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Advances in Applied Clifford Algebras

Clifford Algebra-Valued Wavelet Transform on Multivector Fields

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Abstract. This paper presents a construction of the $n = 2 \pmod 4$ Clifford algebra $Cl_{n,0}$-valued admissible wavelet transform using the admissible similitude group *SIM(n)*, a subgroup of the affine group of $\mathbb{R}^n$. We express the admissibility condition in terms of the $Cl_{n,0}$ Clifford Fourier transform (CFT). We show that its fundamental properties such as inner product, norm relation, and inversion formula can be established whenever the Clifford admissible wavelet satisfies a particular admissibility condition. As an application we derive a Heisenberg type uncertainty principle for the Clifford algebra $Cl_{n,0}$-valued admissible wavelet transform. Finally, we provide some basic examples of these extended wavelets such as Clifford Morlet wavelets and Clifford Hermite wavelets.

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1. Introduction

Recently it has become popular to generalize classical wavelets to Clifford algebra. The generalization can be found in several publications. These publications deal for example with the discrete Clifford wavelet transform [5, 16, 18] and the continuous Clifford wavelet transform [4, 6, 20]. Their approaches use the classical Fourier transform (FT) to investigate some properties of these extended wavelets.

In [9, 10, 15], the Clifford Fourier transform (CFT) on $Cl_{n,0}$ for $n = 2$, $3 \pmod 4$ has been introduced. Based on the basic concepts of Clifford algebra and its Fourier transform, we constructed Clifford algebra $Cl_{3,0}$-valued wavelet transform[1]. The commutativity of the $n = 3 \pmod 4$ Clifford $Cl_{n,0}$ Fourier kernel with every element of $Cl_{n,0}$ is a unique advantage over the

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[1]Based on Clifford algebra and its Fourier transform, we can easily generalize Clifford algebra-valued wavelet in $Cl_{3,0}$ to $Cl_{n,0}$ for $n = 3 \pmod 4$.

$n = 2 \pmod 4$ Clifford $Cl_{n,0}$ Fourier kernel. Therefore, the construction of the $n = 3 \pmod 4$ Clifford algebra $Cl_{n,0}$-valued wavelet transform is simpler than the $n = 2 \pmod 4$ Clifford algebra $Cl_{n,0}$-valued wavelet transform.

The main aim of this paper is to expand the generalization of the Clifford algebra $Cl_{3,0}$-valued wavelet transform in [17] to Clifford algebra $Cl_{n,0}$ for $n = 2 \pmod 4$ using the admissible similitude group *SIM(n)*. We investigate the properties of the extended wavelets using the CFT. Special attention is devoted to inner product, norm relation, and inversion formula. We show that these fundamental properties can be established whenever the Clifford admissible wavelet satisfies a particular admissibility condition. We apply the uncertainty principle for the CFT and properties of the Clifford algebra $Cl_{n,0}$ wavelets to establish a Heisenberg type uncertainty principle for the $n = 2 \pmod 4$ Clifford algebra $Cl_{n,0}$-valued admissible wavelet transform.

2. Preliminaries

This preliminary section introduces the basic knowledge [1, 3, 8, 14] of Clifford algebra $Cl_{n,0}$ and its Fourier transform. We also recall the similitude group $SIM(n)$ and its properties from the viewpoint of wavelets.

2.1. Clifford (Geometric) Algebra

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_n\}$ be an orthonormal vector basis of the real n-dimensional Euclidean vector space $\mathbb{R}^n$. The Clifford algebra over $\mathbb{R}^n$ denoted by $Cl_{n,0}$ then has the graded 2^n-dimensional basis

$$\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n, \boldsymbol{e}_{12}, \boldsymbol{e}_{31}, \boldsymbol{e}_{23}, \ldots, i_n = \boldsymbol{e}_1\boldsymbol{e}_2 \cdots \boldsymbol{e}_n\}, \tag{1}$$

where i_n is called the unit oriented pseudoscalar. Observe that $i_n^2 = -1$ for $n = 2, 3 \pmod 4$.

The associative geometric multiplication of the basis vectors is governed by the rules:

$$\begin{aligned} \boldsymbol{e}_k\, \boldsymbol{e}_l &= -\boldsymbol{e}_l\, \boldsymbol{e}_k \qquad &&\text{for} \quad k \neq l, \qquad 1 \leq k, l \leq n, \\ \boldsymbol{e}_k^2 &= 1 &&\text{for} \quad 1 \leq k \leq n. \end{aligned} \tag{2}$$

We may express Clifford algebra $Cl_{n,0}$ as the sum of an odd part $Cl_{n,0}^-$ and an even part $Cl_{n,0}^+$, i.e.

$$Cl_{n,0} = Cl_{n,0}^+ \oplus Cl_{n,0}^-. \tag{3}$$

It is straightforward to verify that $Cl_{n,0}^+$ is closed under multiplication, but $Cl_{n,0}^-$ is not. For this reason $Cl_{n,0}^+$ is often used to represent a rotation-dilation in n-dimensions.

The general elements of Clifford algebra are called multivectors[2]. Every multivector $f \in Cl_{n,0}$ can be represented in the form

$$f = \sum_A f_A \boldsymbol{e}_A, \tag{4}$$

where $f_A \in \mathbb{R}$, $\boldsymbol{e}_A = \boldsymbol{e}_{\alpha_1\alpha_2\cdots\alpha_k} = \boldsymbol{e}_{\alpha_1}\boldsymbol{e}_{\alpha_2}\cdots\boldsymbol{e}_{\alpha_k}$, and $1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq n$ with $\alpha_j \in \{1, 2, \ldots n\}$. For convenience, we introduce $\langle f\rangle_k = \sum_{|A|=k} f_A \boldsymbol{e}_A$ to denote *k-vector part* of f $(k = 0, 1, 2, \ldots, n)$, then

$$f = \sum_{k=0}^{k=n} \langle f\rangle_k = \langle f\rangle + \langle f\rangle_1 + \langle f\rangle_2 + \cdots + \langle f\rangle_n, \tag{5}$$

where $\langle \ldots\rangle_0 = \langle \ldots\rangle$.

According to (3) and (5), a multivector f may be decomposed as a sum of even part f_{even} and odd part f_{odd}. Thus

$$f = f_{even} \oplus f_{odd}, \tag{6}$$

where

$$\begin{aligned} f_{even} &= \langle f\rangle + \langle f\rangle_2 + \cdots + \langle f\rangle_r, \qquad && r = 2s, s \in \mathbb{N}, s \leq \frac{n}{2}, \\ f_{odd} &= \langle f\rangle_1 + \langle f\rangle_3 + \cdots + \langle f\rangle_r, \qquad && r = 2s+1, s \in \mathbb{N}, s < \frac{n}{2}. \end{aligned} \tag{7}$$

Later, we shall see that the distinction between even and odd grade multivectors is very important, because the even multivectors commute with i_n but the odd multivectors anti-commute with i_n.

The reverse $\tilde{f}$ of a multivector f is an anti-automorphism given by

$$\widetilde{f} = \sum_{k=0}^{k=n} (-1)^{k(k-1)/2} \langle f\rangle_k, \tag{8}$$

which satisfies $\widetilde{fg} = \tilde{g}\tilde{f}$ for every $f, g \in Cl_{n,0}$. The multivector $f \in Cl_{n,0}$ is called a paravector if (5) takes the form

$$f = \langle f\rangle + \langle f\rangle_1 = f_0 + \sum_{i=1}^{n} f_i\, \boldsymbol{e}_i. \tag{9}$$

From equation (9) it is not difficult to check that the geometric product $f\tilde{f}$ is real valued.

The scalar product of multivectors $f, \tilde{g}$ is defined as the scalar part of the geometric product $f\tilde{g}$ of multivectors

$$\langle f\tilde{g}\rangle = f * \tilde{g} = \sum_A f_A g_A, \tag{10}$$

which leads to a cyclic product symmetry

$$\langle pqr\rangle = \langle qrp\rangle, \quad \forall p, q, r \in Cl_{n,0}. \tag{11}$$

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[2]In the geometric algebra literature, multivectors are represented in capital letters, but in this paper we adopt small letters to represent multivectors: f for a multivector and $f(\boldsymbol{x})$ for a multivector-function.

In particular, if $f = g$ in (10), we obtain the modulus (or magnitude) $|f|$ of a multivector $f \in Cl_{n,0}$ defined as

$$|f|^2 = f * \tilde{f} = \sum_A f_A^2 \,. \tag{12}$$

It is convenient to introduce an inner product for two functions $f, g : \mathbb{R}^n \longrightarrow Cl_{n,0}$ as follows:

$$(f,g)_{L^2(\mathbb{R}^n;Cl_{n,0})} = \int_{\mathbb{R}^n} f(\boldsymbol{x})\widetilde{g(\boldsymbol{x})}\; d^n\boldsymbol{x} = \sum_{A,B} \boldsymbol{e}_A \widetilde{\boldsymbol{e}_B} \int_{\mathbb{R}^n} f_A(\boldsymbol{x}) g_B(\boldsymbol{x})\; d^n\boldsymbol{x}. \tag{13}$$

One can check that this inner product satisfies the following rules:

$$\begin{aligned}
(f, g+h)_{L^2(\mathbb{R}^n;Cl_{n,0})} &= (f,g)_{L^2(\mathbb{R}^n;Cl_{n,0})} + (f,h)_{L^2(\mathbb{R}^n;Cl_{n,0})}, \\
(f, \lambda g)_{L^2(\mathbb{R}^n;Cl_{n,0})} &= (f,g)_{L^2(\mathbb{R}^n;Cl_{n,0})}\tilde{\lambda}, \\
(f\lambda, g)_{L^2(\mathbb{R}^n;Cl_{n,0})} &= (f, g\tilde{\lambda})_{L^2(\mathbb{R}^n;Cl_{n,0})}, \\
(f,g)_{L^2(\mathbb{R}^n;Cl_{n,0})} &= \widetilde{(g,f)}_{L^2(\mathbb{R}^n;Cl_{n,0})},
\end{aligned} \tag{14}$$

where $f, g, h \in L^2(\mathbb{R}^n; Cl_{n,0})$ and multivector constant $\lambda \in Cl_{n,0}$. In particular, if $f = g$, then we obtain the associated norm

$$\begin{aligned}
\|f\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})} = \langle (f,f)_{L^2(\mathbb{R}^n;Cl_{n,0})} \rangle &= \int_{\mathbb{R}^n} |f|^2 d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} f(\boldsymbol{x}) * \tilde{f}(\boldsymbol{x}) d^n\boldsymbol{x} \overset{(10)}{=} \int_{\mathbb{R}^n} \sum_A f_A^2(\boldsymbol{x}) d^n\boldsymbol{x}.
\end{aligned} \tag{15}$$

Definition 1. *Let $Cl_{n,0}$ be the real Clifford algebra of n-dimensional Euclidean space. A Clifford algebra module $L^2(\mathbb{R}^n; Cl_{n,0})$ is defined by*

$$L^2(\mathbb{R}^n; Cl_{n,0}) = \{f : \mathbb{R}^n \longrightarrow Cl_{n,0}, \boldsymbol{x} \to \sum_A f_A(\boldsymbol{x}) e_A \mid \|f\|_{L^2(\mathbb{R}^n;Cl_{n,0})} < \infty\}. \tag{16}$$

As an easy consequence of the inner product (13) we obtain the *Clifford Cauchy-Schwarz* inequality

$$\left|\int_{\mathbb{R}^n} f\tilde{g}\, d^n\boldsymbol{x}\right| \leq \left(\int_{\mathbb{R}^n} |f|^2 d^n\boldsymbol{x}\right)^{1/2} \left(\int_{\mathbb{R}^n} |g|^2 d^n\boldsymbol{x}\right)^{1/2}, \quad \forall f, g \in L^2(\mathbb{R}^n; Cl_{n,0}). \tag{17}$$

2.2. Clifford Fourier Transform (CFT)

It is natural to extend the classical Fourier transform to Clifford algebra $Cl_{n,0}$, $n = 2,\, 3 \pmod 4$. This extension is often called the Clifford Fourier transform (CFT). For detailed discussions of the properties of the CFT and their proofs, see e.g. [9, 10, 15].

Definition 2. *The CFT of* $f \in L^1(\mathbb{R}^n; Cl_{n,0})$ *is the function* $\mathcal{F}\{f\}\colon \mathbb{R}^n \to Cl_{n,0}$ *given by*

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{x}, \tag{18}$$

where we can write in component

$$\boldsymbol{\omega} = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \cdots + \omega_n \boldsymbol{e}_n \text{ and } \boldsymbol{\omega}\cdot\boldsymbol{x} = \omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n. \tag{19}$$

The Clifford exponential $e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}$ is often called Clifford Fourier kernel. It should be remembered that

$$d^n\boldsymbol{x} = \frac{d\boldsymbol{x_1} \wedge d\boldsymbol{x_2} \wedge \cdots \wedge d\boldsymbol{x_n}}{i_n}, \tag{20}$$

is scalar valued ($d\boldsymbol{x}_k = dx_k \boldsymbol{e}_k$, $k = 1, 2, \ldots, n$, no summation). For the dimension $n = 3 \pmod 4$ $e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}$ commutes with all elements of $Cl_{n,0}$, but for $n = 2 \pmod 4$ it does not. As we will see later, the different commutation rules of $e^{-i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}$ play a crucial rule in establishing the properties of the Clifford algebra $Cl_{n,0}$-valued wavelet transform.

Theorem 1. *Suppose that* $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ *and* $\mathcal{F}\{f\} \in L^1(\mathbb{R}^n; Cl_{n,0})$. *Then the CFT of* $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ *is invertible and its inverse is calculated by*

$$\mathcal{F}^{-1}[\mathcal{F}\{f\}](\boldsymbol{x}) = f(\boldsymbol{x}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\boldsymbol{\omega})\, e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n\boldsymbol{\omega}. \tag{21}$$

For the sake of simplicity, if not otherwise stated, n is assumed to be $n = 2$ (mod 4) in the rest of this section.

2.3. Similitude Group

We consider *the similitude* group *SIM (n)* denoted by $\mathcal{G}$, a subgroup of the affine group of motion on $\mathbb{R}^n$ associated with wavelets as follows

$$\mathcal{G} = \mathbb{R}^+ \times SO(n) \rtimes \mathbb{R}^n = \{(a, r_{\boldsymbol{\theta}}, \boldsymbol{b}) | a \in \mathbb{R}^+, r_{\boldsymbol{\theta}} \in SO(n), \boldsymbol{b} \in \mathbb{R}^n\}, \tag{22}$$

where $SO(n)$ is the special orthogonal group of $\mathbb{R}^n$. Instead of $(a, r_{\boldsymbol{\theta}}, \boldsymbol{b})$ we often write simply $(a, \boldsymbol{\theta}, \boldsymbol{b})$. More precisely, we represent $SO(n)$ of $\mathbb{R}^n$ by rotors R in the *spin group*, i.e.

$$\begin{aligned} Spin(n) &= \{R \in Cl^+_{n,0}, \tilde{R}R = R\tilde{R} = 1\} \\ SO(n) &= \{r \,|r(\boldsymbol{x}) = \tilde{R}\boldsymbol{x}R, R \in Spin(n)\}, \end{aligned} \tag{23}$$

where $r(\boldsymbol{x})$ denotes the rotation of a vector $\boldsymbol{x} \in \mathbb{R}^n$. We observe that

$$SO(n) \cong Spin(n)/\{\pm 1\}. \tag{24}$$

Clearly, the group $\mathcal{G}$ includes dilations, rotations and translations. The representation defined by (22) is consistent with the group action of dilation, rotation and translation $(a, \boldsymbol{\theta}, \boldsymbol{b})$ on $\mathbb{R}^n$ as follows

$$\begin{aligned} (a, \boldsymbol{\theta}, \boldsymbol{b}) :\ & \mathbb{R}^n \to \mathbb{R}^n \\ & \boldsymbol{x} \mapsto a\tilde{R}(\boldsymbol{\theta})\boldsymbol{x}R(\boldsymbol{\theta}) + \boldsymbol{b}. \end{aligned} \tag{25}$$

The above leads to two important propositions.

Proposition 1. *With respect to the representation defined by* (22), $\mathcal{G}$ *is a non-abelian group in which* $(1, 1, 0)$ *and* $(a^{-1}, r^{-1}, -a^{-1}r^{-1}(\boldsymbol{b}) = -R\boldsymbol{b}\tilde{R}/a)$ *are its identity element and inverse element, respectively.*

Proposition 2. *The left Haar measure on* $\mathcal{G}$ *is given by*

$$d\lambda(a, \boldsymbol{\theta}, \boldsymbol{b}) = d\mu(a, \boldsymbol{\theta}) d^n\boldsymbol{b},$$
$$d\mu(a, \boldsymbol{\theta}) = \frac{dad\boldsymbol{\theta}}{a^{n+1}}, \tag{26}$$

where $d\boldsymbol{\theta}$ *is the Haar measure on SO(n)* (*see* [11]).

We often abbreviate $d\mu = d\mu(a, \boldsymbol{\theta})$ and $d\lambda = d\lambda(a, \boldsymbol{\theta}, \boldsymbol{b})$. Similar to (13), the inner product of $f, g \in L^2(\mathcal{G}; Cl_{n,0})$ is defined by

$$(f, g)_{L^2(\mathcal{G};Cl_{n,0})} = \int_{\mathcal{G}} f(a, \boldsymbol{\theta}, \boldsymbol{b}) g\widetilde{(a, \boldsymbol{\theta}, \boldsymbol{b})}\, d\lambda, \tag{27}$$

and its associated scalar norm

$$\|f\|_{L^2(\mathcal{G};Cl_{n,0})} = \langle (f, f)_{L^2(\mathcal{G};Cl_{n,0})} \rangle = \int_{\mathcal{G}} f(a, \boldsymbol{\theta}, \boldsymbol{b}) * f\widetilde{(a, \boldsymbol{\theta}, \boldsymbol{b})}\, d\lambda. \tag{28}$$

3. Construction of Clifford Algebra-Valued Wavelet Transform in $Cl_{n,0}$, $n = 2 \pmod 4$

This section extends classical wavelet transform to Clifford algebra $Cl_{n,0}$, $n = 2 \pmod 4$ using the admissible similitude group $SIM(n)$. Using the spectral representation of the CFT, we investigate some of its fundamental properties. Due to the non-commutative multiplication rule of Clifford algebra, a number properties of classical wavelet transform are modified in the new construction. In particular we look at the admissibility condition, inner product, norm relation, and inversion formula.

3.1. Clifford Admissible Wavelet and its Wavelet Transform

Definition 3 (Clifford admissible wavelet). *A Clifford admissible wavelet is a multivector function* $\psi \in L^2(\mathbb{R}^n; Cl_{n,0})$ *which satisfies the following admissibility condition, i.e.*

$$C_\psi = \int_{\mathbb{R}^n} \frac{\{\widehat{\psi}(\boldsymbol{\omega})\}^{\sim}\, \widehat{\psi}(\boldsymbol{\omega})}{|\boldsymbol{\omega}|^n}\, d^n\boldsymbol{\omega}, \tag{29}$$

such that C_ψ *is an invertible multivector constant.*

Similar to classical wavelets in [7, 12], the zero *th* moment of the Clifford admissible wavelet $\psi \in L^2(\mathbb{R}^n; Cl_{n,0})$ vanishes

$$\int_{\mathbb{R}^n} \psi(\boldsymbol{x})\, d^n\boldsymbol{x} = \int_{\mathbb{R}^n} \psi_A(\boldsymbol{x})\boldsymbol{e}_A\, d^n\boldsymbol{x} = 0, \tag{30}$$

where $\psi_A(\boldsymbol{x}) = \psi * \widetilde{\boldsymbol{e}}_A$ are 2^n real-valued functions. Thus, the integral of every component ψ_A of the Clifford mother wavelet is zero:

$$\int_{\mathbb{R}^n} \psi_A(\boldsymbol{x}) d^n\boldsymbol{x} = 0. \tag{31}$$

If $\psi_A \neq 0$, then it is a real valued admissible wavelet. With the help of equation (5), we can decompose (29) to the following form[3]

$$C_\psi = \langle C_\psi \rangle + \langle C_\psi \rangle_1 + \langle C_\psi \rangle_2 + \cdots + \langle C_\psi \rangle_n, \tag{32}$$

where ψ determines the nonzero and zero terms.

Definition 4. *For $\psi \in L^2(\mathbb{R}^n; Cl_{n,0}), a \in \mathbb{R}^+, \boldsymbol{b} \in \mathbb{R}^n$, and $\boldsymbol{\theta} \in SO(n)$. We define the unitary linear operator*

$$U_{a,\boldsymbol{\theta},\boldsymbol{b}} : L^2(\mathbb{R}^n; Cl_{n,0}) \longrightarrow L^2(\mathcal{G}; Cl_{n,0}),$$

given by

$$\left(U_{a,\boldsymbol{\theta},\boldsymbol{b}}(\psi)\right) = \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x}) = \frac{1}{a^{\frac{n}{2}}} \psi\left(r_{\boldsymbol{\theta}}^{-1}\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)\right). \tag{33}$$

The family of wavelets $\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}$ are so-called *daughter Clifford wavelets* with a as the dilation parameter, $\boldsymbol{b}$ as the translation vector parameter, and $\boldsymbol{\theta}$ as the $SO(n)$ rotation parameters.

Lemma 1. *If $\psi \in L^2(\mathbb{R}^n; Cl_{n,0})$, then the norm of $\psi_{a,\boldsymbol{\theta},\boldsymbol{b}} \in L^2(\mathbb{R}^n; Cl_{n,0})$ is independent on a, equivalently,*

$$\|\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\|_{L^2(\mathbb{R}^n;Cl_{n,0})} = ||\psi||_{L^2(\mathbb{R}^n;Cl_{n,0})}. \tag{34}$$

Lemma 2. *For $\psi_{a,\boldsymbol{\theta},\boldsymbol{b}} \in L^2(\mathbb{R}^n; Cl_{n,0})$, we have*

$$\mathcal{F}\{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}\}(\boldsymbol{\omega}) = a^{\frac{n}{2}} \widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\, e^{-i_n \boldsymbol{b}\cdot\boldsymbol{\omega}}. \tag{35}$$

Definition 5 (Clifford admissible wavelet transform). *Let $\psi \in L^2(\mathbb{R}^n; Cl_{n,0})$ be the Clifford admissible mother wavelet. The transformation T_ψ given by*

$$\begin{aligned} T_\psi : L^2(\mathbb{R}^n; Cl_{n,0}) &\to L^2(\mathcal{G}; Cl_{n,0}) \\ f &\longmapsto T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) \\ &= (f, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^n;Cl_{n,0})} \\ &= \int_{\mathbb{R}^n} f(\boldsymbol{x}) \frac{1}{a^{\frac{n}{2}}} \left[\psi\left(r_{\boldsymbol{\theta}}^{-1}\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)\right)\right]^{\sim} d^n\boldsymbol{x} \end{aligned} \tag{36}$$

is called the Clifford admissible wavelet transform.

Please note that the order in (36) is fixed because of the non-commutativity of the product of Clifford algebra. Since $e^{-i_n \boldsymbol{b}\cdot\boldsymbol{\omega}}$ does not commute with any element of $Cl_{n,0}$ we get the following result (compare to [17]).

[3] It is obvious that $C_\psi = \widetilde{C_\psi}$. Therefore, we can also write $C_\psi = \int_{\mathbb{R}^n} \frac{\widehat{\psi}(\boldsymbol{\omega})\,\{\widehat{\psi}(\boldsymbol{\omega})\}^{\sim}}{|\boldsymbol{\omega}|^n}\, d^n\boldsymbol{\omega}$.

Lemma 3. *The Clifford admissible wavelet transform* (36) *has a Clifford Fourier representation of the form*

$$T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f}(\boldsymbol{\omega})\, e^{i_n \boldsymbol{b}\cdot\boldsymbol{\omega}} a^{\frac{n}{2}} \{\widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim}\, d^n\boldsymbol{\omega}. \tag{37}$$

By reversing to (37) we obtain

$$\widetilde{T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} a^{\frac{n}{2}} \widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega})) e^{-i_n \boldsymbol{b}\cdot\boldsymbol{\omega}}\, \widetilde{\widehat{f}}(\boldsymbol{\omega})\, d^n\boldsymbol{\omega}. \tag{38}$$

Remark 3.1. *Note that formula* (37) *can not be written in the form*

$$\mathcal{F}(T_\psi f(a,\boldsymbol{\theta},\cdot))(\boldsymbol{\omega}) = a^{\frac{n}{2}} \widehat{f}(\boldsymbol{\omega})\{\widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim}. \tag{39}$$

However, formula (39) *is valid if the Clifford mother wavelet ψ is the even grade multivector.*

If the CFT of a multivector $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ is the odd grade multivector, then equation (38) can be expressed as

$$\begin{aligned}
\widetilde{T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})} &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} a^{\frac{n}{2}} \widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega})) e^{-i_n \boldsymbol{b}\cdot\boldsymbol{\omega}}\, \widetilde{\widehat{f}}(\boldsymbol{\omega})\, d^n\boldsymbol{\omega} \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} a^{\frac{n}{2}} \widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\widetilde{\widehat{f}}(\boldsymbol{\omega})\, e^{i_n \boldsymbol{b}\cdot\boldsymbol{\omega}} d^n\boldsymbol{\omega} \\
&= \mathcal{F}^{-1}\left\{ a^{\frac{n}{2}} \widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\widetilde{\widehat{f}}(\boldsymbol{\omega}) \right\}(\boldsymbol{b}).
\end{aligned} \tag{40}$$

Or, equivalently,

$$\mathcal{F}(\widetilde{T_\psi f(a,\boldsymbol{\theta},\cdot)})(\boldsymbol{\omega}) = a^{\frac{n}{2}} \widehat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\widetilde{\widehat{f}}(\boldsymbol{\omega}). \tag{41}$$

3.2. Its Properties

Let us start this subsection by listing the properties of the Clifford algebra $Cl_{n,0}$-valued wavelet transform, which correspond to classical wavelet transform properties

Proposition 3. *Let ψ and ϕ be two Clifford admissible wavelets. If f and g are two Clifford module functions belong to $L^2(\mathbb{R}^n; Cl_{n,0})$. Then for every $(a,\boldsymbol{b}) \in \mathbb{R}^+ \times \mathbb{R}^n$*

(i) *(Left linearity)*

$$[T_\psi(\lambda f + \mu g)](a,\boldsymbol{\theta},\boldsymbol{b}) = \lambda T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}) + \mu T_\psi g(a,\boldsymbol{\theta},\boldsymbol{b}),$$

where λ and μ are multivector constants in $Cl_{n,0}$.

(ii) *(Translation covariance)*

$$[T_\psi f(\cdot - \boldsymbol{x}_0)](a,\boldsymbol{\theta},\boldsymbol{b}) = T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}-\boldsymbol{x}_0),$$

for any constant $\boldsymbol{x}_0 \in \mathbb{R}^n$.

(iii) *(Dilation covariance)*

$$[T_\psi f(c\,\cdot)](a,\boldsymbol{\theta},\boldsymbol{b}) = \frac{1}{c^{\frac{n}{2}}} T_\psi f(ac,\boldsymbol{\theta},\boldsymbol{b}c),$$

where c is a real positive constant.

(iv) *(Rotation covariance)*

$$[T_\psi f(r_{\boldsymbol{\theta}_0}\cdot)](a,\boldsymbol{\theta},\boldsymbol{b}) = T_\psi f(a,\boldsymbol{\theta}', r_{\boldsymbol{\theta}_0}\boldsymbol{b}),$$

with rotors $R_{\boldsymbol{\theta}'} = R_{\boldsymbol{\theta}_0} R_{\boldsymbol{\theta}}$.

(v) *(Parity)*

$$[T_{P\psi} P f](a,\boldsymbol{\theta},\boldsymbol{b}) = T_\psi f(a,\boldsymbol{\theta},-\boldsymbol{b}),$$

where P *is the parity operator defined by* $Pf(x) = f(-x)$.

(vi) *(Antilinearity)*

$$[T_{\lambda\psi+\mu\phi} f](a,\boldsymbol{\theta},\boldsymbol{b}) = T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b})\,\tilde{\lambda} + T_\phi f(a,\boldsymbol{\theta},\boldsymbol{b})\,\tilde{\mu},$$

for any multivector constants λ, μ *in* $Cl_{n,0}$.

(vii) *If we introduce the translation operator* $M_{\boldsymbol{x}_0}\psi(\boldsymbol{x}) = \psi(\boldsymbol{x}-\boldsymbol{x}_0)$, *then*

$$[T_{M_{\boldsymbol{x}_0}\psi} f](a,\boldsymbol{\theta},\boldsymbol{b}) = T_\psi f(a,\boldsymbol{\theta},\boldsymbol{b}+\boldsymbol{x}_0 a).$$

(viii) *Consider the dilation operator* $D^c\psi(\boldsymbol{x}) = \frac{1}{c^n}\psi\left(\frac{\boldsymbol{x}}{c}\right), c > 0$. *We have*

$$[T_{D^c\psi} f](a,\boldsymbol{\theta},\boldsymbol{b}) = \frac{1}{c^{n/2}} T_\psi f(ac,\boldsymbol{\theta},\boldsymbol{b}).$$

Proof.

(v) Direct calculations give for every $f \in L^2(\mathbb{R}^n; Cl_{n,0})$

$$\begin{aligned}
[T_{P\psi} P f](a,\boldsymbol{\theta},\boldsymbol{b}) &= \int_{\mathbb{R}^n} f(-\boldsymbol{x})\widetilde{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(-\boldsymbol{x})}\, d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} f(-\boldsymbol{x})\frac{1}{a^{n/2}}\left[\psi\left(r_{\boldsymbol{\theta}}^{-1}(\frac{-\boldsymbol{x}+\boldsymbol{b}}{a})\right)\right]^\sim d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} f(-\boldsymbol{x})\frac{1}{a^{n/2}}\left[\psi\left(r_{\boldsymbol{\theta}}^{-1}(\frac{-\boldsymbol{x}-(-\boldsymbol{b})}{a})\right)\right]^\sim d^n\boldsymbol{x},
\end{aligned}$$

and the proof is complete. □

(vi) Application of Definition 5 and the inner product (14) gives

$$\begin{aligned}
[T_{\lambda\psi+\mu\phi} f](a,\boldsymbol{\theta},\boldsymbol{b}) &= (f, \lambda\psi_{a,\boldsymbol{\theta},\boldsymbol{b}} + \mu\phi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^n;Cl_{n,0})} \\
&= (f, \lambda\psi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^n;Cl_{n,0})} + (f, \mu\phi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^n;Cl_{n,0})} \\
&= (f, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^n;Cl_{n,0})}\tilde{\lambda} + (f, \mu\phi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^n;Cl_{n,0})}\tilde{\mu}.
\end{aligned} \quad (42)$$

This finishes the proof. □

(vii) Equation (36) gives

$$\begin{aligned}
[T_{M_{\boldsymbol{x}_0}\psi} f](a,\boldsymbol{\theta},\boldsymbol{b}) &= \int_{\mathbb{R}^n} f(\boldsymbol{x})\{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x}-\boldsymbol{x}_0)\}^\sim d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} f(\boldsymbol{x})\frac{1}{a^{n/2}}\left[\psi\left(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{x}-\boldsymbol{b}}{a} - \boldsymbol{x}_0)\right)\right]^\sim d^n\boldsymbol{x} \\
&= \int_{\mathbb{R}^n} f(\boldsymbol{x})\frac{1}{a^{n/2}}\left[\psi\left(r_{\boldsymbol{\theta}}^{-1}\left(\frac{(\boldsymbol{x}-(\boldsymbol{b}+a\boldsymbol{x}_0)}{a}\right)\right)\right]^\sim d^n\boldsymbol{x},
\end{aligned}$$

which was to be proved. □

(viii) Equation (36) gives again

$$\begin{aligned}[T_{M_{\boldsymbol{x}_0}\psi} f](a,\boldsymbol{\theta},\boldsymbol{b}) &= \int_{\mathbb{R}^n} f(\boldsymbol{x}) \frac{1}{c^n} \{\psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x}/c)\}^{\sim} \, d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} f(\boldsymbol{x}) \frac{1}{c^n} \, \frac{1}{a^{n/2}} \left[\psi\left(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{x}-\boldsymbol{b}}{ac})\right)\right]^{\sim} d^n\boldsymbol{x} \\ &= \frac{1}{c^{n/2}} \int_{\mathbb{R}^n} f(\boldsymbol{x}) \frac{1}{(ac)^{n/2}} \left[\psi\left(r_{\boldsymbol{\theta}}^{-1}(\frac{\boldsymbol{x}-\boldsymbol{b}}{ac})\right)\right]^{\sim} d^n\boldsymbol{x},\end{aligned}$$

which completes the proof.

□

The proof of the remaining properties in the above-mentioned proposition was verified by straightforward calculations and exploited in [17]. Now we study the differences between the Clifford and classical continuous wavelet transforms.

Theorem 2 (Inner product relation). *Let $\psi \in L^2(\mathbb{R}^n; Cl_{n,0})$ be a Clifford admissible mother wavelet and let C_ψ be the admissibility condition defined by*

$$C_\psi = \int_{SO(n)} \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^n} \{\hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega}))\}^{\sim} \, \hat{\psi}(ar_\theta^{-1}(\boldsymbol{\omega})) \, d^n\boldsymbol{\omega} \right) \frac{da d\boldsymbol{\theta}}{a}, \quad |\boldsymbol{\omega}| = 1, \tag{43}$$

and (43) is independent of $\boldsymbol{\omega}$.

(i) *If $\{\widehat{\psi}(\boldsymbol{\omega})\}^{\sim}\widehat{\psi}(\boldsymbol{\omega})$ is an even grade multivector for all $\boldsymbol{\omega} \in \mathbb{R}^n$, then for any multivector functions $f, g \in L^2(\mathbb{R}^n; Cl_{n,0})$ we get*

$$(T_\psi f, T_\psi g)_{L^2(\mathcal{G};Cl_{n,0})} = (f C_{\psi_{even}}, g)_{L^2(\mathbb{R}^n;Cl_{n,0})}, \tag{44}$$

where for all $r = 2s, s \in \mathbb{N}, s \leq n/2$

$$\begin{aligned}(f C_{\psi_{even}}, g)_{L^2(\mathbb{R}^n;Cl_{n,0})} &= \langle C_\psi \rangle (f,g)_{L^2(\mathbb{R}^n;Cl_{n,0})} + (f\langle C_\psi\rangle_2, g)_{L^2(\mathbb{R}^n;Cl_{n,0})} + \\ &\quad (f\langle C_\psi\rangle_4, g)_{L^2(\mathbb{R}^n;Cl_{n,0})} + \cdots + (f\langle C_\psi\rangle_r, g)_{L^2(\mathbb{R}^n;Cl_{n,0})}.\end{aligned}$$

(ii) *If the CFT of a Clifford mother wavelet $\psi \in L^2(\mathbb{R}^n; Cl_{n,0})$ is the odd grade multivector, we obtain*

$$(T_\psi f, T_\psi g)_{L^2(\mathcal{G};Cl_{n,0})} = (f C_\psi, g)_{L^2(\mathbb{R}^n;Cl_{n,0})} \tag{45}$$

where $C_\psi = C_{\psi_{even}}$ or $C_\psi = C_{\psi_{odd}}$

Proof.

(i) Suppose that $\{\widehat{\psi}(\boldsymbol{\omega})\}^{\sim}\widehat{\psi}(\boldsymbol{\omega})$ is an even grade multivector. Then, by inserting (37) and (38) into the left-hand side of (44), we immediately obtain

$$
\begin{aligned}
&(T_\psi f, T_\psi g)_{L^2(\mathcal{G};Cl_{n,0})} \\
&= \int_{SO(n)}\int_{\mathbb{R}^+} \frac{a^n}{(2\pi)^{2n}} \left(\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\omega}) e^{i_n \boldsymbol{b}\cdot\boldsymbol{\omega}} \{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim} d^n\boldsymbol{\omega} \right.\right. \\
&\qquad\qquad \left.\left. \times \int_{\mathbb{R}^n} \left\{ \hat{g}(\boldsymbol{\omega}') e^{i_n \boldsymbol{b}\cdot\boldsymbol{\omega}'} \{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}'))\}^{\sim} \right\}^{\sim} d^n\boldsymbol{\omega}' \right] d^n\boldsymbol{b} \right) d\mu \\
&= \int_{SO(n)}\int_{\mathbb{R}^+} \frac{a^n}{(2\pi)^{2n}} \left(\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\omega}) e^{i_n \boldsymbol{b}\cdot\boldsymbol{\omega}} \{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim} \hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}'))\, d^n\boldsymbol{\omega} \right.\right. \\
&\qquad\qquad \left.\left. \times \int_{\mathbb{R}^n} e^{-i_n \boldsymbol{b}\cdot\boldsymbol{\omega}'} \widetilde{\hat{g}(\boldsymbol{\omega}')} d^n\boldsymbol{\omega}' \right] d^n\boldsymbol{b} \right) d\mu \\
&= \int_{SO(n)}\int_{\mathbb{R}^+} \frac{a^n}{(2\pi)^{2n}} \left(\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\omega}) \{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim} \hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}')) \int_{\mathbb{R}^n} e^{i_n \boldsymbol{b}\cdot\boldsymbol{\omega}} \right. \\
&\qquad\qquad \left. \times\ e^{-i_n \boldsymbol{b}\cdot\boldsymbol{\omega}'} d^n\boldsymbol{b} \int_{\mathbb{R}^n} \widetilde{\hat{g}(\boldsymbol{\omega}')} d^n\boldsymbol{\omega}'\, d^n\boldsymbol{\omega} \right) d\mu \\
&= \int_{SO(n)}\int_{\mathbb{R}^+} \frac{a^n}{(2\pi)^{n}} \left(\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\omega}) \{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim} \hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}')) \right. \\
&\qquad\qquad \left. \times\ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i_n \boldsymbol{b}\cdot(\boldsymbol{\omega}'-\boldsymbol{\omega})} d^n\boldsymbol{b} \int_{\mathbb{R}^n} \widetilde{\hat{g}(\boldsymbol{\omega}')} d^n\boldsymbol{\omega}'\, d^n\boldsymbol{\omega} \right) d\mu.
\end{aligned}
\tag{46}
$$

Second, from assumption we obtain the admissibility condition (43) is an even grade multivector, then application of the orthogonality of harmonic exponential function of the above equation leads to

$$
\begin{aligned}
&(T_\psi f, T_\psi g)_{L^2(\mathcal{G};Cl_{n,0})} \\
&= \frac{1}{(2\pi)^n} \int_{SO(n)}\int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\omega}) \{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim} \right. \\
&\qquad \left. \times \int_{\mathbb{R}^n} \hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}')) \delta(\boldsymbol{\omega}'-\boldsymbol{\omega}) \widetilde{\hat{g}(\boldsymbol{\omega}')}\, d^n\boldsymbol{\omega}'\, d^n\boldsymbol{\omega} \right) \frac{da d\boldsymbol{\theta}}{a} \\
&= \frac{1}{(2\pi)^n} \int_{SO(n)}\int_{\mathbb{R}^+} \left(\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\omega}) \{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim}\, \hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\, \widetilde{\hat{g}(\boldsymbol{\omega})}\, d^n\boldsymbol{\omega} \right) \frac{da d\boldsymbol{\theta}}{a} \\
&\overset{(29)}{=} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\omega}) C_{\psi_{even}}\, \widetilde{\hat{g}(\boldsymbol{\omega})}\, d^n\boldsymbol{\omega} \\
&\overset{\text{Plancherel Th.}}{=} \int_{\mathbb{R}^n} f(\boldsymbol{x}) C_{\psi_{even}}\, \widetilde{g(\boldsymbol{x})}\, d^n\boldsymbol{x} \\
&= (f C_{\psi_{even}}, g)_{L^2(\mathbb{R}^n;Cl_{n,0})}.
\end{aligned}
\tag{47}
$$

Hence, by applying the linearity of the inner product (14), the identity (44) follows immediately. □

(ii) Assume that $\tilde{\psi} \in L^2(\mathbb{R}^n; Cl_{n,0})$ is the odd grade multivector. Then

$$\begin{aligned}
&(T_\psi f, T_\psi g)_{L^2(\mathcal{G};Cl_{n,0})} \\
&= \int_{SO(n)} \int_{\mathbb{R}^+} \frac{a^n}{(2\pi)^{2n}} \left(\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\omega}) e^{i_n \boldsymbol{b}\cdot\boldsymbol{\omega}} \{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim} d^n\boldsymbol{\omega} \right.\right. \\
&\qquad \left.\left. \times \int_{\mathbb{R}^n} \left\{ \hat{g}(\boldsymbol{\omega}') e^{i_n \boldsymbol{b}\cdot\boldsymbol{\omega}'} \{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}'))\}^{\sim} \right\}^{\sim} d^n\boldsymbol{\omega}' \right] d^n\boldsymbol{b} \right) d\mu \\
&= \int_{SO(n)} \int_{\mathbb{R}^+} \frac{a^n}{(2\pi)^{2n}} \left(\int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\omega}) \{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim} e^{-i_n \boldsymbol{b}\cdot\boldsymbol{\omega}} \hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}')) d^n\boldsymbol{\omega} \right.\right. \\
&\qquad \left.\left. \times \int_{\mathbb{R}^n} e^{-i_n \boldsymbol{b}\cdot\boldsymbol{\omega}'} \widetilde{\hat{g}(\boldsymbol{\omega}')} d^n\boldsymbol{\omega}' \right] d^n\boldsymbol{b} \right) d\mu \\
&= \int_{SO(n)} \int_{\mathbb{R}^+} \frac{a^n}{(2\pi)^{n}} \left(\int_{\mathbb{R}^n} \hat{f}(\boldsymbol{\omega}) \{\hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i_n \boldsymbol{b}\cdot(\boldsymbol{\omega}-\boldsymbol{\omega}')} d^n\boldsymbol{b} \right. \\
&\qquad \left. \times \int_{\mathbb{R}^n} \hat{\psi}(ar_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}')) \widetilde{\hat{g}(\boldsymbol{\omega}')} \, d^n\boldsymbol{\omega}' \, d^n\boldsymbol{\omega} \right) d\mu. \qquad (48)
\end{aligned}$$

Here we have used the fact that $\tilde{\psi}$ anti-commutes with $e^{i_n \boldsymbol{b}\cdot\boldsymbol{\omega}}$ (see [10] for more details), i.e.

$$\tilde{\psi}\, e^{i_n \boldsymbol{b}\cdot\boldsymbol{\omega}} = e^{-i_n \boldsymbol{b}\cdot\boldsymbol{\omega}} \tilde{\psi}, \qquad \forall \tilde{\psi} \in L^2(\mathbb{R}^n; Cl_{n,0}). \qquad (49)$$

Next, we follow the steps of the proof of (47) by replacing $C_{\psi_{odd}}$ with C_ψ, then we arrive at equation (45). □

Taking the scalar part of Theorem 2, for $f = g$ we immediately obtain the following corollary.

Corollary 1 (Norm relation). *Let $\psi \in L^2(\mathbb{R}^n; Cl_{n,0})$ be a Clifford admissible mother wavelet.*

(i) *Under the assumptions stated in the first part of Theorem 2, we get*

$$\begin{aligned}
\|T_\psi f\|^2_{L^2(\mathcal{G};Cl_{n,0})} &= C_{\psi_{even}} * (\tilde{f}, \tilde{f})_{L^2(\mathbb{R}^n;Cl_{n,0})}. \\
&= \langle C_\psi \rangle \|f\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})} + \langle C_\psi \rangle_2 * \langle (\tilde{f}, \tilde{f})_{L^2(\mathbb{R}^n;Cl_{n,0})} \rangle_2 \\
&\quad + \cdots + \langle C_\psi \rangle_r * \langle (\tilde{f}, \tilde{f})_{L^2(\mathbb{R}^n;Cl_{n,0})} \rangle_r, r = 2s, s \in \mathbb{N}, s \le n/2. \qquad (50)
\end{aligned}$$

(ii) *With the assumptions as the second part of Theorem 2, we obtain*

$$\|T_\psi f\|^2_{L^2(\mathcal{G};Cl_{n,0})} = C_\psi * (\tilde{f}, \tilde{f})_{L^2(\mathbb{R}^n;Cl_{n,0})}. \qquad (51)$$

*By reversing to (*51*) we have*

$$\|\widetilde{T_\psi f}\|^2_{L^2(\mathcal{G};Cl_{n,0})} = \langle (f, f C_\psi)_{L^2(\mathbb{R}^n;Cl_{n,0})} \rangle = C_\psi * (\tilde{f}, \tilde{f})_{L^2(\mathbb{R}^n;Cl_{n,0})}. \qquad (52)$$

Corollary 2 (Inversion formula). *Let $\psi \in L^2(\mathbb{R}^n; Cl_{n,0})$ be a Clifford admissible mother wavelet.*

(i) *Under the assumptions in the first part of Theorem 2, any $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ can be decomposed as*

$$f(\boldsymbol{x}) = \int_{\mathcal{G}} (f, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^n;Cl_{n,0})} \psi_{a,\boldsymbol{\theta},\boldsymbol{b}} \, C_\psi^{-1} \, d\lambda, \tag{53}$$

where the equality holds a.e. and $C_\psi = C_{\psi_{even}}$.

(ii) *With the assumptions as the second part of Theorem 2, any $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ can be decomposed as*

$$f(\boldsymbol{x}) = \int_{\mathcal{G}} (f, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^n;Cl_{n,0})} \psi_{a,\boldsymbol{\theta},\boldsymbol{b}} \, C_\psi^{-1} \, d\lambda, \tag{54}$$

where the equality holds a.e. and $C_\psi = C_{\psi_{even}}$ or $C_\psi = C_{\psi_{odd}}$.

Proof. The proof of this identity is the same as that of Theorem 3. 7 in the paper [17]. □

Theorem 3 (Reproducing kernel). *Let $\psi \in L^2(\mathbb{R}^n; Cl_{n,0})$ be the Clifford admissible wavelet. If*

$$\mathbb{K}_\psi(a, \boldsymbol{\theta}, \boldsymbol{b}; a', \boldsymbol{\theta}', \boldsymbol{b}') = (\psi_{a,\boldsymbol{\theta},\boldsymbol{b}} C_\psi^{-1}; \psi_{a',\boldsymbol{\theta}',\boldsymbol{b}'})_{L^2(\mathbb{R}^n;Cl_{n,0})}. \tag{55}$$

Then $\mathbb{K}_\psi(a, \boldsymbol{\theta}, \boldsymbol{b}; a', \boldsymbol{\theta}', \boldsymbol{b}')$ is a reproducing kernel in $L^2(\mathcal{G}, d\lambda)$, i.e.

$$T_\psi f(a', \boldsymbol{\theta}', \boldsymbol{b}') = \int_{\mathcal{G}} T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) \mathbb{K}_\psi(a, \boldsymbol{\theta}, \boldsymbol{b}; a', \boldsymbol{\theta}', \boldsymbol{b}') \, d\lambda. \tag{56}$$

Proof. Substituting (54) into Definition 5 yields

$$\begin{aligned} T_\psi f(a', \boldsymbol{\theta}', \boldsymbol{b}') &= \int_{\mathbb{R}^n} f(\boldsymbol{x}) \widetilde{\psi_{a',\boldsymbol{\theta}',\boldsymbol{b}'}(\boldsymbol{x})} \, d^n\boldsymbol{x} \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathcal{G}} (f, \psi_{a,\boldsymbol{\theta},\boldsymbol{b}})_{L^2(\mathbb{R}^n;Cl_{n,0})} \psi_{a,\boldsymbol{\theta},\boldsymbol{b}} \, d\lambda \, C_\psi^{-1} \right) \widetilde{\psi_{a',\boldsymbol{\theta}',\boldsymbol{b}'}(\boldsymbol{x})} \, d^n\boldsymbol{x} \\ &= \int_{\mathcal{G}} T_\psi f(a, \boldsymbol{\theta}, \boldsymbol{b}) \left(\int_{\mathbb{R}^n} \psi_{a,\boldsymbol{\theta},\boldsymbol{b}}(\boldsymbol{x}) C_\psi^{-1} \{\psi_{a',\boldsymbol{\theta}',\boldsymbol{b}'}(\boldsymbol{x})\}^\sim \, d^n\boldsymbol{x} \right) d\lambda \\ &= \int_{\mathcal{G}} T_\psi f(a, \boldsymbol{b}, \boldsymbol{\theta}) \mathbb{K}_\psi(a, \boldsymbol{\theta}, \boldsymbol{b}; a', \boldsymbol{\theta}', \boldsymbol{b}') \, d\lambda. \end{aligned} \tag{57}$$

This achieves the proof. □

4. Uncertainty Principle for Clifford Algebra $Cl_{n,0}$ wavelets

The classical uncertainty principle of harmonic analysis states that a nontrivial function and its FT cannot both be simultaneously sharply localized [19]. In quantum mechanics an uncertainty principle asserts that one cannot at the same time be certain of the position and of the velocity of an electron (or any particle). That is, increasing the knowledge of the position decreases the knowledge of the velocity or momentum of an electron. This section extends the uncertainty principle which is valid for the CFT to the setting of the Clifford algebra $Cl_{n,0}$-valued wavelets.

Let us now formulate an uncertainty principle for Clifford algebra $Cl_{n,0}$-valued wavelets. This principle describes how the *reversion* of the Clifford algebra $Cl_{n,0}$-valued wavelet transform relates to the CFT of an odd grade multivector function.

Theorem 4. *Let ψ be a Clifford admissible wavelet that satisfies the admissibility condition (43). If the CFT of a multivector $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ is the odd grade multivector, then we have the inequality*

$$\begin{aligned} &\|\boldsymbol{b} T_\psi \widetilde{f(a,\boldsymbol{\theta},\boldsymbol{b})}\|_{L^2(\mathcal{G};Cl_{n,0})} \left(C_\psi * (\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^n;Cl_{n,0})} \right)^{\frac{1}{2}} \\ &\geq \frac{\sqrt{n}(2\pi)^{n/2}}{2} \left[C_\psi * (\tilde{f}, \tilde{f})_{L^2(\mathbb{R}^n;Cl_{n,0})} \right]. \end{aligned} \tag{58}$$

In order to prove this theorem, we need to introduce the following lemma.

Lemma 4.

$$\int_{SO(n)} \int_{\mathbb{R}^+} \|\boldsymbol{\omega}\ \mathcal{F}\{T_\psi \widetilde{f(a,\boldsymbol{\theta},\cdot)}\}\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})}\, d\mu = C_\psi * (\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^n;Cl_{n,0})}. \tag{59}$$

Proof. We observe that

$$\begin{aligned} &\int_{SO(n)} \int_{\mathbb{R}^+} \|\boldsymbol{\omega}\ \mathcal{F}\{T_\psi \widetilde{f(a,\boldsymbol{\theta},\cdot)}\}\|^2_{L^2(\mathbb{R}^3;Cl_{3,0})}\, d\mu \\ &\overset{(15)}{=} \int_{\mathbb{R}^n} \int_{SO(n)} \int_{\mathbb{R}^+} \boldsymbol{\omega} \mathcal{F}\{\widetilde{T_\psi f}\} * [\mathcal{F}\{\widetilde{T_\psi f}\}]^{\sim} \boldsymbol{\omega}\, d\mu\, d^n\boldsymbol{\omega} \\ &\overset{(41)}{=} \int_{\mathbb{R}^n} \int_{SO(n)} \int_{\mathbb{R}^+} a^n \boldsymbol{\omega}\, \widehat{\psi}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega})) \widetilde{\widehat{f}}(\boldsymbol{\omega}) * \widehat{f}(\boldsymbol{\omega}) \{\widehat{\psi}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim} \boldsymbol{\omega}\, \frac{da d\boldsymbol{\theta}}{a^{n+1}}\, d^n\boldsymbol{\omega} \\ &\overset{(11)}{=} \int_{SO(n)} \int_{\mathbb{R}^+} \boldsymbol{\omega}^2 \{\widehat{\psi}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega}))\}^{\sim} \widehat{\psi}(a r_{\boldsymbol{\theta}}^{-1}(\boldsymbol{\omega})) \frac{da d\boldsymbol{\theta}}{a} * \int_{\mathbb{R}^n} \widetilde{\widehat{f}}(\boldsymbol{\omega}) \widehat{f}(\boldsymbol{\omega})\, d^n\boldsymbol{\omega} \\ &= C_\psi * (\widetilde{\boldsymbol{\omega}\hat{f}}, \widetilde{\boldsymbol{\omega}\hat{f}})_{L^2(\mathbb{R}^n;Cl_{n,0})}. \end{aligned} \tag{60}$$

□

Let us now begin with the proof of Theorem 4.

Proof. Using the uncertainty principle for multivector functions (in Theorem 5. 5 of [10]), we get

$$\begin{aligned} &\left[\|\boldsymbol{b} T_\psi \widetilde{f(a,\boldsymbol{\theta},\cdot)}\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})} \right]^{\frac{1}{2}} \times \left[\|\boldsymbol{\omega}\ \mathcal{F}\{T_\psi \widetilde{f(a,\boldsymbol{\theta},\cdot)}\}\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})} \right]^{\frac{1}{2}} \\ &\geq \frac{\sqrt{n}(2\pi)^{n/2}}{2} \|T_\psi \widetilde{f(a,\boldsymbol{\theta},\cdot)}\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})}. \end{aligned} \tag{61}$$

Now we integrate both sides of (61) with respect to Haar measure $d\mu$, we obtain

$$\int_{SO(n)}\int_{\mathbb{R}^+}\left(\left[\|\boldsymbol{b}T_\psi \widetilde{f(a,\boldsymbol{\theta},\cdot)}\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})}\right]^{\frac{1}{2}}\left[\|\boldsymbol{\omega}\ \mathcal{F}\{T_\psi \widetilde{f(a,\boldsymbol{\theta},\cdot)}\}\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})}\right]^{\frac{1}{2}}\right)d\mu \geq \frac{\sqrt{n}(2\pi)^{n/2}}{2}\int_{SO(n)}\int_{\mathbb{R}^+}\|T_\psi \widetilde{f(a,\boldsymbol{\theta},\cdot)}\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})}\,d\mu. \tag{62}$$

By applying the Clifford Cauchy-Schwartz inequality (17) into the left-hand side of (62). We see that

$$\left(\int_{SO(n)}\int_{\mathbb{R}^+}\|\boldsymbol{b}T_\psi \widetilde{f(a,\boldsymbol{\theta},\cdot)}\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})}\,d\mu\right)^{\frac{1}{2}}\left(\int_{SO(n)}\int_{\mathbb{R}^+}\|\boldsymbol{\omega}\ \mathcal{F}\{T_\psi \widetilde{f(a,\boldsymbol{\theta},\cdot)}\}\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})}\,d\mu\right)^{\frac{1}{2}} \geq \frac{\sqrt{n}(2\pi)^{n/2}}{2}\int_{SO(n)}\int_{\mathbb{R}^+}\|T_\psi \widetilde{f(a,\boldsymbol{\theta},\cdot)}\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})}\,d\mu. \tag{63}$$

Then by inserting (59) into the second term of (63), it follows that

$$\left(\int_{SO(n)}\int_{\mathbb{R}^+}\|\boldsymbol{b}T_\psi \widetilde{f(a,\boldsymbol{\theta},\cdot)}\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})}\,d\mu\right)^{\frac{1}{2}}\left(C_\psi * (\widetilde{\boldsymbol{\omega}\widehat{f}},\widetilde{\boldsymbol{\omega}\widehat{f}})_{L^2(\mathbb{R}^n;Cl_{n,0})}\right)^{\frac{1}{2}} \geq \frac{\sqrt{n}(2\pi)^{n/2}}{2}\int_{SO(n)}\int_{\mathbb{R}^+}\|T_\psi \widetilde{f(a,\boldsymbol{\theta},\cdot)}\|^2_{L^2(\mathbb{R}^n;Cl_{n,0})}\,d\mu. \tag{64}$$

We recognize that the first and third terms of (64) are the $L^2(\mathcal{G};Cl_{n,0})$-norms. This implies that

$$\|\boldsymbol{b}T_\psi \widetilde{f(a,\boldsymbol{\theta},\boldsymbol{b})}\|_{L^2(\mathcal{G};Cl_{n,0})}\left(C_\psi * (\widetilde{\boldsymbol{\omega}\widehat{f}},\widetilde{\boldsymbol{\omega}\widehat{f}})_{L^2(\mathbb{R}^n;Cl_{n,0})}\right)^{\frac{1}{2}} \geq \frac{\sqrt{n}(2\pi)^{n/2}}{2}\|\widetilde{T_\psi f}\|^2_{L^2(\mathcal{G};Cl_{n,0})}. \tag{65}$$

Substituting (52) into the right-hand side of (65) we finally get

$$\|\boldsymbol{b}T_\psi \widetilde{f(a,\boldsymbol{\theta},\boldsymbol{b})}\|_{L^2(\mathcal{G};Cl_{n,0})}\left(C_\psi * (\widetilde{\boldsymbol{\omega}\widehat{f}},\widetilde{\boldsymbol{\omega}\widehat{f}})_{L^2(\mathbb{R}^n;Cl_{n,0})}\right)^{\frac{1}{2}} \geq \frac{\sqrt{n}(2\pi)^{n/2}}{2}\left[C_\psi * (\tilde{f},\tilde{f})_{L^2(\mathbb{R}^n;Cl_{n,0})}\right], \tag{66}$$

which concludes the proof of Theorem 4. □

5. Examples of Clifford-Valued Wavelets

In this section we discuss the examples of Clifford admissible wavelets on multivector fields.

5.1. Clifford Morlet Wavelets

Recall that the two-dimensional complex (2D) Morlet wavelets which are sometimes called Gabor wavelet are defined by

$$\psi_M = e^{i(u_{01}x_1 + u_{02}x_2) - \frac{1}{2}(x_1^2 + x_2^2)}. \tag{67}$$

We extend the 2D Morlet wavelets into multivector fields by replacing the complex kernel $e^{i(u_{01}x_1+u_{02}x_2)}$ in (67) with the Clifford Fourier kernel $e^{i_n \boldsymbol{\omega}\cdot\boldsymbol{x}}$ (compare to [17]) . So let $\boldsymbol{\omega}_0 = u_{01}\boldsymbol{e}_1 + u_{02}\boldsymbol{e}_2 + \cdots + u_{0n}\boldsymbol{e}_n$ be an arbitrary frequency vector and $\xi(\boldsymbol{x}) = e^{-\frac{1}{2}\boldsymbol{\omega}_0^2}\, e^{-\frac{1}{2}\boldsymbol{x}^2}$ a correction term in order for equation (30) to be satisfied (see [2]). Then multivector-valued wavelets are given by

$$\psi^c(\boldsymbol{x}) = \left(e^{i_n \boldsymbol{\omega}_0\cdot\boldsymbol{x}} e^{-\frac{1}{2}\boldsymbol{x}^2}\right) - \xi(\boldsymbol{x}). \tag{68}$$

Applying the shift and scaling properties of the CFT, we can rewrite the Clifford Morlet wavelets (68) in terms of the $Cl_{n,0}$ Clifford Fourier transform as

$$\begin{aligned}\mathcal{F}\{\psi^c\}(\boldsymbol{\omega}) &= e^{-\frac{1}{2}\left((\omega_1-u_{01})^2+(\omega_2-u_{02})^2+\cdots+(\omega_n-u_{0n})^2\right)} - \\ &\quad e^{-\frac{1}{2}\left((\omega_1^2+u_{01}^2)+(\omega_2^2+u_{02}^2)+\cdots+(\omega_n^2+u_{0n}^2)\right)}.\end{aligned} \tag{69}$$

We see that $\mathcal{F}\{\psi^c\}$ is a real positive scalar constant or the even grade multivector. Hence, the first part of Theorem 2 holds.

The representation of the Clifford Morlet wavelets (68) show that they are formally analogous to the n-dimensional Morlet wavelets. We can apply the Euler formula to the Clifford exponential (see Hestenes [1]) to express (68) in the form

$$\begin{aligned}\psi^c(\boldsymbol{x}) &= e^{-\frac{1}{2}\boldsymbol{x}^2}\cos(\boldsymbol{\omega}_0\cdot\boldsymbol{x}) + i_n\, e^{-\frac{1}{2}\boldsymbol{x}^2}\sin(\boldsymbol{\omega}_0\cdot\boldsymbol{x}) - \xi(\boldsymbol{x}) \\ &= \langle\psi^c\rangle + \langle\psi^c\rangle_n.\end{aligned} \tag{70}$$

This shows that the resulting wavelets consist of a real scalar and pseudoscalar parts, respectively. Since the equation (69) is a real-valued function, the admissibility condition (43) has to be real-valued too.

5.2. Clifford Hermite Wavelets

Let us now introduce the Clifford Hermite wavelet (see[4]) as follows:

$$\psi_n(\boldsymbol{x}) = \left(e^{-\frac{\boldsymbol{x}^2}{2}}\right) H_n(\boldsymbol{x}) = (-1)^n \frac{\partial^n}{\partial\boldsymbol{x}} \left(e^{-\frac{\boldsymbol{x}^2}{2}}\right), \tag{71}$$

where H_n is the radial Hermite polynomials in $\mathbb{R}^n$ given by

$$H_n(\boldsymbol{x}) = (-1)^n \left(e^{-\frac{\boldsymbol{x}^2}{2}}\right) \frac{\partial^n}{\partial\boldsymbol{x}} \left(e^{-\frac{\boldsymbol{x}^2}{2}}\right). \tag{72}$$

Depending upon n, the Clifford Hermite wavelets (71) are alternately real or vector-valued. Their Clifford Fourier transform is given by

$$\mathcal{F}\{\psi_n\}(\boldsymbol{\omega}) = \pi^{n/2}(-i_n)^n \boldsymbol{\omega}^n \left(e^{-\frac{\boldsymbol{\omega}^2}{2}}\right). \tag{73}$$

So we can now see that in Clifford domain, the Clifford Hermite wavelets are alternately real-valued or pseudoscalar-valued. In particular, for $n = 2$ (mod 4) it is not difficult to see that $\mathcal{F}\{\psi_n\}$ is a real-valued so that the first part of Theorem 2 holds too.

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