The Space-time Transformations Between Accelerated Systems

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Abstract

We determine transformations between coordinate systems which are mutually in linear accelerated motion. We also determine the transformations for rotating systems. In case of the symmetrical linear mutual acceleration, we immediately get the maximal acceleration limit which was derived by Caianiello from quantum mechanics. Maximal acceleration is an analogue of maximal velocity in special relativity. We discuss the possible verification of derived formulae by the measurement with ultracentrifuge. It is argued that the derived results can play crucial role in modern particle physics and cosmology.
The problem of acceleration of charged particles or systems of particles is the permanent
and the most prestige problem in the accelerator physics. Particles can be accelerated
by different ways. Usually by the classical electromagnetic fields, or, by light pressure of
latter method is the permanent problem of the laser physics for many years.

However, the theoretical problem is not only to find the mechanisms of acceleration
but also the space-time relations between systems which are mutually in the accelerated
motion. The uniformly accelerated systems are well known, also the rotating systems
represented for instance by the centrifuges are also known and theoretically investigated
by many authors.

Here, we determine transformations between coordinate systems which moves mutually
with acceleration. We determine transformations between nonrelativistic and relativistic
uniformly accelerated systems and rotating systems. We derive also some consequences
following from the nonlinearity of motion of these systems.

We show that the transformation laws between accelerated systems can be derived
from the infinitesimal Lorentz transformation on the one hand, or, by postulation some
kinematical symmetries between these systems on the other hand. These two approaches
gives different results. In the case of the first approach which is based on the original
Lorentz transformation, the derived results can be taken for sure-footed.

We do not consider in this article the problem of accelerated strings, which is solved for
instance by Bachas (2002), because string theory is under the permanent reconstruction
and according to Witten (Witten, 2002) it is not clear that the present form of the string
theory is correct.

2 The infinitesimal form of the Lorentz transformation

We know, that the Lorentz transformation between two inertial coordinate systems
$S(0, x, y, z)$ and $S'(0, x', y', z')$ (where system $S'$ moves in such a way that $x$-axes converge,
while $y$ and $z$-axes run parallel and at time $t = t' = 0$ for the beginning of the systems $O$
and $O'$ it is $O \equiv O'$) is as follows:

$$x' = \gamma(v)(x - vt), \quad y' = y, \quad z' = z', \quad t' = \gamma(v)\left(t - \frac{v^2 x}{c^2}\right),$$

where

$$\gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

The infinitesimal form of this transformation is evidently given by differentiation of
the every equation. Or,

$$dx' = \gamma(v)(dx - vdt), \quad dy' = dy, \quad dz' = dz, \quad dt' = \gamma(v)\left(dt - \frac{v}{c^2}dx\right).$$
If we put $dt = 0$ in the first equation of system (3), then the Lorentz length contraction follows in the infinitesimal form $dx' = \gamma(v)dx$. Or, in other words, if in the system $S'$ the infinitesimal length is $dx'$, then the relative length with regard to the system $S$ is $\gamma^{-1}dx'$. Similarly, from the last equation of (3) it follows the time dilatation for $dx = 0$. Historical view on this effect is in the Selleri (1997) article.

If the velocity depends on time, which is for instance in the case of the nonlinear motion, then we write

\[
dx' = \gamma(v(t))(dx - v(t)dt), \quad dy' = dy, \quad dz' = dz, \quad dt' = \gamma(v(t)) \left( dt - \frac{v(t)}{c^2}dx \right).
\]

(4)

This infinitesimal form enables the integration and if we know the dependence of $v$ on time, then there is no obstacles to get the Lorentz-like transformation between two nonlinear systems. At the same time the transformations (4) does not change the so called Minkowski metric element, the square of which is

\[
ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.
\]

(5)

The Lorentz transformation for coordinates and time referring to two inertial systems harbours the assumption that the following expression holds good: $(x = ct) \iff (x' = ct')$, where the invariant function $x = ct$ being considered in the special theory of relativity as the mathematical expression of the principle of constant light velocity. From the mathematical point of view the relation is the formal mathematical requirement for unambiguous determination of the Lorentz transformation and it follows from the theory of the continuous group of transformations (Eisenhart, 1943). The physical meaning of $ds$ is usually defined as a distance between two infinitesimal events, however, some authors consider $ds$ as a formal mathematical object with no physical meaning (Brillouin, 1970).

3 Uniformly accelerated systems

According to Einstein (1965), Fok (1961) and Logunov (1987), space and time, or, space-time is a form of the existence of matter. To study space-time means to study form of the existence of matter. Special theory of relativity investigates behavior of matter and fields in case of the inertial systems in the inertial motion. The behavior of space-time in case that the systems are mutually or individually accelerated is investigated here.

Let us suppose that in the finite time interval the system $S'$ is accelerated by the constant acceleration in such a way that the motion is nonrelativistic one. So, for the velocity of the system $S'$ we have $v = at$ and from the equations (4) we have for the infinitesimal Lorentz transformation:

\[
dx' = \gamma(at)(dx - atdt), \quad dy' = dy, \quad dz' = dz, \quad dt' = \gamma(at) \left( dt - \frac{at}{c^2}dx \right),
\]

(6)

where

\[
\gamma(at) = \left(1 - \frac{a^2t^2}{c^2}\right)^{-1/2}.
\]

(7)
After integration of equation (6) we get for the coordinate and time components:

\[ x' = x \left(1 - \frac{a^2 t^2}{c^2}\right)^{-1/2} - a \int t \, dt \left(1 - \frac{a^2 t^2}{c^2}\right)^{-1/2} = x \left(1 - \frac{a^2 t^2}{c^2}\right)^{-1/2} - \frac{a}{2} \left(1 - \frac{a^2 t^2}{c^2}\right)^{1/2} \]

(8)

and

\[ t' = \frac{c}{a} \arcsin \left(\frac{at}{c}\right) - x \frac{at}{c^2} \left(1 - \frac{a^2 t^2}{c^2}\right)^{-1/2} \]

(9)

In case of the relativistic motion of a body with mass \( m \) which is caused by the action of the constant force \( F \) on this body, the dependence of velocity on time is (Landau et al. 1962)

\[ v = \frac{at}{\sqrt{1 + \left(\frac{at}{c}\right)^2}}. \]

(10)

Then, the relativistic coefficient \( \gamma(v) \) is given by the relation

\[ \gamma(v) = \left(1 + \frac{a^2 t^2}{c^2}\right)^{1/2}. \]

(11)

In this situation we get for the coordinate and time transformation:

\[ x' = x \left(1 + \frac{a^2 t^2}{c^2}\right)^{1/2} - a \int t \, dt = x \left(1 + \frac{a^2 t^2}{c^2}\right)^{1/2} - \frac{a}{2} t^2 \]

(12)

and

\[ t' = \frac{1}{2ac} \left[ at \sqrt{a^2 t^2 + c^2} + c^2 \arg \sinh \left(\frac{at}{c}\right) \right] - x \frac{at}{c^2}, \]

(13)

where \( \arg \sinh(at/c) \) can be expressed in the logarithmic form according to the following formula:

\[ \arg \sinh \left(\frac{at}{c}\right) = \ln \left[ \left(\frac{at}{c}\right) + \sqrt{\left(\frac{at}{c}\right)^2 + 1} \right]. \]

(14)

Let us remark, that the problem of transformations between nonlinear systems was also discussed by Møller (1943) and Fok (1961). The transformations derived by Møller are not identical with ones derived by us. According to Fok, the nonlinear transformation of space-time is as follows:

\[ x' = x \cosh \left(\frac{at}{c}\right) - c^2 \frac{a}{a} \left(\cosh \left(\frac{at}{c}\right) - 1\right) \]

(15a)

and
\[ t' = \frac{c}{a} \sinh \left( \frac{at}{c} \right) - \frac{x}{c} \sinh \left( \frac{at}{c} \right). \] (15b)

The transformations between system \( S \) and uniformly accelerated system \( S' \) was also derived by Logunov (1987), but in the substantially different form:

\[ x' = x \frac{c^2}{a} \left[ \sqrt{1 + \frac{a^2 t^2}{c^2}} - 1 \right]; \quad t' = t. \] (16)

where equation \( t' = t \) can be chosen according to Logunov as the free decision of a physicist. The Logunov approach is not in the final form.

The transformations derived reduces for

\[ \frac{at}{c} \ll 1 \] (17)

to the Galilean transformation in the form

\[ x' = x - \frac{1}{2} at^2; \quad t' = t. \] (18)

After insertion of transformation (15) into space-time element (4), we get:

\[ ds^2 = \left( c - \frac{ax}{c} \right)^2 dt^2 - dx^2 - dy^2 - dz^2 \] (19)

which is approximately (for \( |ax| \ll c^2 \)) the same as the square of the space-time element in the homogenous gravitational potential \( U = ax \). It means that the uniformly accelerated system form some analogue with the homogenous gravitational field. The analogy is usually defined as the principle of equivalence.

However, the principle of equivalence can be easily derived from the special theory relativity from the well known relation \( E = mc^2 \). This relation means that to every mass corresponds energy. And the energy has the same inertia in the arbitrary acceleration field. It means, there is no difference between gravitational and inertial mass. So, the principle of equivalence between inertial and gravitational mass follows from the Einstein relation \( E = mc^2 \). From this proof it follows, that no future experiment will reveal the difference between inertial and gravitational mass. To our knowledge, this elementary theorem was not published in any journal.

We can say, that there is some different nonequivalent possibilities in the derivation of the transformations for nonlinearly moving systems. It signalizes that the theory of the space-time transformation between nonlinear systems is not in the definite form.

The inverse transformations to the derived ones are evidently of the different form (excepting the Logunov transformation) than the original transformations. We show, In the following section that it is possible to find such transformations between coordinates and time, that they are symmetrical. The physical meaning of such transformation is open.
Let us take two systems \( S(0, x, y, z) \) and \( S'(0, x', y', z') \), where system \( S' \) moves in such a way that \( x \)-axes converge, while \( y \) and \( z \)-axes run parallel and at time \( t = t' = 0 \) for the beginning of the systems \( O \) and \( O' \) it is \( O \equiv O' \). Let us suppose that system \( S' \) moves relative to some basic system \( B \) with acceleration \( a/2 \) and system \( S' \) moves relative to system \( B \) with acceleration \(-a/2\). It means that both systems moves one another with acceleration \( a \) and are equivalent because in every system it is possibly to observe the force caused by the acceleration \( a/2 \). In other word no system is inertial.

Now, let us consider the formal transformation equations between two systems. At this moment we give to this transform only formal meaning because at this time, the physical meaning of such transformation is not known. On the other hand, the consequences of the transformation will be shown very interesting.

We write the transformation equations in the form:

\[
x' = a_1 x + a_2 t^2, \quad y' = y, \quad z' = z, \quad t' = \sqrt{b_1 x + b_2 t^2},
\]

(20)

where constants involved in the equations will be determined from the viewpoint of kinematics. Since from the viewpoint of kinematics, both systems are equivalent, for the inverse transformation to the transformation (20) it must hold:

\[
x = a_1 x' - a_2 t'^2, \quad y' = y, \quad z' = z, \quad t = \sqrt{-b_1 x' + b_2 t'^2}.
\]

(21)

The minus sign with coefficients \( a_2 \) and \( b_1 \) appearing for the reason that constant \( a_2 \) has the rate of acceleration while constant \( b_1 \) the rate of inverse value of acceleration.

Similarly as in inertial systems, the hypothetical requirement can be now expressed that the transformation equations for system moving relative to themselves with acceleration include a suitable invariant function. Let us now define such transformations as follows:

\[
x = \frac{1}{2} \alpha t^2, \quad (22a)
\]

\[
x' = \frac{1}{2} \alpha t'^2, \quad (22b)
\]

where \( \alpha \) is the constant having the rate of acceleration.

If we now substitute (21) into (20) we obtain

\[
x' = x'(a_1^2 - a_2 b_1) + t'^2(a_2 b_2 - a_1 a_2),
\]

(23)

\[
t'^2 = x'(a_1 b_1 - b_1 b_2) + t'^2(b_2^2 - b_1 a_2).
\]

(24)

After comparing the left and right sides in the relations (23), we get

\[
a_1 = b_2, \quad a_1^2 - b_1 a_2 = 1.
\]

(25)
If we put in the relation (20) \( x' = 0 \), we obtain \( x = -(a_2/a_1)t^2 \). In accordance with the assumption the motion of the beginning of the system \( S' \) relative to system \( S \) is described by the \( x = \frac{1}{2}at^2 \), we thus obtain

\[
a_2 = -\frac{1}{2}a_1a. \tag{26}
\]

From \([x' = (1/2)at^2] \iff [x = (1/2)at^2]\), we get

\[
\frac{1}{4}ab_2 - a_2 \quad \frac{a}{a_1 - \frac{1}{2}\alpha b_1} = \frac{1}{2}\alpha. \tag{27}
\]

Through solving the equations (26) and (27), we obtain

\[
a_1 = b_2 = \frac{1}{\sqrt{1 - \frac{a^2}{\alpha^2}}}, \quad a_2 = -\frac{1}{2} \frac{a}{\sqrt{1 - \frac{a^2}{\alpha^2}}}, \quad b_1 = -\frac{2}{\alpha^2} \frac{a}{\sqrt{1 - \frac{a^2}{\alpha^2}}}. \tag{28}
\]

Using (28), we can rewrite the transformation (20) in the definite form:

\[
x' = \Gamma(a)(x - \frac{1}{2}at^2), \quad y' = y, \quad z' = z, \quad t'^2 = \Gamma(a)\left(t^2 - \frac{2a}{\alpha^2}x\right) \tag{29}
\]

with

\[
\Gamma(a) = \frac{1}{\sqrt{1 - \frac{a^2}{\alpha^2}}}. \tag{30}
\]

Let us remark that the more simple derivation of the last transformation can be obtained if we perform in the Lorentz transformation the elementary change of variables as follows: \( t \to t^2, \quad t' \to t'^2, \quad v \to \frac{1}{2}a, \quad c \to \frac{1}{2}\alpha \).

After performing such elementary transition which is practically redenotation of variables, we really get the Lorentz-like transformation (29) between accelerated systems.

The physical interpretation of this nonlinear transformations is the same as in the case of the Lorentz transformation only the physical interpretation of the invariant function \( x = \frac{1}{2}at^2 \) is open.

However, we know from history, that Lorentz transformation was taken first as physically meaningless by Lorentz himself and later only Einstein decided to put the physical meaning to this transformation and to the invariant function \( x = ct \). We hope that the derived transformation will appear as physically meaningful.

Now, let us prove the following assumption. The transformation (29) forms one-parametric group with parameter \( a \). To prove it we must prove by the direct calculations the four requirements involving in the definition of group. However, we know, that using relations \( t \to t^2, \quad t' \to t'^2, \quad v \to \frac{1}{2}a, \quad c \to \frac{1}{2}\alpha \), the nonlinear transformation is expressed as the Lorentz transformation forming the one-parametric group. And this is a proof. Such proof is equivalent to the proof by direct calculation. The integral part of the group properties is the so called addition theorem for acceleration.

\[
w_3 = \frac{w_1 + w_2}{1 + \frac{w_1 w_2}{a^2}}. \tag{31}
\]
where $w_1$ is the acceleration of the system $S'$ with regard to the system $S$, $w_2$ is the acceleration of the system $S''$ with regard to the system $S'$ and $w_3$ is the acceleration of the system $S''$ with regard to the system $S$.

The relation (31), expresses the law of acceleration addition theorem on the understanding that the events are marked according to the relation (29). In this formula as well as in the transformation equation (29) appears constant $\alpha$ which cannot be calculated from the theoretical considerations, or, from the theory. What is its magnitude and whether there exists such a physical field that is consistent with the designation of the events given by the relations (29) can be established only by experiments. On the other hand the constant $\alpha$ has physical meaning of the maximal acceleration and its meaning is similar to the maximal velocity $c$ in special relativity. The notion maximal acceleration is not new physics, because Caianiello (1981, 1992) introduced it as some consequence of quantum mechanics and Landau theory of fluctuations. Revisiting view on the maximal acceleration was given by Papini (2003). At recent time it was argued by Lambiase et al. (1998, 1999) that maximal acceleration determines the upper limit of the Higgs boson and that it gives also the relation which links the mass of $W$ boson with the mass of the Higgs boson. The LHC Experiments probably give the answer to this problem.

5 Dependence of mass on acceleration

If the maximal acceleration is the physical reality, then it should have the similar consequences in a dynamics as the maximal velocity of motion has consequences in the dependence of mass on velocity. We can suppose in analogy with the special relativity that mass depends on the acceleration for small velocities, in the similar way as it depends on velocity in case of uniform motion. Of course such assumption must be experimentally verified and in no case it follows from special theory of relativity, or, general theory of relativity (Okun, 2001). So, we postulate ad hoc, in analogy with special theory of relativity:

$$m(a) = \frac{m_0}{\sqrt{1 - \frac{a^2}{c^2}}}; \quad v \ll c, \quad a = \frac{dv}{dt}. \quad (32)$$

Let us derive as an example the law of motion when the constant force $F$ acts on the body with the rest mass $m_0$. Then, the Newton law reads (Landau et al., 1962):

$$F = \frac{dp}{dt} = m_0 \frac{dv}{dt} \sqrt{1 - \frac{a^2}{c^2}}. \quad (33)$$

The first integral of this equation can be written in the form:

$$\frac{Ft}{m_0} = \frac{v}{\sqrt{1 - \frac{a^2}{c^2}}}; \quad a = \frac{dv}{dt}, \quad F = const.. \quad (34)$$

Let us introduce quantities

$$v = y, \quad a = y', \quad A(t) = \frac{F^2 t^2}{m_0^2 \alpha^2}. \quad (35)$$
Then, the quadratic form of the equation (34) can be written as the following differential equation:

$$A(t)y'^2 + y^2 - A(t)\alpha^2 = 0,$$

which is nonlinear differential equation of the first order. The solution of it is of the form $y = Dt$, where $D$ is some constant, which can be easily determined. Then, we have the solution in the form:

$$y = v = \frac{t}{\sqrt{\frac{m_0^2}{r^2} + \frac{1}{\alpha^2}}}.$$  (37)

For $F \to \infty$, we get $v = \alpha^2 t$. This relation can play substantial role at the beginning of the big-bang, where the accelerating forces can be considered as infinite, however the law of acceleration has finite nonsingular form. At this moment it is not clear if the dependence of the mass on acceleration can be related to the energy dependence on acceleration similarly to the Einstein relation uniting energy, mass and velocity (Sachs, 1973; Okun, 2001).

## 6 The rotating systems

According to Einstein, there is an analogy between gravitational fields and noninertial reference system. Therefore, when studying properties of gravitational fields in relativistic mechanics, we can start from this analogy.

The description of the rotation system can be described in the Cartesian coordinate system (Landau et al., 1962), or, in the more appropriate form using the polar coordinate $r, \varphi$ (Goy and Selleri, 1997). Then, we write

$$x = r \cos(\varphi + \omega t), \quad y = r \sin(\varphi + \omega t).$$  (38)

The corresponding space-time element is as follows:

$$ds^2 = \left(1 - \frac{\omega^2 r^2}{c^2}\right)(cdt)^2 - \frac{2\omega r^2}{c}d\varphi(cdt) - dz^2 - dr^2 - r^2 d\varphi^2.$$  (39)

We see from the time term in (39) that if we suppose that the velocity of light in the rotating system is constant, then the elapsing of time depends on acceleration.

The equations (38) does not involve the transformation between $t'$ and $t$, so we have motivation for find such transformation.

Such transformation between inertial and rotating system can be expressed in the Lorentz form if we insert into the original Lorentz transformation the following formulae $\Delta x = \Delta \varphi r, v = \omega r$. Then we have for $0 \leq \varphi \leq 2\pi$:

$$\Delta\varphi' = \frac{\Delta\varphi - \omega \Delta t}{\sqrt{1 - \frac{\omega^2 r^2}{c^2}}}, \quad \Delta t' = \frac{\Delta t - \Delta \varphi \frac{\omega r^2}{c}}{\sqrt{1 - \frac{\omega^2 r^2}{c^2}}}.$$  (40)

It follows from this transformation that for every $r > c/\omega$ and given $\omega$, it has no physical meaning. It also means that the infinite rotating system as a whole has no
relativistic meaning and it is not clear how to solve this problem. We also see that from equation (40) the time dilatation follows and it is the same as in equation (39).

Although the rotating system cannot be considered as equivalent to the linear accelerated system, nevertheless, the radial component of every part of this system is in the permanent acceleration.

If the element 1 of the rotating plane at the radial coordinate $r_1$ has acceleration $w_1$ and if the element of the rotating plane at the radial coordinate $r_2$ has acceleration $w_2$, then the relative acceleration $w_r$ of the element 2 with regard to the element 1 is not $w_2 - w_1$, but must be determined according to the formula

$$w_r = \frac{w_2 - w_1}{1 - \frac{w_1 w_2}{\alpha^2}}. \tag{41}$$

The last formula is an analogue of the formula which determines the relative velocities in case of the inertial motion in the special theory of relativity. The last formula is true only if the transversal effect do not influence the radial effects. It can be verified optically, because we know that the optical frequency of the emission source is influenced by acceleration, or, equivalently by the gravitational field.

Similarly, it is possible to verify the dependence of mass on acceleration, also by the ultracentrifuge.

7 Discussion

We have derived transformations between accelerated systems moving mutually uniformly. We have discussed also the rotating systems. We have derived some consequences following from the nonlinearity of motion of these systems. In case, when we used the symmetry principle in derivation of the space-time transformation, we derived by the formal way so called maximal acceleration which was derived using quantum mechanics by Caianiello (1981, 1992). Our derivation of the maximal acceleration is not equivalent to the Caianiello derivation and at the same time it is not in the contradiction with his approach because the heuristical ways to the maximal acceleration were substantially different.

If some experiment will confirm the existence of maximal acceleration $\alpha$, then it will have certainly crucial consequences for Einstein theory of gravity because this theory does not involve this factor. Also the cosmological theories constructed on the basis of the original Einstein equations will require modifications. In such a way, Einstein equations can play a role only in the specific conditions where the maximal acceleration can be neglected. Maximal acceleration does not allow the existence of black holes with arbitrary big mas. Also standard model in particle physics will require generalization because it does not involve the maximal acceleration.

We did not consider the problem of accelerated strings, because string theory is under permanent reconstruction and according to Witten (2002) it is not sure that the present form of this theory is correct. It also does not involve the Penrose nonlinear graviton (Penrose, 1976) and it does not involve the Gassendi string model of gravity (Fraser et al. 1998).
One of the prestige problems in modern theoretical physics is the Unruh effect, or, the existence of thermal radiation detected by accelerated observers. The theory of the Unruh effect is unfortunately under reconstruction (Fedotov et al., 2002) and to say serious statement or comment to the relation of this effect to the maximal acceleration is premature.

So, at this moment, we study only the accelerated classical systems, and after some time, when string theory will be in the perfect form, we will study the situation where the string is accelerated by quantized laser field, following the author articles concerning laser acceleration (Pardy, 1998, 2001, 2002).

REFERENCES


