.1007/s10114-011-9655-0
Http://www.ActaMath.com

Real Clifford Windowed Fourier Transform

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Abstract We study the windowed Fourier transform in the framework of Clifford analysis, which we call the Clifford windowed Fourier transform (CWFT). Based on the spectral representation of the Clifford Fourier transform (CFT), we derive several important properties such as shift, modulation, reconstruction formula, orthogonality relation, isometry, and reproducing kernel. We also present an example to show the differences between the classical windowed Fourier transform (WFT) and the CWFT. Finally, as an application we establish a Heisenberg type uncertainty principle for the CWFT.

Keywords Multivector-valued function, Clifford analysis, Clifford Fourier transform, uncertainty principle

MR(2000) Subject Classification 15A66, 42B10, 30G35

1 Introduction

The classical windowed Fourier transform (WFT) was originally introduced by Gabor [1], which may be called the Gabor transform. He concluded that the WFT is a powerful tool for the analysis of signals. The effectiveness of the WFT is a result of its providing a unique representation for the signals in terms of the windowed Fourier kernel. Recently, some authors [2–7] have extensively studied the WFT and its properties from a mathematical point of view. Nowadays the WFT has effectively been applied in many fields of science and engineering, such as image analysis and image compression, object and pattern recognition, computer vision, optics, and filter banks (see e.g. [6, 8, 9]).

Received November 16, 2009, accepted May 6, 2010
The second author is supported by the Malaysian Research Grant (Fundamental Research Grant Scheme) from the Universiti Sains Malaysia; the third author is supported by SRF for ROCS, SEM, China and supported by NNSF of China (Grant Nos. 10871048 and 10931001) and Scientific Research Foundation of Beijing Normal University

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The WFT on n-dimensional Euclidean space of a function $f \in L^2(\mathbb{R}^n)$ with respect to the window function $g \in L^2(\mathbb{R}^n) \setminus \{0\}$ is given by

$$\mathcal{G}_g f(\boldsymbol{\omega}, \boldsymbol{b}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\boldsymbol{x}) \, \overline{g_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x})} \, d^n\boldsymbol{x}, \tag{1.1}$$

where the window daughter functions $g_{\boldsymbol{\omega},\boldsymbol{b}}$ are called the windowed Fourier kernel defined by

$$g_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x}) = g(\boldsymbol{x} - \boldsymbol{b}) \mathrm{e}^{\sqrt{-1}\,\boldsymbol{\omega}\cdot\boldsymbol{x}}. \tag{1.2}$$

Clearly, the window function g is first translated by the parameter $\boldsymbol{b}$ and then modulated by the complex Fourier kernel $\mathrm{e}^{\sqrt{-1}\,\boldsymbol{\omega}\cdot\boldsymbol{x}}$. In the case if the window function g is the Gaussian function, we obtain complex Gabor filters. These complex Gabor filters are fairly successful for many applications because they were well localized in both the spatial and frequency domains.

In this paper, we construct higher dimensional windowed Fourier transform in the framework of Clifford analysis. This generalization enables us to establish the Clifford Gabor filters [10–12], which can extend the applications of the complex Gabor filters. Based on the basic properties of Clifford algebra and its Fourier transform, we demonstrate the properties of the CWFT. We subsequently apply some of these properties and the uncertainty principle for the CFT [13, 14] to establish a generalized CWFT uncertainty principle.

This paper is organized as follows. In Section 2, we briefly review the basic knowledge of the Clifford algebra and its Fourier transform used in the paper. In Section 3, we construct the CWFT and derive some of its important properties. We then give an example of the CWFT to show the differences between the CWFT and the WFT. In Section 4, we formulate and prove a Heisenberg type uncertainty principle for the CWFT. The proof of this principle involves the properties of the CWFT and the uncertainty principle for the CFT.

2 Preliminaries

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_n\}$ be an orthonormal vector basis of the real n-dimensional Euclidean vector space $\mathbb{R}^n$. Clifford algebra (see [15–17]) over $\mathbb{R}^n$ denoted by $\mathcal{G}_n$ then has the graded 2^n-dimensional basis

$$\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n, \boldsymbol{e}_{12}, \boldsymbol{e}_{31}, \boldsymbol{e}_{23}, \ldots, i_n = \boldsymbol{e}_1\boldsymbol{e}_2 \cdots \boldsymbol{e}_n\}, \tag{2.1}$$

where i_n is called the unit oriented pseudoscalar. Observe that $i_n^2 = -1$ for $n = 2, 3 \pmod 4$.

The associative geometric multiplication of the basis vectors is governed by the rules:

$$\begin{aligned} &\boldsymbol{e}_k \, \boldsymbol{e}_l = -\boldsymbol{e}_l \, \boldsymbol{e}_k \quad \text{for } k \neq l, \ 1 \leq k, l \leq n, \\ &\boldsymbol{e}_k^2 = 1 \quad \text{for } 1 \leq k \leq n. \end{aligned} \tag{2.2}$$

We may decompose $\mathcal{G}_n$ as the sum of an odd part $\mathcal{G}_n^-$ and an even part $\mathcal{G}_n^+$, i.e.,

$$\mathcal{G}_n = \mathcal{G}_n^+ \oplus \mathcal{G}_n^-. \tag{2.3}$$

It is straightforward to verify that $\mathcal{G}_n^+$ is closed under multiplication, but $\mathcal{G}_n^-$ is not. For this reason $\mathcal{G}_n^+$ is often used to represent a rotation-dilation in n-dimensions.

The general elements of Clifford algebra are called multivectors. Every multivector $f \in \mathcal{G}_n$ can be represented in the form

$$f = \sum_A f_A \boldsymbol{e}_A, \tag{2.4}$$

where $f_A \in \mathbb{R}$, $\boldsymbol{e}_A = \boldsymbol{e}_{\alpha_1\alpha_2\cdots\alpha_k} = \boldsymbol{e}_{\alpha_1}\boldsymbol{e}_{\alpha_2}\cdots\boldsymbol{e}_{\alpha_k}$, and $1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_k \le n$ with $\alpha_j \in \{1, 2, \ldots, n\}$. For convenience, we introduce $\langle f \rangle_k = \sum_{|A|=k} f_A \boldsymbol{e}_A$ to denote the *k-vector part* of f $(k = 0, 1, 2, \ldots, n)$, then

$$f = \sum_{k=0}^{k=n} \langle f \rangle_k = \langle f \rangle + \langle f \rangle_1 + \langle f \rangle_2 + \cdots + \langle f \rangle_n, \tag{2.5}$$

where $\langle \cdots \rangle_0 = \langle \cdots \rangle$.

According to (2.3) and (2.5), a multivector $f \in \mathcal{G}_n, n = 2 \pmod 4$ may be decomposed as a sum of even part f_{even} and odd part f_{odd}. Thus

$$f = f_{\text{even}} \oplus f_{\text{odd}}, \tag{2.6}$$

where

$$\begin{aligned} f_{\text{even}} &= \langle f \rangle + \langle f \rangle_2 + \cdots + \langle f \rangle_r, \quad r = 2s, s \in \mathbb{N}, s \le \frac{n}{2}, \\ f_{\text{odd}} &= \langle f \rangle_1 + \langle f \rangle_3 + \cdots + \langle f \rangle_r, \quad r = 2s+1, s \in \mathbb{N}, s < \frac{n}{2}. \end{aligned} \tag{2.7}$$

Later, we shall see that the distinction between even and odd grade multivectors is very important, because the even multivectors commute with i_n but the odd multivectors anti-commute with i_n.

The reverse $\tilde{f}$ of a multivector f is an anti-automorphism given by

$$\tilde{f} = \sum_{k=0}^{k=n} (-1)^{k(k-1)/2} \langle f \rangle_k, \tag{2.8}$$

which satisfies $\widetilde{fg} = \tilde{g}\tilde{f}$ for every $f, g \in \mathcal{G}_n$. The multivector $f \in \mathcal{G}_n$ is called a paravector if (2.5) takes the form

$$f = \langle f \rangle + \langle f \rangle_1 = f_0 + \sum_{i=1}^{n} f_i\, \boldsymbol{e}_i. \tag{2.9}$$

From Equation (2.9) it is not difficult to check that the geometric product $f\tilde{f}$ is scalar valued.

The scalar product of multivectors $f, \tilde{g}$ is defined as the scalar part of the geometric product $f\tilde{g}$ of multivectors

$$\langle f\tilde{g} \rangle = f * \tilde{g} = \sum_A f_A g_A, \tag{2.10}$$

which leads to a cyclic product symmetry

$$\langle pqr \rangle = \langle qrp \rangle, \quad \forall\, p, q, r \in \mathcal{G}_n. \tag{2.11}$$

In particular, if $f = g$ in (2.10), then we obtain the modulus (or magnitude) $|f|$ of a multivector $f \in \mathcal{G}_n$ defined as

$$|f|^2 = f * \tilde{f} = \sum_A f_A^2. \tag{2.12}$$

Definition 2.1 *Let $\mathcal{G}_n$ be the real Clifford algebra of n-dimensional Euclidean space. A Clifford algebra module $L^2(\mathbb{R}^n;\mathcal{G}_n)$ is defined by*

$$L^2(\mathbb{R}^n;\mathcal{G}_n) = \left\{ f : \mathbb{R}^n \longrightarrow \mathcal{G}_n, \boldsymbol{x} \to \sum_A f_A(\boldsymbol{x}) e_A \;\middle|\; \|f\|_{L^2(\mathbb{R}^n;\mathcal{G}_n)} < \infty \right\}. \tag{2.13}$$

It is convenient to introduce an inner product for two multivector functions $f, g : \mathbb{R}^n \to \mathcal{G}_n$ as follows:

$$(f,g)_{L^2(\mathbb{R}^n;\mathcal{G}_n)} = \int_{\mathbb{R}^n} f(\boldsymbol{x}) \widetilde{g(\boldsymbol{x})}\, d^n\boldsymbol{x} = \sum_{A,B} \boldsymbol{e}_A \widetilde{\boldsymbol{e}_B} \int_{\mathbb{R}^n} f_A(\boldsymbol{x}) g_B(\boldsymbol{x})\, d^n\boldsymbol{x}. \tag{2.14}$$

Thus for $f = g$ we obtain the associated norm

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)} &= \langle (f,f)_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \rangle = \int_{\mathbb{R}^n} |f|^2 d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} f(\boldsymbol{x}) * \tilde{f}(\boldsymbol{x}) d^n\boldsymbol{x} \overset{(2.10)}{=} \int_{\mathbb{R}^n} \sum_A f_A^2(\boldsymbol{x}) d^n\boldsymbol{x}. \end{aligned} \tag{2.15}$$

As an easy consequence of the inner product (2.14) we obtain the *Clifford Cauchy–Schwarz* inequality

$$\left| \int_{\mathbb{R}^n} f\tilde{g}\, d^n\boldsymbol{x} \right| \le \left(\int_{\mathbb{R}^n} |f|^2 d^n\boldsymbol{x} \right)^{1/2} \left(\int_{\mathbb{R}^n} |g|^2 d^n\boldsymbol{x} \right)^{1/2}, \quad \forall f, g \in L^2(\mathbb{R}^n;\mathcal{G}_n). \tag{2.16}$$

In the following, we introduce the Clifford Fourier transform (CFT). For the detailed discussions of the properties of the CFT and their proofs, see e.g. [12–14].

Definition 2.2 *The* CFT *of $f \in L^1(\mathbb{R}^n;\mathcal{G}_n)$ is the function $\mathcal{F}\{f\} : \mathbb{R}^n \to \mathcal{G}_n, n = 2, 3 \pmod 4$ given by*

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x})\, \mathrm{e}^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x}, \tag{2.17}$$

with $\boldsymbol{x}, \boldsymbol{\omega} \in \mathbb{R}^n$.

The Clifford exponential $\mathrm{e}^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}$ is often called Clifford Fourier kernel. For the dimension $n = 3 \pmod 4$ $\mathrm{e}^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}$ commutes with all elements of $\mathcal{G}_n$, but for $n = 2 \pmod 4$ it does not. As we will see later, the different commutation rules of the pseudoscalar i_n play a crucial rule in establishing the properties of the CWFT.

Theorem 2.3 *Suppose that $f \in L^2(\mathbb{R}^n;\mathcal{G}_n)$ and $\mathcal{F}\{f\} \in L^1(\mathbb{R}^n;\mathcal{G}_n)$. Then the* CFT *of $f \in L^2(\mathbb{R}^n;\mathcal{G}_n)$ is invertible and its inverse is calculated by*

$$\mathcal{F}^{-1}[\mathcal{F}\{f\}](\boldsymbol{x}) = f(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega})\, \mathrm{e}^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{\omega}. \tag{2.18}$$

For the sake of simplicity, if not otherwise stated, n is assumed to be $n = 2,\ 3 \pmod 4$ in the rest of this section.

3 Construction of WFT in Clifford Algebra $\mathcal{G}_n$

In [12–14] the CFT has been introduced. This enables us to establish the CWFT. We see that some properties of the WFT can be established in the new construction with some modifications. We begin with the definition of the CWFT.

3.1 Definition of CWFT

Definition 3.1 *A Clifford window function is a function* $\phi \in L^2(\mathbb{R}^n;\mathcal{G}_n) \setminus \{0\}$ *such that* $|\boldsymbol{x}|^{1/2}\phi(\boldsymbol{x}) \in L^2(\mathbb{R}^n;\mathcal{G}_n)$. *Denote*

$$\phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x}) = \mathrm{e}^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\phi(\boldsymbol{x}-\boldsymbol{b}), \tag{3.1}$$

which are called the Clifford window daughter functions.

Lemma 3.2 *For* $\phi \in L^2(\mathbb{R}^n;\mathcal{G}_n)$, $n = 3 \pmod 4$, *the* CFT *of* (3.1) *can be represented in the form*

$$\mathcal{F}\{\phi_{\boldsymbol{\omega},\boldsymbol{b}}\}(\boldsymbol{\omega}') = \mathrm{e}^{-i_n(\boldsymbol{\omega}'-\boldsymbol{\omega})\cdot\boldsymbol{b}}\,\widehat{\phi}(\boldsymbol{\omega}'-\boldsymbol{\omega}), \tag{3.2}$$

and for $n = 2 \pmod 4$ *its* CFT *takes the form*

$$\mathcal{F}\{\phi_{\boldsymbol{\omega},\boldsymbol{b}}\}(\boldsymbol{\omega}') = \widehat{\phi_{\mathrm{odd}}}(\boldsymbol{\omega}'+\boldsymbol{\omega})\,\mathrm{e}^{(\boldsymbol{\omega}'+\boldsymbol{\omega})\cdot\boldsymbol{b}} + \mathrm{e}^{-i_n(\boldsymbol{\omega}'-\boldsymbol{\omega})\cdot\boldsymbol{b}}\,\widehat{\phi_{\mathrm{even}}}(\boldsymbol{\omega}'-\boldsymbol{\omega}), \tag{3.3}$$

where $\phi_{\mathrm{even}}(\phi_{\mathrm{odd}})$ *is the even (odd) grade part of* ϕ.

Proof We only prove (3.3) of Lemma 3.2. Taking its CFT we have

$$\begin{aligned}
\mathcal{F}\{\phi_{\boldsymbol{\omega},\boldsymbol{b}}\}(\boldsymbol{\omega}') &= \mathcal{F}\{\mathrm{e}^{i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\phi(\boldsymbol{x}-\boldsymbol{b})\}(\boldsymbol{\omega}') \\
&= \mathcal{F}\{\mathrm{e}^{i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}(\phi_{\mathrm{odd}}(\boldsymbol{x}-\boldsymbol{b}) + \phi_{\mathrm{even}}(\boldsymbol{x}-\boldsymbol{b}))\}(\boldsymbol{\omega}') \\
&= \mathcal{F}\{\phi_{\mathrm{odd}}(\boldsymbol{x}-\boldsymbol{b})\,\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\}(\boldsymbol{\omega}') + \mathcal{F}\{\phi_{\mathrm{even}}(\boldsymbol{x}-\boldsymbol{b})\,\mathrm{e}^{i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\}(\boldsymbol{\omega}').
\end{aligned} \tag{3.4}$$

Application of the shift and modulation properties of the CFT to (3.4) finishes the proof of equation (3.3). □

Lemma 3.3 *For* $\phi_{a,\boldsymbol{b}} \in L^2(\mathbb{R}^n;\mathcal{G}_n)$, *we have*

$$\|\phi_{a,\boldsymbol{b}}\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)} = \|\phi\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)}. \tag{3.5}$$

Proof Note that $i_n^2 = -1$ for $n = 2, 3 \pmod 4$ and $i_n\,\boldsymbol{\omega}\cdot\boldsymbol{x} = \boldsymbol{\omega}\cdot\boldsymbol{x}\,i_n$. Furthermore, we may apply the generalization of the Euler formula for Clifford algebra (see [17, 18]) to get

$$\begin{aligned}
\|\phi_{a,\boldsymbol{b}}\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)} &= \|\mathrm{e}^{i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\phi(\boldsymbol{x}-\boldsymbol{b})\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \\
&= \|\phi(\boldsymbol{x}-\boldsymbol{b})\cos(\boldsymbol{\omega}\cdot\boldsymbol{x}) + i_n\sin(\boldsymbol{\omega}\cdot\boldsymbol{x})\phi(\boldsymbol{x}-\boldsymbol{b})\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \\
&= \int_{\mathbb{R}^n}\sum_A \phi_A^2(\boldsymbol{y})d^n\boldsymbol{y}, \qquad \boldsymbol{y} = \boldsymbol{x}-\boldsymbol{b}.
\end{aligned} \tag{3.6}$$

Applying (2.15) to the last line of (3.6) proves the lemma. □

Example 1 Consider the Clifford Gaussian window $\phi \in L^2(\mathbb{R}^2;\mathcal{G}_2)$ given by

$$\phi(\boldsymbol{x}) = (1 + 2\boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_{12})\,\mathrm{e}^{-(x_1^2+x_2^2)}. \tag{3.7}$$

Thus we obtain the Clifford window daughter functions (3.1) of the form:

$$\phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x}) = \big\{(1+\boldsymbol{e}_{12})\mathrm{e}^{i_2(\omega_1x_1+\omega_2x_2)} + (2\boldsymbol{e}_1 - \boldsymbol{e}_2)\,\mathrm{e}^{-i_2(\omega_1x_1+\omega_2x_2)}\big\}\mathrm{e}^{-((x_1-b_1)^2+(x_2-b_2)^2)}. \tag{3.8}$$

According to (3.3) we easily get

$$\mathcal{F}\{\phi_{\boldsymbol{\omega},\boldsymbol{b}}\}(\boldsymbol{\omega}') = (1+\boldsymbol{e}_{12})\pi\mathrm{e}^{-\frac{(\boldsymbol{\omega}'-\boldsymbol{\omega})}{4}}\mathrm{e}^{-i_2(\boldsymbol{\omega}'-\boldsymbol{\omega})\cdot\boldsymbol{b}} + (2\boldsymbol{e}_1-\boldsymbol{e}_2)\pi\mathrm{e}^{-\frac{(\boldsymbol{\omega}'+\boldsymbol{\omega})}{4}}\mathrm{e}^{-i_2(\boldsymbol{\omega}'+\boldsymbol{\omega})\cdot\boldsymbol{b}}, \tag{3.9}$$

where $\boldsymbol{\omega} = \omega_1\boldsymbol{e}_1 + \omega_2\boldsymbol{e}_2, \boldsymbol{\omega}' = \omega_1'\boldsymbol{e}_1 + \omega_2'\boldsymbol{e}_2$ and $\boldsymbol{b} = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2$.

Definition 3.4 (CWFT) *Denote the* CWFT *on* $L^2(\mathbb{R}^n;\mathcal{G}_n)$ *by* G_ϕ. *Then the* CWFT *of* $f \in L^2(\mathbb{R}^n;\mathcal{G}_n)$ *is defined by*

$$\begin{aligned} f(\boldsymbol{x}) \longrightarrow G_\phi f(\boldsymbol{\omega},\boldsymbol{b}) &= (f,\phi_{\boldsymbol{\omega},\boldsymbol{b}})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \\ &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\ \{\mathrm{e}^{i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\phi(\boldsymbol{x}-\boldsymbol{b})\}^{\sim}\, d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\ \widetilde{\phi(\boldsymbol{x}-\boldsymbol{b})}\,\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}} d^n\boldsymbol{x}. \end{aligned} \tag{3.10}$$

It should be remembered that the order of Equation (3.10) is fixed because of the non-commutativity of the product of Clifford algebra.

Lemma 3.5 *For* $n = 2 \pmod 4$, $f,\phi \in L^2(\mathbb{R}^n;\mathcal{G}_n)$, *the* CWFT (3.10) *has a Clifford Fourier representation of the form*

$$\begin{aligned} G_\phi f(\boldsymbol{\omega},\boldsymbol{b}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} &\hat{f}(\boldsymbol{\omega}')\{\widetilde{\widehat{\phi}}_{\rm odd}(\boldsymbol{\omega}'+\boldsymbol{\omega})\,\mathrm{e}^{-i_n\boldsymbol{\omega}'\cdot\boldsymbol{b}}\,\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}} \\ &+\widetilde{\widehat{\phi}}_{\rm even}(\boldsymbol{\omega}'-\boldsymbol{\omega})\,\mathrm{e}^{i_n\boldsymbol{\omega}'\cdot\boldsymbol{b}}\,\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}}\}^{\sim}\, d^n\boldsymbol{\omega}'; \end{aligned} \tag{3.11}$$

and for $n = 3 \pmod 4$ *we have*

$$G_\phi f(\boldsymbol{\omega},\boldsymbol{b}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\omega}')\widetilde{\widehat{\phi}}(\boldsymbol{\omega}-\boldsymbol{\omega}')\,\mathrm{e}^{i_n\boldsymbol{\omega}'\cdot\boldsymbol{b}}\,\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}}\, d^n\boldsymbol{\omega}'. \tag{3.12}$$

Proof A simple calculation gives

$$\begin{aligned} G_\phi f(\boldsymbol{\omega},\boldsymbol{b}) &= (f,\phi_{\boldsymbol{\omega},\boldsymbol{b}})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \\ &\overset{\text{P. T.}}{=} \frac{1}{(2\pi)^n}(\widehat{f},\widehat{\phi_{\boldsymbol{\omega},\boldsymbol{b}}})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \\ &\overset{(3.3)}{=} \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\omega}')\{\widehat{\phi}_{\rm odd}(\boldsymbol{\omega}'+\boldsymbol{\omega})\mathrm{e}^{-i_n\boldsymbol{\omega}'\cdot\boldsymbol{b}}\,\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}} \\ &\quad +\widehat{\phi}_{\rm even}(\boldsymbol{\omega}'-\boldsymbol{\omega})\mathrm{e}^{-i_n\boldsymbol{\omega}'\cdot\boldsymbol{b}}\mathrm{e}^{i_n\boldsymbol{\omega}\cdot\boldsymbol{b}}\}^{\sim} d^n\boldsymbol{\omega}' \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\omega}')\{\widetilde{\widehat{\phi}}_{\rm odd}(\boldsymbol{\omega}'+\boldsymbol{\omega})\mathrm{e}^{-i_n\boldsymbol{\omega}'\cdot\boldsymbol{b}}\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}} \\ &\quad +\widetilde{\widehat{\phi}}_{\rm even}(\boldsymbol{\omega}'-\boldsymbol{\omega})\mathrm{e}^{i_n\boldsymbol{\omega}'\cdot\boldsymbol{b}}\,\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}}\}^{\sim} d^n\boldsymbol{\omega}', \end{aligned} \tag{3.13}$$

where P. T. denotes the Plancherel theorem for the CFT. From (3.13) it is easy to see that equation (3.12) holds. □

With the inverse CFT, equation (3.12) becomes

$$G_\phi f(\boldsymbol{\omega},\boldsymbol{b}) = \mathcal{F}^{-1}\big\{\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}}\widehat{f}(\cdot)\widetilde{\widehat{\phi}}(\cdot-\boldsymbol{\omega})\big\}(\boldsymbol{b}), \tag{3.14}$$

or equivalently

$$\mathcal{F}(G_\phi f(\boldsymbol{\omega},\boldsymbol{b}))(\boldsymbol{\omega}') = \mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}}\widehat{f}(\boldsymbol{\omega}')\widetilde{\widehat{\phi}}(\boldsymbol{\omega}'-\boldsymbol{\omega}). \tag{3.15}$$

Next, we discuss the following consequences of the above definition:

(1) For a fixed $\boldsymbol{b}$, we have

$$G_\phi f(\boldsymbol{\omega},\boldsymbol{b}) = \mathcal{F}\{f_{\boldsymbol{b}}(\boldsymbol{x})\}(\boldsymbol{\omega}), \tag{3.16}$$

where $f_{\boldsymbol{b}}(\boldsymbol{x}) = f(\boldsymbol{x})\widetilde{\phi(\boldsymbol{x}-\boldsymbol{b})}$. It thus means that the CWFT can be regarded as the CFT of the product of a multivector-valued function f and a shifted and Clifford reversion version of the Clifford window function.

(2) The energy density is defined as the modulus square given by

$$|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2 = \left| \int_{\mathbb{R}^n} f(\boldsymbol{x})\ \widetilde{\phi(\boldsymbol{x}-\boldsymbol{b})} e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x} \right|^2. \tag{3.17}$$

Equation (3.17) is often called a spectrogram which measures the energy of a Clifford-valued function f in the position-frequency neighborhood of $(\boldsymbol{\omega},\ \boldsymbol{b})$.

(3) Applying the Clifford Cauchy–Schwarz inequality (2.16) to (3.17) yields

$$\|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)} = \langle (f,\phi_{\boldsymbol{\omega},\boldsymbol{b}})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\rangle^2 \leq \|f\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\|\phi_{\boldsymbol{\omega},\boldsymbol{b}}\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)}, \tag{3.18}$$

which shows that the CWFT is a bounded linear operator on $L^2(\mathbb{R}^n;\mathcal{G}_n)$.

(4) Taking the Gaussian function as the Clifford window function of (3.1), then for $\boldsymbol{\omega} = \boldsymbol{\omega}_0$ fixed we obtain Clifford Gabor filters, i.e.

$$g_c(\boldsymbol{x},\sigma_1,\sigma_2,\ldots,\sigma_n) = \frac{1}{(2\pi)^n} e^{i_n\boldsymbol{\omega}_0\cdot\boldsymbol{x}} e^{-\left[(x_1/\sigma_1)^2+(x_2/\sigma_2)^2+\cdots+(x_n/\sigma_n)^2\right]/2}, \tag{3.19}$$

where $\sigma_1,\sigma_2,\ldots,\sigma_n$ are standard deviations of the Gaussian functions and the parameter $\boldsymbol{b}=0$. In terms of the CFT, equation (3.19) can be expressed as

$$\mathcal{F}\{g_c\}(\boldsymbol{\omega}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sigma_1\cdots\sigma_n} e^{-\frac{1}{2}\left[(\sigma_1^2(\omega_1-u_{01})^2+\sigma_2^2(\omega_2-u_{02})^2+\cdots+\sigma_n^2(\omega_n-u_{0n})^n\right]}. \tag{3.20}$$

From Equations (3.19) and (3.20) we see that the Clifford Gabor filters were well localized in both the spatial and Clifford Fourier domains.

For illustrative purpose, we shall discuss a simple example of the CWFT. We then compute its energy density.

Example 2 Consider the Clifford Gabor filters (see [11]) defined by

$$f(\boldsymbol{x}) = e^{-\boldsymbol{x}^2+i_n\boldsymbol{\omega}_0\cdot\boldsymbol{x}}. \tag{3.21}$$

Obtain the CWFT of f with respect to the Clifford Gaussian window function given by

$$\phi(\boldsymbol{x}) = C\, e^{-\boldsymbol{x}^2}, \tag{3.22}$$

where $C \in \mathcal{G}_n$ is a multivector constant.

First, if $C \in \mathcal{G}_n$, $n = 3 \pmod 4$ then by the definition of the CWFT (3.10) we have

$$G_\phi f(\boldsymbol{\omega},\boldsymbol{b}) = \int_{\mathbb{R}^n} e^{-\boldsymbol{x}^2+i_n\boldsymbol{\omega}_0\cdot\boldsymbol{x}}\, C\, e^{-(\boldsymbol{x}-\boldsymbol{b})^2} e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}} d^n\boldsymbol{x}. \tag{3.23}$$

Putting $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{b}/2$ and applying the fact that C commutes with the Clifford Fourier kernel $e^{i_n\boldsymbol{\omega}_0\cdot\boldsymbol{x}}$ we can rewrite (3.23) in the form

$$\begin{aligned} G_\phi f(\boldsymbol{\omega},\boldsymbol{b}) &= C\int_{\mathbb{R}^n} e^{-(\boldsymbol{y}+\boldsymbol{b}/2)^2+i_n\boldsymbol{\omega}_0\cdot(\boldsymbol{y}+\boldsymbol{b}/2)} e^{-(\boldsymbol{y}-\boldsymbol{b}/2)^2} e^{-i_n\boldsymbol{\omega}\cdot(\boldsymbol{y}+\boldsymbol{b}/2)}\, d^n\boldsymbol{y} \\ &= C\, e^{-\boldsymbol{b}^2/2}\int_{\mathbb{R}^n} e^{-2\boldsymbol{y}^2} e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{y}}\, e^{i_n\boldsymbol{\omega}_0\cdot\boldsymbol{y}} d^n\boldsymbol{y}\, e^{-i_n(\boldsymbol{\omega}-\boldsymbol{\omega}_0)\cdot\boldsymbol{b}/2} \end{aligned}$$

$$= C(\pi/2)^{n/2}\,\mathrm{e}^{-\boldsymbol{b}^2/2}\,\mathrm{e}^{-(\boldsymbol{\omega}-\boldsymbol{\omega}_0)^2/8}\,\mathrm{e}^{-i_n(\boldsymbol{\omega}-\boldsymbol{\omega}_0)\cdot\boldsymbol{b}/2}. \tag{3.24}$$

Second, if $C_0 \in \mathcal{G}_n$, $n = 2 \pmod 4$, then the definition of the CWFT (3.10) gives

$$\begin{aligned} G_\phi f(\boldsymbol{\omega},\boldsymbol{b}) &= \int_{\mathbb{R}^n} \mathrm{e}^{-\boldsymbol{x}^2+i_n\boldsymbol{\omega}_0\cdot\boldsymbol{x}}(C_{\text{even}}+C_{\text{odd}})\,\mathrm{e}^{-(\boldsymbol{x}-\boldsymbol{b})^2}\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}d^n\boldsymbol{x} \\ &= C_{\text{even}}\int_{\mathbb{R}^n} \mathrm{e}^{-\boldsymbol{x}^2+i_n\boldsymbol{\omega}_0\cdot\boldsymbol{x}}\,\mathrm{e}^{-(\boldsymbol{x}-\boldsymbol{b})^2}\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\,d^n\boldsymbol{x} \\ &\quad + C_{\text{odd}}\int_{\mathbb{R}^n} \mathrm{e}^{-\boldsymbol{x}^2-i_n\boldsymbol{\omega}_0\cdot\boldsymbol{x}}\,\mathrm{e}^{-(\boldsymbol{x}-\boldsymbol{b})^2}\,\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}d^n\boldsymbol{x}, \end{aligned} \tag{3.25}$$

where we have used that C_{odd} anti-commutes with $\mathrm{e}^{i_n\boldsymbol{\omega}_0\cdot\boldsymbol{x}}$, i.e.

$$C_{\text{odd}}\,\mathrm{e}^{i_n\boldsymbol{\omega}\cdot\boldsymbol{x}} = \mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\,C_{\text{odd}}, \quad \forall\, C_{\text{odd}} \in L^2(\mathbb{R}^n;\mathcal{G}_n). \tag{3.26}$$

Next, we follow the steps of (3.24), then we immediately obtain

$$\begin{aligned} G_\phi f(\boldsymbol{\omega},\boldsymbol{b}) = (\pi/2)^{n/2}\,\mathrm{e}^{-\boldsymbol{b}^2/2}\{&C_{\text{even}}\,\mathrm{e}^{-(\boldsymbol{\omega}-\boldsymbol{\omega}_0)^2/8}\mathrm{e}^{-i_n(\boldsymbol{\omega}-\boldsymbol{\omega}_0)\cdot\boldsymbol{b}/2} \\ &+ C_{\text{odd}}\,\mathrm{e}^{-(\boldsymbol{\omega}+\boldsymbol{\omega}_0)^2/8}\mathrm{e}^{-i_n(\boldsymbol{\omega}+\boldsymbol{\omega}_0)\cdot\boldsymbol{b}/2}\}. \end{aligned} \tag{3.27}$$

According to (3.17), we easily obtain the energy density of (3.23)

$$|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2 = k_0(\pi/2)^n\,\mathrm{e}^{-\boldsymbol{b}^2/2}\,\mathrm{e}^{-(\boldsymbol{\omega}-\boldsymbol{\omega}_0)^2/4}, \tag{3.28}$$

where $k_0 = C * C$ is a scalar constant. Similarly, the energy density of (3.25) is

$$|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2 = (\pi/2)^n\,\mathrm{e}^{-\boldsymbol{b}^2/2}\{k_1\mathrm{e}^{-(\boldsymbol{\omega}-\boldsymbol{\omega}_0)^2/4} + k_2\mathrm{e}^{-(\boldsymbol{\omega}+\boldsymbol{\omega}_0)^2/4}\}, \tag{3.29}$$

where $k_1 = C_{\text{even}} * C_{\text{even}}$ and $k_2 = C_{\text{odd}} * C_{\text{odd}}$ are scalar constants.

3.2 Basic Properties of CWFT

In this subsection, we will discuss the properties of the CWFT (compare [7, 19]). We find that many of the properties of the WFT are still valid for the CWFT, however with certain modifications.

Proposition 3.6 *Let $\phi \in L^2(\mathbb{R}^n;\mathcal{G}_n)$ be a Clifford window function.*

(i) (*Left linearity*)

$$[G_\phi(\lambda f + \mu g)](\boldsymbol{\omega},\boldsymbol{b}) = \lambda G_\phi f(\boldsymbol{\omega},\boldsymbol{b}) + \mu G_\phi g(\boldsymbol{\omega},\boldsymbol{b}), \tag{3.30}$$

with Clifford constants $\lambda,\mu \in \mathcal{G}_n$.

(ii) (*Parity*)

$$G_{P\phi}Pf(\boldsymbol{\omega},\boldsymbol{b}) = G_\phi f(-\boldsymbol{\omega},-\boldsymbol{b}), \tag{3.31}$$

where P is the parity operator defined by $Pf(x) = f(-x)$.

(iii) (*Delay property*)

$$G_\phi T_{\boldsymbol{x}_0}f(\boldsymbol{\omega},\boldsymbol{b}) = (G_\phi f(\boldsymbol{\omega},\boldsymbol{b}-\boldsymbol{x}_0))\,\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}_0}, \tag{3.32}$$

where the translation operator is given by $T_{\boldsymbol{x}_0}f(\boldsymbol{x}) = f(\boldsymbol{x}-\boldsymbol{x}_0)$.

(iv) (*Modulation property*) *Let ϕ be a Clifford window function in $L^2(\mathbb{R}^n;\mathcal{G}_n), n = 3 \pmod 4$. If $\boldsymbol{\omega}_0 \in \mathbb{R}^n$ and $f_0(\boldsymbol{x}) = f(\boldsymbol{x})\mathrm{e}^{i_n\boldsymbol{\omega}_0\cdot\boldsymbol{x}}$, then*

$$G_\phi f_0(\boldsymbol{\omega},\boldsymbol{b}) = G_\phi f(\boldsymbol{\omega}-\boldsymbol{\omega}_0,\boldsymbol{b}). \tag{3.33}$$

For $n = 2 \pmod 4$ *the modulation property takes the form*

$$G_\phi f_0(\boldsymbol{\omega}, \boldsymbol{b}) = G_{\phi_{\rm even}} f(\boldsymbol{\omega} - \boldsymbol{\omega}_0, \boldsymbol{b}) + G_{\phi_{\rm odd}} f(\boldsymbol{\omega} + \boldsymbol{\omega}_0, \boldsymbol{b}). \tag{3.34}$$

Theorem 3.7 (Orthogonality relation) *Let* $\phi \in L^2(\mathbb{R}^n; \mathcal{G}_n)$ *be a Clifford window function. If two Clifford functions* $f, g \in L^2(\mathbb{R}^n; \mathcal{G}_n)$*, then we have*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\, \widetilde{G_\phi g(\boldsymbol{\omega}, \boldsymbol{b})} d^n\boldsymbol{\omega}\, d^n\boldsymbol{b} = (2\pi)^n (f(\tilde{\phi}, \tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}, g)_{L^2(\mathbb{R}^n;\mathcal{G}_n)}. \tag{3.35}$$

Proof We first notice that

$$G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) = \mathcal{F}\{f_{\boldsymbol{b}}(\boldsymbol{x})\}(\boldsymbol{\omega}) = \mathcal{F}\{f(\boldsymbol{x})\phi_{\boldsymbol{b}}(\boldsymbol{x})\}(\boldsymbol{\omega}), \tag{3.36}$$

where $\phi_{\boldsymbol{b}}(\boldsymbol{x}) = \widetilde{\phi(\boldsymbol{x} - \boldsymbol{b})}$.

Next, the Plancherel theorem for the CFT (see [13] for more details) gives

$$\begin{aligned}\int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\, \widetilde{G_\phi g(\boldsymbol{\omega}, \boldsymbol{b})}\, d^n\boldsymbol{\omega} &= (\mathcal{F}\{f\phi_{\boldsymbol{b}}\}, \mathcal{F}\{g\phi_{\boldsymbol{b}}\})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \\ &= (2\pi)^n (f\phi_{\boldsymbol{b}}, g\phi_{\boldsymbol{b}})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \\ &= (2\pi)^n \int_{\mathbb{R}^n} f(\boldsymbol{x}) \widetilde{\phi(\boldsymbol{x} - \boldsymbol{b})} \phi(\boldsymbol{x} - \boldsymbol{b}) \widetilde{g(\boldsymbol{x})}\, d^n\boldsymbol{x}. \end{aligned} \tag{3.37}$$

If we assume that $f\tilde{\phi}$ and $\phi\tilde{g}$ are in $L^2(\mathbb{R}^n; \mathcal{G}_n)$, then integrating (3.37) with respect to $d^n\boldsymbol{b}$ yields

$$\begin{aligned}\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\, \widetilde{G_\phi g(\boldsymbol{\omega}, \boldsymbol{b})}\, d^n\boldsymbol{\omega}\, d^n\boldsymbol{b} &= (2\pi)^n \int_{\mathbb{R}^n} f(\boldsymbol{x}) \int_{\mathbb{R}^n} \widetilde{\phi(\boldsymbol{x} - \boldsymbol{b})} \phi(\boldsymbol{x} - \boldsymbol{b}) \widetilde{g(\boldsymbol{x})}\, d^n\boldsymbol{x}\, d^n\boldsymbol{b} \\ &= (2\pi)^n \int_{\mathbb{R}^n} f(\boldsymbol{x}) \int_{\mathbb{R}^n} \widetilde{\phi(\boldsymbol{x}')} \phi(\boldsymbol{x}')\, d^n\boldsymbol{x}' \widetilde{g(\boldsymbol{x})}\, d^n\boldsymbol{x}, \end{aligned} \tag{3.38}$$

where we have used Fubini's theorem to interchange the order of integration. This proves the theorem. □

The scalar part of Theorem 3.7 gives us the following corollary which will be necessary to prove the uncertainty principle for the CWFT.

Corollary 3.8 *If* $f, \phi \in L^2(\mathbb{R}^n; \mathcal{G}_n)$ *are two Clifford-valued signals, then*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})|^2\, d^n\boldsymbol{b}\, d^n\boldsymbol{\omega} = (2\pi)^n (\tilde{f}, \tilde{f})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} * (\tilde{\phi}, \tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}. \tag{3.39}$$

Especially, if we assume that $C_{\tilde{\phi}} = (\tilde{\phi}, \tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}$ *is a multivector constant, then* (3.39) *will reduce to*

$$\begin{aligned}\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})|^2\, d^n\boldsymbol{b}\, d^n\boldsymbol{\omega} &= (2\pi)^n C_{\tilde{\phi}} * (\tilde{f}, \tilde{f})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}. \\ &= (2\pi)^n \langle C_{\tilde{\psi}}\rangle \|f\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)} + (2\pi)^n \langle C_{\tilde{\psi}}\rangle_1 * \langle(\tilde{f}, \tilde{f})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\rangle_1 \\ &\quad + \cdots + (2\pi)^n \langle C_{\tilde{\psi}}\rangle_n * \langle(\tilde{f}, \tilde{f})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\rangle_n. \end{aligned} \tag{3.40}$$

Theorem 3.9 (Reconstruction formula) *Let* $\phi \in L^2(\mathbb{R}^n; \mathcal{G}_n)$ *be a Clifford window function. Then every Clifford signal* $f \in L^2(\mathbb{R}^n; \mathcal{G}_n)$ *can be fully reconstructed by*

$$f(\boldsymbol{x})(\tilde{\phi}, \tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \phi_{\boldsymbol{\omega}, \boldsymbol{b}}(\boldsymbol{x})\, d^n\boldsymbol{b}\, d^n\boldsymbol{\omega}. \tag{3.41}$$

Assuming that $C_{\tilde{\phi}} = (\tilde{\phi}, \tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}$ *is an invertible multivector constant, then* (3.41) *becomes*

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x})\, C_{\tilde{\phi}}^{-1}\, d^n\boldsymbol{b}\, d^n\boldsymbol{\omega}. \tag{3.42}$$

Proof Direct calculation gives for every $g \in L^2(\mathbb{R}^n; \mathcal{G}_n)$

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\, \widetilde{G_\phi g(\boldsymbol{\omega}, \boldsymbol{b})}\, d^n\boldsymbol{\omega}\, d^n\boldsymbol{b} \\ &\quad = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\, \phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x}) \tilde{g}(\boldsymbol{x})\, d^n\boldsymbol{\omega}\, d^n\boldsymbol{b}\, d^n\boldsymbol{x} \\ &\quad = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \phi_{\boldsymbol{\omega},\boldsymbol{b}}\, d\boldsymbol{\omega}\, d^n\boldsymbol{b}, g \right)_{L^n(\mathbb{R}^n;\mathcal{G}_n)}. \end{aligned} \tag{3.43}$$

Applying (3.35) of Theorem 3.7 to the left-hand side of (3.43) gives for every $g \in L^2(\mathbb{R}^n; \mathcal{G}_n)$

$$(2\pi)^n (f(\tilde{\phi}, \tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}, g)_{L^2(\mathbb{R}^n;\mathcal{G}_n)} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \phi_{\boldsymbol{\omega},\boldsymbol{b}}\, d\boldsymbol{\omega}\, d^n\boldsymbol{b}, g \right)_{L^n(\mathbb{R}^n;\mathcal{G}_n)}. \tag{3.44}$$

Taking the scalar part of (3.44), we obtain

$$\begin{aligned} &\langle (2\pi)^n (f(\tilde{\phi}, \tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}, g)_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \rangle \\ &\quad = \left\langle \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \phi_{\boldsymbol{\omega},\boldsymbol{b}}\, d\boldsymbol{\omega}\, d^n\boldsymbol{b}, g \right)_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \right\rangle. \end{aligned} \tag{3.45}$$

Because the inner product identity (3.45) holds for every $g \in L^2(\mathbb{R}^n; \mathcal{G}_n)$ we conclude that

$$f(\boldsymbol{x}) (\tilde{\phi}, \tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x})\, d^n\boldsymbol{b}\, d^n\boldsymbol{\omega}, \tag{3.46}$$

or equivalently, because of the assumed invertibility of $C_{\tilde{\phi}}$

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x})\, C_{\tilde{\phi}}^{-1}\, d^n\boldsymbol{b}\, d^n\boldsymbol{\omega} \tag{3.47}$$

which concludes the proof. □

Theorem 3.10 (Reproducing kernel) *Let* $\phi \in L^2(\mathbb{R}^n; \mathcal{G}_n)$ *be a Clifford window function so that* $C_{\tilde{\phi}} = (\tilde{\phi}, \tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}$ *is an invertible multivector constant. If*

$$\mathbb{K}_\phi(\boldsymbol{\omega}, \boldsymbol{b}; \boldsymbol{\omega}', \boldsymbol{b}') = \frac{1}{(2\pi)^n} (\phi_{\boldsymbol{\omega},\boldsymbol{b}} C_{\tilde{\phi}}^{-1}; \phi_{\boldsymbol{\omega}',\boldsymbol{b}'})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}, \tag{3.48}$$

then $\mathbb{K}_\phi(\boldsymbol{\omega}, \boldsymbol{b}; \boldsymbol{\omega}', \boldsymbol{b}')$ *is a reproducing kernel, i.e.*

$$G_\phi f(\boldsymbol{\omega}', \boldsymbol{b}') = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \mathbb{K}_\phi(\boldsymbol{\omega}, \boldsymbol{b}; \boldsymbol{\omega}', \boldsymbol{b}')\, d^n\boldsymbol{\omega}\, d^n\boldsymbol{b}. \tag{3.49}$$

Proof Substituting (3.42) into the definition of the CWFT yields

$$\begin{aligned} G_\phi f(\boldsymbol{\omega}', \boldsymbol{b}') &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\, \widetilde{\phi_{\boldsymbol{\omega}',\boldsymbol{b}'}(\boldsymbol{x})}\, d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\, \phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x})\, C_{\tilde{\phi}}^{-1} d^n\boldsymbol{b} d^n\boldsymbol{\omega} \right) \widetilde{\phi_{\boldsymbol{\omega}',\boldsymbol{b}'}(\boldsymbol{x})} d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \left(\int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} \phi_{\boldsymbol{\omega},\boldsymbol{b}}(\boldsymbol{x})\, C_{\tilde{\phi}}^{-1} \widetilde{\phi_{\boldsymbol{\omega}',\boldsymbol{b}'}(\boldsymbol{x})}\, d^n\boldsymbol{x} \right) d^n\boldsymbol{b}\, d^n\boldsymbol{\omega} \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}) \mathbb{K}_\phi(\boldsymbol{\omega}, \boldsymbol{b}; \boldsymbol{\omega}', \boldsymbol{b}')\, d^n\boldsymbol{b}\, d^n\boldsymbol{\omega}, \end{aligned} \tag{3.50}$$

which completes the proof. □

4 Uncertainty Principle for CWFT

Let us now formulate an uncertainty principle for the CWFT, which involves a Clifford-valued function and its CWFT simultaneously. From a mathematical point of view this principle describes how a Clifford-valued function relates to its CWFT.

Before going on the main theorem, we rewrite the uncertainty principle for the CFT (see [13] and [14] for more details) as we will use it to prove the uncertainty principle for the CWFT.

Theorem 4.1 *Let $f \in L^2(\mathbb{R}^n; \mathcal{G}_n)$ be a multivector-valued function such that $\boldsymbol{x}f \in L^2(\mathbb{R}^n; \mathcal{G}_n)$. Then we have inequality*

$$\|\boldsymbol{x}f\|_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \|\boldsymbol{\omega}\mathcal{F}\{f\}\|_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \geq \frac{\sqrt{n}(2\pi)^{n/2}}{2}\|f\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)}. \tag{4.1}$$

Substituting the Parseval theorem for the CFT into the right-hand side of (4.1) we get

$$\|\boldsymbol{x}\mathcal{F}^{-1}[\mathcal{F}\{f\}]\|_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \|\boldsymbol{\omega}\mathcal{F}\{f\}\|_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \geq \frac{\sqrt{n}}{2(2\pi)^{n/2}}\|\mathcal{F}\{f\}\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)}. \tag{4.2}$$

Theorem 4.2 (CWFT uncertainty principle) *Let $\phi \in L^2(\mathbb{R}^n; \mathcal{G}_n)$ be a Clifford window function such that $\boldsymbol{\omega}G_\phi f \in L^2(\mathbb{R}^n; \mathcal{G}_n)$. Then the following uncertainty inequality holds*

$$\begin{aligned}&\left((\tilde{\phi},\tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} * (\widetilde{\boldsymbol{x}f},\widetilde{\boldsymbol{x}f})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\right)^{1/2}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\boldsymbol{\omega}G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\,d^n\boldsymbol{\omega}\,d^n\boldsymbol{b}\right)^{1/2}\\&\geq \frac{\sqrt{n}}{2(2\pi)^{3n/2}}(\tilde{f},\tilde{f})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} * (\tilde{\phi},\tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}.\end{aligned} \tag{4.3}$$

To facilitate the proof of Theorem 4.2 we need to introduce the following lemma.

Lemma 4.3 *Under the assumptions of Theorem 4.2, we have*

$$(\tilde{\phi},\tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} * (\widetilde{\boldsymbol{x}f},\widetilde{\boldsymbol{x}f})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\boldsymbol{x}\mathcal{F}^{-1}[G_\phi f(\cdot,\boldsymbol{b})]|^2\,d^n\boldsymbol{x}\,d^n\boldsymbol{b}. \tag{4.4}$$

Proof Using the cyclic product symmetry (2.11) we have

$$\begin{aligned}(\tilde{\phi},\tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} * (\widetilde{\boldsymbol{x}f},\widetilde{\boldsymbol{x}f})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} &= \int_{\mathbb{R}^n}\widetilde{\phi(\boldsymbol{x}-\boldsymbol{b})}\phi(\boldsymbol{x}-\boldsymbol{b})\,d^n\boldsymbol{b} * \int_{\mathbb{R}^n}\boldsymbol{x}^2\widetilde{f(\boldsymbol{x})}f(\boldsymbol{x})\,d^n\boldsymbol{x}\\&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\boldsymbol{x}^2 f(\boldsymbol{x})\widetilde{\phi(\boldsymbol{x}-\boldsymbol{b})}\,d^n\boldsymbol{b} * \phi(\boldsymbol{x}-\boldsymbol{b})\widetilde{f(\boldsymbol{x})}\,d^n\boldsymbol{x}\\&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\boldsymbol{x}^2 f(\boldsymbol{x})\widetilde{\phi(\boldsymbol{x}-\boldsymbol{b})}\,d^n\boldsymbol{b} * \{f(\boldsymbol{x})\widetilde{\phi(\boldsymbol{x}-\boldsymbol{b})}\}^{\sim}\,d^n\boldsymbol{x}\\&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\boldsymbol{x}^2\,\frac{1}{(2\pi)^n}\mathcal{F}^{-1}\{G_\phi f(\cdot,\boldsymbol{b})\}(\boldsymbol{x}) * \frac{1}{(2\pi)^n}\\&\quad\times[\mathcal{F}^{-1}\{G_\phi f(\cdot,\boldsymbol{b})\}(\boldsymbol{x})]^{\sim}\,d^n\boldsymbol{x}\,d^n\boldsymbol{b},\end{aligned}$$

which was to be proved. □

Remark 4.4 It is important to notice that if the Clifford window function ϕ is a paravector-valued function, the above lemma will reduce to

$$\|\phi\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\|\boldsymbol{x}f\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)} = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n}|\boldsymbol{x}\mathcal{F}^{-1}[G_\phi f(\cdot,\boldsymbol{b})]|^2\,d^n\boldsymbol{x}\,d^n\boldsymbol{b}. \tag{4.5}$$

Let us now begin with the proof of Theorem 4.2.

Proof Replacing the CFT of f by the CWFT of f on both sides of (4.2) we obtain

$$\|\boldsymbol{x}\mathcal{F}^{-1}[G_\phi f(\boldsymbol{\omega}\cdot,\boldsymbol{b})]\|_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\,\|\boldsymbol{\omega}G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\|_{L^2(\mathbb{R}^n;\mathcal{G}_n)} \geq \frac{\sqrt{n}}{2(2\pi)^{n/2}}\int_{\mathbb{R}^n}|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\,d^n\boldsymbol{\omega}. \tag{4.6}$$

Integrating both sides of (4.6) with respect to $d^n\boldsymbol{b}$ yields

$$\begin{aligned}&\int_{\mathbb{R}^n}\left\{\|\boldsymbol{x}\mathcal{F}^{-1}[G_\phi\, f(\cdot,\boldsymbol{b})(\boldsymbol{x})]\|_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\|\boldsymbol{\omega}G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\|_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\right\}d^n\boldsymbol{b}\\&\quad\geq\frac{\sqrt{n}}{2(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\,d^n\boldsymbol{\omega}\,d^n\boldsymbol{b}.\end{aligned} \tag{4.7}$$

Now applying the inequality of the Cauchy–Schwarz for multivector functions (2.16) to line 1 of (4.7) we further get

$$\begin{aligned}&\left(\int_{\mathbb{R}^n}\|\boldsymbol{x}\mathcal{F}^{-1}[G_\phi\, f(\cdot,\boldsymbol{b})]\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\,d^n\boldsymbol{b}\right)^{1/2}\left(\int_{\mathbb{R}^n}\|\boldsymbol{\omega}G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\,d^n\boldsymbol{b}\right)^{1/2}\\&\quad\geq\frac{\sqrt{n}}{2(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\,d^n\boldsymbol{\omega}\,d^n\boldsymbol{b}.\end{aligned} \tag{4.8}$$

Inserting Lemma 4.3 into the first term of (4.8), we have

$$\begin{aligned}&\left((2\pi)^{2n}\,(\tilde{\phi},\tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}*(\widetilde{\boldsymbol{x}f},\widetilde{\boldsymbol{x}f)_{L^2(\mathbb{R}^n;\mathcal{G}_n)}}\right)^{1/2}\left(\int_{\mathbb{R}^n}\|\boldsymbol{\omega}G_\phi f(\boldsymbol{\omega},\boldsymbol{b})\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\,d^n\boldsymbol{b}\right)^{1/2}\\&\quad\geq\frac{\sqrt{n}}{2(2\pi)^{n/2}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\,d^n\boldsymbol{\omega}\,d^n\boldsymbol{b}.\end{aligned} \tag{4.9}$$

Finally, substituting (3.39) in Corollary 3.8 into the right-hand side of (4.9) we see that

$$\begin{aligned}&\left((\tilde{\phi},\tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}*(\widetilde{\boldsymbol{x}f},\widetilde{\boldsymbol{x}f})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\right)^{1/2}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\boldsymbol{\omega}G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\,d^n\boldsymbol{\omega}\,d^n\boldsymbol{b}\right)^{1/2}\\&\quad\geq\frac{\sqrt{n}}{2(2\pi)^{3n/2}}(\tilde{f},\tilde{f})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}*(\tilde{\phi},\tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)},\end{aligned} \tag{4.10}$$

and this proves the theorem. □

The following result is a straightforward consequence of the previous theorem.

Corollary 4.5 *If the Clifford window function is a paravector-valued function, then equation* (4.3) *takes the form*

$$\begin{aligned}&\|\boldsymbol{x}f\|_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\boldsymbol{\omega}G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\,d^n\boldsymbol{\omega}\,d^n\boldsymbol{b}\right)^{1/2}\\&\quad\geq\frac{\sqrt{n}}{2(2\pi)^{3n/2}}\|f\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\|\phi\|_{L^2(\mathbb{R}^n;\mathcal{G}_n)}.\end{aligned} \tag{4.11}$$

In a similar way we can prove the following uncertainty principle, its proof is omitted here.

Theorem 4.6 *Let* $\phi\in L^2(\mathbb{R}^n;\mathcal{G}_n), n=3 \pmod 4$ *be a Clifford window function such that* $\boldsymbol{b}G_\phi f\in L^2(\mathbb{R}^n;\mathcal{G}_n)$*. Then the following uncertainty inequality holds*

$$\begin{aligned}&\left((\tilde{\phi},\tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}*(\widetilde{\boldsymbol{\omega}'\hat{f}},\widetilde{\boldsymbol{\omega}'\hat{f}})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\right)^{1/2}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\boldsymbol{b}\,G_\phi f(\boldsymbol{\omega},\boldsymbol{b})|^2\,d^n\boldsymbol{\omega}\,d^n\boldsymbol{b}\right)^{1/2}\\&\quad\geq\frac{\sqrt{n}}{2(2\pi)^{3n/2}}(\tilde{f},\tilde{f})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}*(\tilde{\phi},\tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}.\end{aligned} \tag{4.12}$$

The following lemma is useful to prove Theorem 4.6.

Lemma 4.7 *Under the assumptions of Theorem* 4.6*, we have*

$$\int_{\mathbb{R}^n} \|\boldsymbol{\omega}'\ \mathcal{F}\{G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\}\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\, d^n\boldsymbol{\omega} = (\tilde{\phi}, \tilde{\phi})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} * (\widetilde{\boldsymbol{\omega}'\hat{f}}, \widetilde{\boldsymbol{\omega}'\hat{f}})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}. \tag{4.13}$$

Proof We first observe that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \|\boldsymbol{\omega}'\ \mathcal{F}\{G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\}\|^2_{L^2(\mathbb{R}^n;\mathcal{G}_n)}\, d^n\boldsymbol{\omega} \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \boldsymbol{\omega}'^2 [\mathcal{F}\{G_\phi f(\boldsymbol{\omega}, \boldsymbol{b})\}(\boldsymbol{\omega}')] * \widetilde{\mathcal{F}}\{G_\phi f(\boldsymbol{\omega}, \boldsymbol{b}\}(\boldsymbol{\omega}')\, d^n\boldsymbol{\omega}'\, d^n\boldsymbol{\omega} \\
&\overset{(3.15)}{=} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \boldsymbol{\omega}'^2 \mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}} \widehat{f}(\boldsymbol{\omega}')\widetilde{\widehat{\phi}}(\boldsymbol{\omega}'-\boldsymbol{\omega}) * \{\mathrm{e}^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{b}} \widehat{f}(\boldsymbol{\omega}')\widetilde{\widehat{\phi}}(\boldsymbol{\omega}'-\boldsymbol{\omega})\}^{\sim}\, d^n\boldsymbol{\omega}'\, d^n\boldsymbol{\omega} \\
&= \int_{\mathbb{R}^n} \widetilde{\widehat{\phi}}(\boldsymbol{\omega}'-\boldsymbol{\omega})\widehat{\phi}(\boldsymbol{\omega}'-\boldsymbol{\omega})\, d^n\boldsymbol{\omega} * \int_{\mathbb{R}^n} \boldsymbol{\omega}'^2\, \widetilde{\widehat{f}}(\boldsymbol{\omega}')\widehat{f}(\boldsymbol{\omega}')\, d^n\boldsymbol{\omega}' \\
&= (\tilde{\hat{\phi}}, \tilde{\hat{\phi}})_{L^2(\mathbb{R}^n;\mathcal{G}_n)} * (\widetilde{\boldsymbol{\omega}'\hat{f}}, \widetilde{\boldsymbol{\omega}'\hat{f}})_{L^2(\mathbb{R}^n;\mathcal{G}_n)}.
\end{aligned} \tag{4.14}$$

Applying the Plancherel theorem for the CFT to the last line of (4.14) finishes the proof of Lemma 4.7. □

Acknowledgements This paper was written while the first author was a postdoctoral fellow at the Universiti Sains Malaysia, Malaysia. He thanks Professor Ryuichi Ashino and Professor Eckhard Hitzer for much of their helpful guidance.

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