Nonadiabatic holonomy operators in classical and quantum completely integrable systems

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Given a completely integrable system, we associate to any connection on a fiber bundle in invariant tori over a parameter manifold the classical and quantum holonomy operator (generalized Berry’s phase factor), without any adiabetic approximation.

I. INTRODUCTION

At present, holonomy operators in quantum systems attract special attention in connection with quantum computation (see, e.g., Refs. [1-3]). They exemplify the non-Abelian generalization of Berry’s geometric phase by means of driving a finite level degenerate eigenstate of a Hamiltonian over a parameter manifold. The key point is that a geometric phase depends only on the geometry of a path executed and, therefore, provides a possibility to perform quantum gate operations in an intrinsically fault-tolerant way. The problem lies in separation of a geometric phase factor from the total evolution operator without using an adiabatic assumption. Firstly, holonomy quantum computation implies exact cyclic evolution, but exact adiabatic cyclic evolution almost never exists. Secondly, an adiabatic condition requires that the evolution time must be long enough.

A nonadiabatic Abelian phase was discovered by Aharonov and Anandan who considered a loop in a projective Hilbert space instead of a parameter space. Non-Abelian generalization of the Aharonov–Anandan phase has been studied under rather particular assumption. Moreover, a non-Abelian Aharonov–Anandan phase fail to be separated from the dynamic one in general. Recently, several schemes using the Aharonov–Anandan phase were proposed for nonadiabatic geometric gates.

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In a general setting, let us consider a linear (not necessarily finite-dimensional) dynamical system \( \partial_t \psi = \hat{S} \psi \) whose linear (time-dependent) dynamic operator \( \hat{S} \) falls into the sum
\[
\hat{S} = \hat{S}_0 + \Delta = \hat{S}_0 + \Delta_\alpha \partial_t \xi^\alpha,
\]
where \( \xi(t) \) is a function of time taking its values in a finite-dimensional smooth real parameter manifold \( \Sigma \) coordinated by \( (\sigma^\alpha) \). Let assume that: (i) the operators \( \hat{S}_0(t) \) and \( \Delta(t') \) commute for all instants \( t \) and \( t' \), and (ii) the operator \( \Delta \) depends on time only through \( \xi(t) \). Then the evolution operator \( U(t) \) can be represented by the product of time-ordered exponentials
\[
U(t) = U_0(t) \circ U_1(t) = T \exp \left[ \int_0^t \hat{S}_0 dt' \right] \circ T \exp \left[ \int_0^t \Delta dt' \right],
\]
where the second one is brought into the ordered exponential
\[
U_1(t) = T \exp \left[ \int_0^t \Delta_\alpha(\xi(t'))\partial_t \xi^\alpha(t') dt' \right] = T \exp \left[ \int_{\xi[0,t]} \Delta_\alpha(\sigma) d\sigma^\alpha \right]
\]
along the curve \( \xi[0,t] \) in the parameter manifold \( \Sigma \). It is a nonadiabatic geometric factor depending only on the trajectory of the parameter function \( \xi \). Accordingly, \( \Delta \) is a holonomy operator. The geometric factor (3) is well defined if \( \Delta_\alpha d\sigma^\alpha \) is an Ehresmann connection on a fiber bundle over a parameter manifold \( \Sigma \). Then this factor is a displacement operator along an arbitrary curve \( \xi[0,t] \subset \Sigma \).

A problem is that the above mentioned commutativity condition (i) is very restrictive. Moreover, it need not be preserved under time-dependent transformations.

For instance, let us consider a Hamiltonian system of dynamic variables \((q, p)\). Written with respect the initial data coordinates \((q_0, p_0)\), its Hamiltonian \( \mathcal{H}(q_0, p_0) \) vanishes. Given these coordinates \((q_0, p_0)\), let one can introduce a perturbed Hamiltonian \( \mathcal{H}_\xi(q_0, p_0, \xi(t)) \) which depends on parameter functions \( \xi(t) \) and generates a holonomy operator \( \Delta \) (1). Then the evolution operator of the perturbed Hamiltonian system reduces to the geometric factor (3). Relative to the original variables \((q, p)\), a Hamiltonian of this perturbed Hamiltonian system is
\[
\mathcal{H}' = \mathcal{H}(q, p, t) + \mathcal{H}_\xi(q_0(t, q, p), p(t, q, p), \xi(t)).
\]
However, the corresponding evolution operator does not fall into the product (2) because a Hamiltonian \( \mathcal{H} \) is not a function under time-dependent transformations and, consequently, the Poisson bracket \( \{\mathcal{H}, \mathcal{H}_\xi\} \) with respect to original variables \((q, p)\) need not vanish.

Nevertheless, basing on this example, we can essentially extend the class of dynamical systems admiting a nonadiabatic geometric phase. We aim to describe dynamical systems...
where the commutativity condition (i) is not satisfied, but a part of dynamic variables is driven only by a holonomy operator. These are completely integrable Hamiltonian systems.

Let us consider a completely integrable Hamiltonian system (henceforth CIS) of $m$ degrees of freedom around its invariant tori $T^m$. We show that, being constant under an internal evolution, its action variables are driven only by a perturbation holonomy operator $\Delta$ which can be associated to an arbitrary connection on a fiber bundle

$$\Sigma \times T^m \to \Sigma.$$ (4)

This holonomy operator is defined with respect to the initial data action-angle coordinates without any adiabatic approximation. Then we return to the original action-angle coordinates. The key point is that both classical evolution of action variables and mean values of quantum action operators relative to original action-angle coordinates are determined in full by the dynamics of initial data action and angle variables.

The plan of the paper is as follows. Section II addresses classical time-dependent CIS. The key point is that any time-dependent CIS of $m$ degrees of freedom is extended to an autonomous CIS of $m + 1$ degrees of freedom and, as a consequence, can be provided with action-angle variables around a regular instantly compact invariant manifold.\textsuperscript{10,11}

In Section III, we introduce the holonomy operator in a classical CIS by use of the fact that a generic Hamiltonian of a mechanical system with time-dependent parameters contains a term which is linear both in momenta and the temporal derivative of a parameter function.\textsuperscript{12,13} This term comes from a connection on the configuration space of the system fibered over a parameter manifold.

Section IV is devoted to geometric quantization of a CIS with respect to the angle polarization. This polarization leads to the Schrödinger representation of action variables in the separable Hilbert space of smooth complex functions on $T^m$.\textsuperscript{10,14} We show that this quantization both with respect to the original action-angle variables and the initial data action-angle variables is equivalent.

In Section V, the classical holonomy operator of Section III is quantized with respect to the initial data action-angle variables.

The symbols $\lrcorner$ and $\lrcorner$ below stand for the left and right interior products of multivector fields and exterior forms, respectively.

Let us recall that, given a fiber bundle $Y \to X$ coordinated by $(x^\lambda, y^i)$, a connection $K$ on $Y \to X$ is defined by a tangent-valued form

$$K = dx^\lambda \otimes (\partial_\lambda + k_\lambda^i \partial_i)$$
on $Y$.\textsuperscript{15} A connection on a fiber bundle $Y \to X$ is said to be an Ehresmann connection if, given an arbitrary smooth curve $\xi([0, 1]) \subset X$, there exists its horizontal lift through any point of $Y$ over $\xi(0)$. 

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Let $X$ be a real axis $\mathbb{R}$ provided with the Cartesian coordinate $t$ possessing transition functions $t' = t + \text{const}$. A connection $K$ on a fiber bundle $Y \to \mathbb{R}$ is uniquely represented by a vector field $K$ on $Y$ such that $K \mid dt = 1$. This is the case of time-dependent mechanics.

II. CLASSICAL TIME-DEPENDENT CIS

Recall that the configuration space of time-dependent mechanics is a fiber bundle $Q \to \mathbb{R}$ over the time axis $\mathbb{R}$. Let it be equipped with the bundle coordinates $(t, q^k), k = 1, \ldots, m$. The corresponding phase space is the vertical cotangent bundle $V^*Q$ of $Q \to \mathbb{R}$ endowed with the induced coordinates $(t, q^k, p_k)$ relative to the holonomic coframes $\{dq^k\}$. The cotangent bundle $T^*Q$ of $Q \to \mathbb{R}$ plays a role of the homogeneous phase space of time-dependent mechanics. It is equipped with the induced coordinates $(t, q^k, p, p_k)$ relative to the holonomic coframes $\{dt, dq^k\}$. With respect to this coordinates, the canonical symplectic form and the corresponding Poisson bracket on $T^*Q$ read

$$\Omega = dp \wedge dt + dp_k \wedge dq^k,$$

$$\{f, f'\}_T = \partial_{p_j} f \partial_{q^j} f' - \partial_{q^j} f \partial_{p_j} f' + \partial^k f \partial_k f' - \partial_k f \partial^k f', \quad f, f' \in C^\infty(T^*Q).$$

There is the one-dimensional trivial affine bundle

$$\zeta : T^*Q \to V^*Q.$$  \hspace{1cm} (5)

As a consequence, the phase space $V^*Q$ of time-dependent mechanics is provided with the canonical Poisson structure

$$\{f, f'\}_V = \partial^k f \partial_k f' - \partial_k f \partial^k f', \quad f, f' \in C^\infty(V^*Q),$$  \hspace{1cm} (6)

given by the relations

$$\zeta^* \{f, f'\}_V = \{\zeta^* f, \zeta^* f'\}_T, \quad f, f' \in C^\infty(V^*Q).$$

The corresponding Poisson bivector on $V^*Q$ reads $w_V = \partial_k \wedge \partial^k$.

A Hamiltonian of time-dependent mechanics is defined as a global section

$$h : V^*Q \to T^*Q, \quad p \circ h = -\mathcal{H}(t, q^i, p_j),$$  \hspace{1cm} (7)

of the affine bundle $\zeta$ (5). Given the pull-back form $h^*\Omega$, the relations $\gamma_H \mid dt = 1$, $\gamma_H \mid h^*\Omega = 0$ define a unique Hamilton vector field

$$\gamma_H = \partial_t + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k$$  \hspace{1cm} (8)
on $V^*Q$ and the corresponding Hamilton equations

$$
\dot{q}^k = \partial^k \mathcal{H}, \quad \dot{p}_k = -\partial_k \mathcal{H}.
$$

Note that, given a connection $\Gamma = \partial_t + \Gamma_i^j \partial_i$ on $Q \to \mathbb{R}$, any Hamiltonian $\mathcal{H}$ admits the decomposition $\mathcal{H} = p_i \Gamma_i^j \partial_j + \tilde{\mathcal{H}}$ where $\tilde{\mathcal{H}}$ is a function on $V^*Q$.

An integral of motion of the Hamilton equations (9) is a smooth real function $F$ on $V^*Q$ whose Lie derivative

$$
\mathbf{L}_{\gamma_H} F = \gamma_H \partial_t F \equiv \partial_t F + \{\mathcal{H}, F\}_V
$$

along the Hamilton vector field $\gamma_H$ (8) vanishes. A time-dependent Hamiltonian system of $m$ degrees of freedom is a CIS if there exist $m$ independent integrals of motion $\{F_k\}$ in involution with respect to the Poisson bracket $\{\cdot,\cdot\}_V$ (6). Their Hamiltonian vector fields

$$
\partial_i = -w_V[dF_i = \partial^k F_i \partial_k - \partial_k F_i \partial^k]
$$

and the Hamilton vector field $\gamma_H$ (8) generate a smooth regular distribution on the phase space $V^*Q$ and the corresponding foliation of $V^*Q$ in invariant manifolds.

One can associate to any time-dependent CIS on $V^*Q$ an autonomous CIS on the homogeneous phase space $T^*Q$ as follows.

Given a Hamiltonian $h$ (7), let us consider an autonomous Hamiltonian system on the symplectic manifold $(T^*Q, \Omega)$ with the Hamiltonian

$$
\mathbf{H} = \partial_t \{\Xi - \zeta^* h^* \Xi\} = p + \mathcal{H}.
$$

Its Hamiltonian vector field

$$
\gamma_T = \partial_t - \partial_t \mathcal{H} \partial_p + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k
$$

is projected onto the Hamilton vector field $\gamma_H$ (8) on $V^*Q$ so that

$$
\zeta^*(\mathbf{L}_{\gamma_H} f) = \{\mathbf{H}, \zeta^* f\}_T, \quad f \in C^\infty(V^*Q).
$$

An immediate consequence of this relation is the following.

(i) Given a time-dependent CIS $(H; F_k)$ on $V^*Q$, the Hamiltonian system $\{\mathbf{H}, \zeta^* F_k\}$ on $T^*Q$ is a CIS.

(ii) If $M \subset V^*Q$ is an invariant manifold of the time-dependent CIS $\{H; F_k\}$, then $h(M) \subset T^*Q$ is an invariant manifold of the homogeneous CIS $(\mathbf{H}, \zeta^* F_k)$.

Hereafter, let the Hamilton vector field $\gamma_H$ (8) be complete, i.e., the Hamilton equations (9) admit a unique global solution (a trajectory of $\gamma_H$) through every point of the phase space $V^*Q$. The trajectories of $\gamma_H$ define a trivial bundle $V^*Q \to V_Q^*Q$ over the fiber $V_Q^*Q$.
of \( V^*Q \to \mathbb{R} \) at \( t = 0 \). Then any invariant manifold \( M \) of \( \{ H; F_k \} \) is also a trivial bundle \( M = \mathbb{R} \times M_0 \) over \( M_0 = M \cap V_0^*Q \).

If \( M_0 \) is compact, one can introduce action-angle coordinates around an invariant manifold \( M \) by use of the action-angle coordinates around the invariant manifold \( h(M) \) of the corresponding autonomous CIS on \( T^*Q \). Namely, \( h(M) \) has an open neighbourhood which is a trivial bundle

\[
U' = V' \times \mathbb{R} \times T^m \to V' \times \mathbb{R} \to V'
\]

over a domain \( V' \subset \mathbb{R}^{m+1} \) with respect to the action-angle coordinates \((I_0, I_i, t, \phi^i)\). Herewith, the following holds. (i) \( I_0 = H \). (ii) The integrals of motion \( \zeta^*F_k \) depend only on the action coordinates \( I_i \). (iii) The symplectic form \( \Omega \) on \( U' \) reads

\[
\Omega = dI_0 \wedge dt + dI_i \wedge d\phi^i.
\]

The symplectic annulus \( U' \) (11) inherits the fibration structure (5) over the toroidal domain

\[
U = V \times \mathbb{R} \times T^m, \quad V \subset \mathbb{R}^m.
\]

Coordinated by \((I_i, t, \phi^i)\) and provided with the Poisson structure (6), the toroidal domain (12) is a phase space of the time-dependent CIS \((H; F_i)\) around its instantly compact invariant manifold \( M \). Since \( H = I_0 \), the Hamilton vector field (10) is \( \gamma_T = \partial_t \), and so is its projection \( \gamma_H \) (8) onto \( U \). Hence, the above-mentioned action-angle coordinates \((I_i, t, \phi^i)\) are the initial data coordinates.

These action-angle coordinates are by no means unique. Let \( \mathcal{H} \) be an arbitrary smooth function on \( \mathbb{R}^m \). Then the canonical transformation

\[
I'_0 = I_0 - \mathcal{H}(I_j), \quad I'_i = I_i, \quad t' = t, \quad \phi^i = \phi^i + t\partial_i \mathcal{H}(I_j)
\]

(13)
gives new action-angle coordinates corresponding to a different trivialization of \( U' \) (11) (and \( U \) (12)). Accordingly, the Hamilton vector field \( \gamma_H \) takes the form (8), and the Hamilton equations (9) read

\[
\dot{\phi}^k = \partial^k \mathcal{H}, \quad \dot{I}_k = 0.
\]

These are the Hamilton equations of an autonomous CIS with a time-independent Hamiltonian \( \mathcal{H} \) on the toroidal domain \( U \) (12).

**III. CLASSICAL HOLOMNY OPERATORS**

The phase space of a Hamiltonian system with time-dependent parameters is a composite fiber bundle \( \Pi \to \Sigma \times \mathbb{R} \to \mathbb{R} \), where \( \Pi \to \Sigma \times \mathbb{R} \) is a symplectic bundle and \( \Sigma \times \mathbb{R} \to \mathbb{R} \).
is a parameter bundle whose sections are parameter functions.\textsuperscript{12,13,17,18} In the case under consideration, all bundles are trivial and their trivializations hold fixed. Namely, the phase space is the product

$$
\Pi = \Sigma \times U = \Sigma \times (V \times \mathbb{R} \times T^m) \to \Sigma \times \mathbb{R} \to \mathbb{R},
$$
equipped with the coordinates \((\sigma^\alpha, I_k, t, \phi^k)\). Let us suppose for a time that parameters are also dynamic variables. The phase space of this system is the fiber bundle

$$
\Pi' = T^*\Sigma \times U \to \Sigma \times \mathbb{R} \times T^m
$$
coordinated by \((\sigma^\alpha, \sigma_\alpha, I_k, t, \phi^k)\). A generic Hamiltonian of such a system is

$$
\mathcal{H}_\Sigma = \sigma_\alpha \Sigma_\alpha^\alpha + I_k (\Lambda_k^k + \Lambda_\alpha^k \Sigma_\alpha^\alpha) + \tilde{\mathcal{H}}(\sigma^\beta, I_j, t, \phi^j),
$$
where

$$
\partial_t + \Sigma_\alpha^\alpha \partial_\alpha + (\Lambda_k^k + \Lambda_\alpha^k \Sigma_\alpha^\alpha) \partial_k
$$
is a composite connection on the fiber bundle \(\Sigma \times \mathbb{R} \times T^m \to \mathbb{R}\) generated by a connection \(\partial_t + \Sigma_\alpha^\alpha \partial_\alpha\) on the parameter bundle \(\Sigma \times \mathbb{R} \to \mathbb{R}\) and a connection

$$
\Lambda = dt \otimes (\partial_t + \Lambda_k^k \partial_k) + d\sigma^\alpha \otimes (\partial_\alpha + \Lambda_\alpha^k \partial_k)
$$
on \(\Sigma \times \mathbb{R} \times T^m \to \Sigma \times \mathbb{R}\).\textsuperscript{12,13,18} Then a Hamiltonian system with a fixed parameter function \(\sigma^\alpha = \xi^\alpha(t)\) is characterized by the Hamiltonian

$$
\mathcal{H}_\xi = I_k [\Lambda_k^k(t, \phi^j) + \Lambda_\alpha^k (\xi^\beta, t, \phi^j) \partial_t \xi^\alpha] + \tilde{\mathcal{H}}(\xi^\beta, I_j, t, \phi^j)
$$
on the pull-back bundle \(U = \xi^*\Pi\) (12).

Let \((I_k, t, \phi^k)\) be the initial data action-angle coordinates of a time-dependent CIS. Its Hamiltonian \(\mathcal{H}\) with respect to these coordinates vanishes. Therefore, we can introduce a desired holonomy operator by the appropriate choice of the connection \(\Lambda\) (15). Let us put \(\Lambda_k^k = 0\) and assume that coefficients \(\Lambda_\alpha^k\) are independent of time, i.e., the part

$$
\Lambda_\Sigma = d\sigma^\alpha \otimes (\partial_\alpha + \Lambda_\alpha^k \partial_k)
$$
of the connection \(\Lambda\) (15) is a connection on the fiber bundle (4). Then the Hamiltonian of a perturbed CIS reads

$$
\mathcal{H}_\xi = I_k \Lambda_\alpha^k (\xi^\beta, \phi^j) \partial_t \xi^\alpha.
$$
Its Hamilton vector field (8) is

$$
\gamma_H = \partial_t + \Lambda_\alpha^i \partial_\alpha \xi^\alpha \partial_i - I_k \partial_t \Lambda_\alpha^k \partial_\alpha \xi^\alpha \partial_i.
$$
It leads to the Hamilton equations

\begin{align}
\partial_t \phi^i &= \Lambda^i_\alpha(\xi(t), \phi^j) \partial_t \xi^\alpha, \\
\partial_t I_i &= -I_k \partial_k \Lambda^k_\alpha(\xi(t), \phi^j) \partial_t \xi^\alpha.
\end{align}

(20) (21)

Note that

\begin{equation}
V^*\Lambda = d\sigma^\alpha \otimes (\partial_\alpha + \Lambda^i_\alpha \partial_i - I_k \partial_i \Lambda^k_\alpha \partial^i)
\end{equation}

(22)

is the lift of the connection \(\Lambda\) (17) onto the fiber bundle \(\Sigma \times (V \times T^m) \to \Sigma\), seen as a subbundle of the vertical cotangent bundle \(V^*(\Sigma \times T^m) = \Sigma \times T^* T^m\) of the fiber bundle (4). It follows that any solution \(I_i(t), \phi^i(t)\) of the Hamilton equations (20) – (21) (i.e., an integral curve of the Hamilton vector field (19)) is a horizontal lift of the curve \(\xi(t) \subset \Sigma\) with respect to the connection \(V^*\Lambda\) (22). i.e., \(I_i(t) = I_i(\xi(t)), \phi^i(t) = \phi(\xi(t))\). Thus, the right-hand side of the Hamilton equations (20) – (21) is the holonomy operator

\(\Delta = (\Lambda^i_\alpha \partial_t \xi^\alpha, -I_k \partial_i \Lambda^k_\alpha \partial_t \xi^\alpha)\)

(cf. (1) where \(\hat{S}_0 = 0\)). It is not a linear operator, but the substitution of a solution \(\phi(\xi(t))\) of the equation (20) into the Hamilton equation (21) results in a linear holonomy operator on the action variables \(I_i\).

Let us show that the holonomy operator (23) is well defined. Since any vector field \(\vartheta\) on \(\mathbb{R} \times T^m\) such that \(\vartheta J dt = 1\) is complete, the Hamilton equation (20) has solutions for any parameter function \(\xi(t)\). It follows that any connection \(\Lambda\) (17) on the fiber bundle (4) is an Ehresmann connection, and so is its lift (22). Therefore, any curve \(\xi([0, 1]) \subset \Sigma\) can play the role of the parameter function in the holonomy operator \(\Delta\) (23).

Now, let us return to the original action-angle coordinates \((I_k, t, \varphi^k)\) by means of the canonical transformation (13). Relative to these coordinates, the perturbed Hamiltonian reads

\[\mathcal{H}' = I_k \Lambda^k_\alpha(\xi(t), \varphi^i - t \partial^i \mathcal{H}(I_j)) \partial_t \xi^\alpha(t) + \mathcal{H}(I_j),\]

and the Hamilton equations (20) – (21) take the form

\begin{align}
\partial_t \varphi^i &= \partial^i \mathcal{H}(I_j) + \Lambda^i_\alpha(\xi(t), \varphi^j - t \partial^j \mathcal{H}(I_j)) \partial_t \xi^\alpha(t) \\
&\quad - t I_k \partial^i \partial^k \mathcal{H}(I_j) \partial_k \Lambda^k_\alpha(\xi(t), \varphi^j) \partial_t \xi^\alpha(t), \\
\partial_t I_i &= -I_k \partial_k \partial_i \Lambda^k_\alpha(\xi(t), \varphi^j - t \partial^j \mathcal{H}(I_j)) \partial_t \xi^\alpha(t).
\end{align}

Their solution is \(I_i(\xi(t)), \varphi^i(t) = \phi^i(\xi(t)) + t \partial^i \mathcal{H}(I_j(\xi(t)))\) where \(I_i(\xi(t)), \phi^i(\xi(t))\) is a solution of the Hamilton equations (20) – (21). It is readily observed that the action variables \(I_k\) are driven only by the holonomy operator, while the angle variables \(\varphi^i\) have a nongeometric summand.

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Let us emphasize that, in the construction of the holonomy operator (23), we did not impose any restriction on the connection $\Lambda_{\Sigma}$ (17). Therefore, any connection on the fiber bundle (4) generates a holonomy operator in a CIS. However, a glance at the expression (23) shows that this operator becomes zero on action variables if all coefficients $\Lambda^k_{\lambda}$ of the connection $\Lambda_{\Sigma}$ (17) are constant, i.e., $\Lambda_{\Sigma}$ is a principal connection on the fiber bundle (4) seen as a principal bundle with the structure group $T^m$.

IV. QUANTUM CIS

There are different approaches to quantization of CISs. Their geometric quantization was studied at first with respect to the polarization spanned by Hamiltonian vector fields of integrals of motion. For example, the well-known Simms quantization of the harmonic oscillator is of this type. In this approach, the problem is that the associated quantum algebra includes affine functions of angle coordinates which are ill defined. As a consequence, elements of the carrier space of this quantization fail to be smooth, but are tempered distributions. In recent works, we have developed a different variant of geometric quantization of CISs by use of the angle polarization spanned by almost-Hamiltonian vector fields $\partial^k$ of angle variables. This quantization is equivalent to geometric quantization of the cotangent bundle $T^*T^m$ of a torus $T^m$ with respect to the vertical polarization. The result is as follows.

Given an autonomous CIS on a symplectic annulus

$$P = V \times T^m, \quad \Omega_P = dI_i \wedge d\phi^i$$

equipped with the action-angle coordinates $(I_i, \phi^i)$, its quantum algebra $\mathcal{A}$ with respect to the above mentioned angle polarization consists of affine functions

$$f = a^k(\phi^j)I_k + b(\phi^j)$$

of action coordinates $I_k$. They are represented by self-adjoint unbounded operators

$$\hat{f} = -ia^k\partial_k - \frac{i}{2}\partial_k a^k - a^k\lambda_k + b$$

in the separable pre-Hilbert space of complex half-forms on $T^m$. If coordinate transformations of $T^m$ are only translations, this space can be identified with the pre-Hilbert space $\mathbb{C}^\infty(T^m)$ of smooth complex functions on $T^m$. Different tuples of real numbers $(\lambda_1, \ldots, \lambda_m)$ and $(\lambda'_1, \ldots, \lambda'_m)$ specify inequivalent representations (24), unless $\lambda_k - \lambda'_k \in \mathbb{Z}$ for all $k = 1, \ldots, m$. These numbers come from the de Rham cohomology group $H^1(T^m) = \mathbb{R}^m$.

In particular, the action operators (24) read $\hat{I}_k = -i\partial_k - \lambda_k$. They are bounded. By virtue of the multidimensional Fourier theorem, an orthonormal basis for $\mathbb{C}^\infty(T^m)$ consists of functions

$$\psi_{(n_r)}(\phi) = \exp[in_r\phi^r], \quad (n_r) = (n_1, \ldots, n_m) \in \mathbb{Z}^m.$$

(25)
With respect to this basis, the action operators are brought into countable diagonal matrices

$$\hat{I}_k \psi_{(n_r)} = (n_k - \lambda_k) \psi_{(n_r)},$$

(26)

while functions \(a^k(\varphi)\) are decomposed in Fourier series of the functions \(\psi_{(n_r)}\), which act on \(\mathbb{C}^\infty(T^m)\) by the law

$$\hat{\psi}_{(n_r)} \psi_{(n'_r)} = \psi_{(n_r + n'_r)}.$$ 

(27)

It should be emphasized that \(\hat{a}^k \hat{I}_k \neq \hat{I}_k \hat{a}^k\).

If a Hamiltonian \(\mathcal{H}(I_k)\) of an autonomous CIS is an analytic function on \(\mathbb{R}^m\), it is uniquely quantized as a Hermitian element \(\hat{\mathcal{H}}(I_k) = \mathcal{H}(\hat{I}_k)\) of the enveloping algebra of \(\mathcal{A}\). It is a bounded self-adjoint operator with the countable spectrum

$$\hat{\mathcal{H}}(I_k) \psi_{(n_r)} = E_{(n_r)} \psi_{(n_r)}, \quad E_{(n_r)} = \mathcal{H}(n_k - \lambda_k), \quad n_k \in (n_r).$$

(28)

In order to quantize a time-dependent CIS on the Poisson toroidal domain \((U, \{J_i\}_V)\) (12) equipped with action-angle coordinates \((I_i, t, \varphi^i)\), one may follow the instantwise geometric quantization of time-dependent mechanics.\(^{21}\) As a result, we can simply replace functions on \(T^m\) with those on \(\mathbb{R} \times T^m\).\(^{10}\) Namely, the corresponding quantum algebra \(\mathcal{A} \subset \mathbb{C}^\infty(U)\) consists of affine functions

$$f = a^k(t, \varphi^j) I_k + b(t, \varphi^j)$$

(29)

of action coordinates \(I_k\) represented by the operators (24) in the space

$$E = \mathbb{C}^\infty(\mathbb{R} \times T^m)$$

(30)

of smooth complex functions \(\psi(t, \varphi)\) on \(\mathbb{R} \times T^m\). This space is provided with the structure of the pre-Hilbert \(\mathbb{C}^\infty(\mathbb{R})\)-module with respect to the nondegenerate \(\mathbb{C}^\infty(\mathbb{R})\)-bilinear form

$$\langle \psi | \psi' \rangle = \left(\frac{1}{2\pi}\right)^m \int_{T^m} \overline{\psi} \psi' \, d^m \varphi, \quad \psi, \psi' \in \mathbb{C}^\infty(\mathbb{R} \times T^m).$$

Its basis consists of the pull-back onto \(\mathbb{R} \times T^m\) of the functions \(\psi_{(n_r)}\) (25).

This quantization of a time-dependent CIS is extended to the associated homogeneous CIS on the symplectic annulus \((U', \Omega)\) (11) by means of the operator \(\hat{I}_0 = -i\partial_t\) in the pre-Hilbert module \(E\) (30). Accordingly, the homogeneous Hamiltonian \(\mathbf{H}\) is quantized as \(\hat{\mathbf{H}} = -i\partial_t + \hat{\mathcal{H}}\). The corresponding Schrödinger equation is

$$\hat{\mathbf{H}} \psi = -i\partial_t \psi + \hat{\mathcal{H}} \psi = 0, \quad \psi \in E.$$

(31)

For instance, the quantum Hamiltonian of the original autonomous CIS is

$$\hat{\mathbf{H}} = -i\partial_t + \mathcal{H}(\hat{I}_j).$$
Its spectrum \( \tilde{H}\psi_{(n_r)} = E_{(n_r)}\psi_{(n_r)} \) relative to the basis \( \{\psi_{(n_r)}\} \) for \( E \) (30) coincides with that of the autonomous Hamiltonian (28). The Schrödinger equation (31) reads

\[
\tilde{H}\psi = -i\partial_t\psi + \mathcal{H}(-i\partial_k - \lambda_k)\psi = 0, \quad \psi \in E.
\]

Its solutions are the Fourier series

\[
\psi = \sum_{(n_r)} B_{(n_r)} \exp[-itE_{(n_r)}]\psi_{(n_r)}, \quad B_{(n_r)} \in \mathbb{C}.
\]

Now, let us quantize this CIS with respect to the initial data action-angle coordinates \( (I_i, \phi^j) \). Its quantum algebra \( \mathcal{A}_0 \subset C^\infty(U) \) consists of affine functions

\[
f = a^k(t, \phi^j)I_k + b(t, \phi^j).
\]

The canonical transformation (13) provides an isomorphism between Poisson algebras \( \mathcal{A} \) and \( \mathcal{A}_0 \). Functions \( f \) (32) are represented by the operators \( \hat{f} \) (24) in the pre-Hilbert module \( E_0 \) of smooth complex functions \( \Psi(t, \phi) \) on \( \mathbb{R} \times T^m \). Given its basis \( \Psi_{(n_r)}(\phi) = [in_r\phi^r] \), the operators \( \hat{I}_k \) and \( \hat{\psi}_{(n_r)} \) take the form (26) and (27), respectively. The Hamiltonian of a quantum CIS with respect to the initial data variables is \( \hat{H}_0 = -i\partial_t \). Then one easily obtains the isometric isomorphism

\[
R(\psi_{(n_r)}) = \exp[itE_{(n_r)}]\Psi_{(n_r)}, \quad (R(\psi)|R(\psi')) = (\psi|\psi'),
\]

between the pre-Hilbert modules \( E \) and \( E_0 \) which provides the equivalence

\[
\hat{I}_i = R^{-1}\hat{I}_iR, \quad \hat{\psi}_{(n_r)} = R^{-1}\hat{\psi}_{(n_r)}R, \quad \hat{H} = R^{-1}\hat{H}_0R \tag{34}
\]

of the quantizations of a CIS with respect to the original and initial data action-angle variables.

V. QUANTUM HOLOMNY OPERATORS

In view of the isomorphism (34), let us first construct a holonomy operator for a quantum CIS \( (\mathcal{A}_0, \hat{H}_0) \) with respect to the initial data action-angle coordinates. Let us consider the perturbed homogeneous Hamiltonian

\[
H_\xi = H_0 + H_1 = I_0 + \partial_t\xi^\alpha(t)\Lambda^k_\alpha(\xi(t), \phi^j)I_k
\]

of the classical perturbed system (18). Its perturbation term \( H_1 \) is of the form (29) and, therefore, is quantized by the operator

\[
\hat{H}_1 = -i\partial_t\xi^\alpha\hat{\Lambda}_\alpha = -i\partial_t\xi^\alpha[\Lambda^k_\alpha\partial_k + \frac{1}{2}\partial_k(\Lambda^k_\alpha) - i\lambda_k\Lambda^k_\alpha].
\]
The quantum Hamiltonian \( \hat{H}_\xi = \hat{H}_0 + \hat{H}_1 \) defines the Schrödinger equation
\[
\partial_t \Psi + \partial_\xi \xi^\alpha [\Lambda_\alpha^k \partial_k + \frac{1}{2} \partial_k (\Lambda_\alpha^k) - i \lambda_k \Lambda_\alpha^k] \Psi = 0.
\]
(35)

If a solution exists, it can be written by means of the evolution operator which reduces to the geometric factor \( U_1 \) (3). The latter can be viewed as a displacement operator along the curve \( \xi[0,1] \subset \Sigma \) with respect to the connection
\[
\hat{\Lambda}_\Sigma = d\sigma^\alpha (\partial_\alpha + \hat{\Delta}_\alpha)
\]
in the \( C^\infty(\Sigma) \)-module \( C^\infty(\Sigma \times T^m) \) of smooth complex functions on \( \Sigma \times T^m \).

Let us study the existence if this displacement operator.

Given a connection \( \Lambda_\Sigma \) (17), let \( \Phi^i(t, \phi) \) denote the flow of the complete vector field \( \partial_t + \Lambda_\alpha^i (\xi, \phi) \partial_\xi \xi^\alpha \partial_i \) on \( \mathbb{R} \times T^m \). It is a solution of the Hamilton equation (20) with the initial data \( \phi \). We need the inverse flow \( (\Phi^{-1})^i(t, \phi) \) which obeys the equation
\[
\partial_t (\Phi^{-1})^i(t, \phi) = -\partial_\xi \xi^\alpha \Lambda_\alpha^k (\xi, (\Phi^{-1})^i(t, \phi)) = -\partial_t \xi^\alpha \Lambda_\alpha^k (\xi, \phi) \partial_k (\Phi^{-1})^i(t, \phi).
\]

Let \( \Psi_0 \) be an arbitrary complex half-form \( \Psi_0 \) on \( T^m \) possessing identical transition functions, and let the same symbol stand for its pull-back onto \( \mathbb{R} \times T^m \). Given its pull-back
\[
(\Phi^{-1})^* \Psi_0 = \det \left( \frac{\partial (\Phi^{-1})^i}{\partial \phi^k} \right)^{1/2} \Psi_0 (\Phi^{-1}(t, \phi)),
\]
(37)
it is readily observed that
\[
\Psi = (\Phi^{-1})^* \Psi_0 \exp[i \lambda_k \phi^k]
\]
obeyes the Schrödinger equation (35) with the initial data \( \Psi_0 \). This function is well defined only if all the numbers \( \lambda_k \) equal 0 or \( \pm 1/2 \). Note that, if some numbers \( \lambda_k \) are equal to \( \pm 1/2 \), then \( \Psi_0 \exp[i \lambda_k \phi^k] \) is a half-density on \( T^m \) whose transition functions equal \( \pm 1 \), i.e., it is a section of a nontrivial metalinear bundle over \( T^m \).

We thus observe that if \( \lambda_k \) equal 0 or \( \pm 1/2 \), then the displacement operator always exists and \( \Delta = i \hat{H}_1 \) is a holonomy operator. A glance at the action law (27) shows that this operator is infinite-dimensional.

For instance, let \( \Lambda_\Sigma \) (17) be the above mentioned principal connection, i.e., \( \Lambda_\alpha^k = \text{const.} \). Then the Schrödinger equation (35) where \( \lambda_k = 0 \) takes the form
\[
\partial_t \Psi(t, \phi^j) + \partial_\xi \xi^\alpha (t) \Lambda_\alpha^k \partial_k \Psi(t, \phi^j) = 0.
\]

Its solution (37) is
\[
\Psi(t, \phi^j) = \Psi_0 (\phi^j - (\xi^\alpha (t) - \xi^\alpha (0)) \Lambda_\alpha^j).
\]
The corresponding evolution operator reduces to Berry’s phase multiplier
\[ U_1 \Psi_{(n_r)} = \exp[-in_j(\xi^\alpha(t) - \xi^\alpha(0))A_j^\alpha)]\Psi_{(n_r)}, \quad n_j \in (n_r). \]

It keeps the eigenvectors of the action operators \( \hat{I}_i \).

In order to return to the original action-angle variables, one can employ the morphism \( R \) (33). The corresponding Hamiltonian reads \( H' = R^{-1}H_R R \). The key point is that, due to the relation (34), the action operators \( \hat{I}_i \) have the same mean values
\[ \langle I_k \psi | \psi \rangle = \langle I_k \Psi | \Psi \rangle, \quad \Psi = R(\psi), \]
with respect both to the original and the initial data action-angle variables. Therefore, these mean values are defined only by the holonomy operator.

VI. CONCLUSIONS

We have shown that any CIS around its compact invariant manifold admits a perturbation dependent on parameters by means of holonomy operator associated to a connection on the fiber bundle (4).

Since action variables are driven only by a holonomy operator, one can use this operator in order to perform a dynamic transition between classical solutions or quantum states of an unperturbed CIS by an appropriate choice of a parameter function \( \xi \). The key point is that this transition can take an arbitrary short time because we are entirely free with time parametrization of \( \xi \) and can choose it quickly changing, in contrast with slowly varying parameter functions in adiabatic models. For instance, one can choose \( \xi \) a step function, then its time derivative is a \( \delta \)-function of time. This fact makes nonadiabatic holonomy operators in CISs promising for several applications, including classical and quantum scattering in integrable Hamiltonian systems,\(^{23}\) quantum control operators,\(^{24,25}\) and the above mentioned quantum computation. It also looks attractive that quantum holonomy operators in CISs are essentially infinite-dimensional, whereas both the existent quantum control theory and the theory of quantum information and computation\(^{26}\) involve only finite-dimensional operators.


