We have investigated the present renormalization prescriptions of Cabibbo-Kobayashi-Maskawa (CKM) matrix. When considering the prescription which is formulated with reference to the case of zero mixing we find the deviation of the CKM counterterm from the unitarity is very small, which can be neglected in actual calculations. We generalize this prescription to all loop level, simultaneously keep the unitarity of the bare CKM matrix. The new prescription also makes the amplitude of an arbitrary physical process involving quark mixing convergent and gauge independent.

Since the exact examination of the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix [1–6] has been developed quickly, the renormalization of CKM matrix becomes very important. This was realized for the Cabibbo angle in the standard model (SM) with two fermion generations in a pioneering paper by Marciano and Sirlin [7] and for the CKM matrix of the three-generation SM by Denner and Sack [8] more than a decade ago. In recent years many people have discussed this issue [9–11], but a completely self-consistent scheme to all loop level has not been obtained. In this paper we try to solve this problem and give some instructive conclusion.

In general, a CKM matrix renormalization prescription needs to satisfy three criterions, as Diener has declared [12]:

1. In order to make the transition amplitude of any physical process involving quark mixing ultraviolet finite, the CKM counterterm must cancel out the ultraviolet divergence left in the loop-corrected amplitudes. On the other hand it must include proper infrared divergence for the sake of infrared finiteness of the final scattering cross-section including soft quanta emission.

2. It must guarantee the transition amplitude of any physical process involving quark mixing gauge parameter independent [13], which is a necessary and fundamental requirement.

3. SM requires the bare CKM matrix $V^0$ be unitary,

$$\sum_k V^0_{ik} V^0_{jk} = \delta_{ij} \quad (1)$$

with $i, j$ and $k$ the generation index, $\delta_{ij}$ the unit matrix element. If we split the bare CKM matrix element $V^0_{ij}$ into the renormalized one and the counterterm $V^0_{ij} = V_{ij} + \delta V_{ij}$ and keep the unitarity of the renormalized CKM matrix, the unitarity of bare CKM matrix requires

$$\sum_k (\delta V_{ik} V^*_{jk} + V_{ik} \delta V^*_{jk} + \delta V_{ik} \delta V^*_{jk}) = 0 \quad (2)$$

Until now there are many papers discussing this problem. The modified minimal subtraction (MS) scheme [14,15] is the simplest one for its manipulation of the divergence, but it introduces the $\mu^2$-dependent term which is very complicated to be dealt with. In the on-shell (OS) renormalization scheme, however there is still not an integrated CKM renormalization prescription. The early prescription [8] was to relate the CKM counterterm with fermion wavefunction renormalization constants (WRC) and the $SU_L(2)$ symmetry of SM has been used [16]. Although it is a delicate prescription, it reduces the physical amplitude involving quark mixing gauge dependent [17–19]. A remedial prescription is to replace the OS fermion WRC in the CKM counterterm with the fermion WRC calculated at zero momentum [17]. Another remedial prescription [19] is to rearrange the off-diagonal quark WRC in a manner similar to the pinch technique [20].

Besides the idea of Ref. [8], another idea is to formulate the CKM renormalization prescription with reference to the case of zero mixing. This has been done in Ref. [21,12] at one loop level. The main idea is to renormalize the

1. This is easy to be understood since the $SU_L(2)$ symmetry of SM is broken by the Higgs mechanism
transition amplitude of $W$ gauge boson decaying into up-type and down-type quarks equal to the amplitude of without generation mixing. The one loop decay amplitude $T_1$ of $W^+ \to u_i d_j$ is [21]

$$T_1 = A_L V_{ij} (F_L + \frac{g}{\sqrt{4} G} \delta Z_W + \frac{1}{2} \delta Z_{ii}^{uu} + \frac{1}{2} \delta Z_{jj}^{dd}) + \sum_{k \neq i} \frac{1}{2} V_{ik} \delta Z_{kj}^{dd} + \sum_{k \neq j} \frac{1}{2} V_{jk} \delta Z_{ki}^{uu} + V_{ij} [A_R F_R + B_L G_L + B_R G_R],$$

where $g$ and $\delta g$ are the SU(2) coupling constant and its counterterm, $\delta Z_W$ is the $W$ boson WRC, $\delta Z_{ii}^{uu}$ and $\delta Z_{jj}^{dd}$ are the left-handed WRC of up-type/down-type quarks, with

$$A_L = \frac{1}{\sqrt{3}} \bar{u}_i(p_l) \gamma_L V_{ij} (q - p_l), \quad B_L = \frac{1}{\sqrt{2}} \bar{u}_i(p_l) \frac{a_{W}}{M_W} \gamma_L V_{ij} (q - p_l).$$

$\varepsilon^\mu$ is the $W$ gauge boson polarization vector, $\gamma_L$ and $\gamma_R$ are the left-handed/right-handed chirality, and $M_W$ is the $W$ boson mass. Similarly, replacing in the above two equations $\gamma_L$ by $\gamma_R$ we define $A_R$ and $B_R$, respectively. $F_L, R$ and $G_L, R$ are the form factors. According to the prescription of Ref. [7], $F_L, R$ and $\delta Z_W$ in Eq. (3) are left unchanged in the renormalized amplitude, only the quark WRC need to be adjusted by CKM counterterm. So it is easy to obtain [21]

$$\delta V_{ij} = -\frac{1}{2} \sum_k (\delta Z_{ki}^{uu} V_{kj} + V_{ik} \delta Z_{kj}^{dd}) + \frac{1}{2} V_{ij} [\delta Z_{ii}^{uu} + \delta Z_{jj}^{dd}],$$

(5)

subscript "[\cdot]\" denotes the quantity is obtained by replacing CKM matrix with unit matrix. Since this counterterm satisfies the unitary criterion as expected. Since the divergence of $\delta V_{ij}$ satisfies the unitary criterion, $\delta V_{ij}$ has the same divergence as $\delta V_{ij}$. On the other hand $\delta V_{ij}$ is gauge independent, which makes the decay amplitude of $W^+ \to u_i d_j$ gauge independent [17,12]. So a CKM matrix renormalization prescription with satisfying the three criterions mentioned previously has been obtained at one loop level.

All of the above prescriptions are only applied to one loop level. A suitable prescription for higher loop level is still not present. In view of the delicacy of Eq. (6) we follow the idea of Ref. [12] to generalize it to be suitable for higher loop level. First of all we need to generalize the method of Ref. [21] to all loop level. In fact Eq. (5) isn’t right when $i \neq j$. At the case of $i \neq j$ the modified fermion WRC $\delta Z_{ii}^{uu}$ in the renormalized amplitude $T_1$ contains the contribution of intermediate state of down-type quark $d_i$, and the modified fermion WRC $\delta Z_{jj}^{dd}$ in the renormalized amplitude $T_1$ contains the contribution of intermediate state of up-type quark $u_j$ [21]. We find these aren’t the expected zero-mixing result, because at zero-mixing case only the external-line fermions appear at the intermediate states. So Eq. (5) will reduce the decay amplitude of $W^+ \to u_i d_j$ ultraviolet divergent and gauge dependent at the case of $i \neq j$. Therefore we find the exact prescription is to change the CKM matrix to unit matrix with keeping the last odd CKM matrix unchanged, simultaneously change all the quarks in intermediate states to the same type external-line quarks. That’s to say

$$\delta V_{ij} = \frac{1}{2} \sum_k (\delta Z_{ki}^{uu} V_{kj} + V_{ik} \delta Z_{kj}^{dd}) + \frac{1}{2} V_{ij} [\delta Z_{ii}^{uu} - m_{d,i} - m_{d,j} + \delta Z_{jj}^{dd} - m_{u,i} - m_{u,j},]$$

where $m_{d,i}$ and $m_{d,j}$ represent the down-type quark’s mass, $m_{u,i}$ and $m_{u,j}$ represent the up-type quark’s mass. Our calculations has shown this CKM counterterm is gauge independent and makes the physical amplitude $T_1$ convergent. When generalizing this method to higher loop level we need some modifications.

Although at an arbitrary loop level the Feynman diagrams of the process of $W^+ \to u_i d_j$ are very complex, it is still very clearly that the difference between the two cases of generation mixing and zero-mixing (without generation mixing) only occurs at the fermion line connecting with the external fermion lines. So the expected zero-mixing amplitude after CKM renormalization can be obtained by modifying the amplitude of $W^+ \to u_i d_j$ as

1. Do such change only at the fermion line which connects with the external fermion lines: changing the CKM matrix to unit matrix and CKM counterterms to zero, but keeping the last odd CKM matrix unchanged (if there have even CKM matrices the modified result will not have CKM matrix), then $m_{u,i} \to m_{u,i}, m_{d,i} \to m_{d,j}$.
After such modification the amplitude of $W^+ \rightarrow u_i d_j$ will be equal to the amplitude of $W^+ \rightarrow u_i d_j$ in the case of zero-mixing timed by a factor $V_{ij}$, which is obviously convergent and gauge independent as it should be. Thus we can define the CKM counterterm equal to the difference between these two amplitudes.

In order to determine the n-loop level CKM counterterm $\delta V_n$, we construct the n-loop level amplitude of $W \rightarrow u_i d_j$ as following (where only the n-loop level counterterms are listed for convenience)

$$T_n = A_L[F_{Ln} + V_{ij}(\frac{\delta g_n}{g} + \frac{1}{2} \delta Z_{Wn}) + \frac{1}{2} \delta Z u_{Ln}^n|V + \frac{1}{2} \delta Z d_{Ln}^n + \delta V_n] + AR_F + B_L G_{Ln} + B_R G_{Ln}$$

(8)

the added denotation "n" represents the n-loop level result. After choosing the CKM counterterm $\delta V_n$ correctly, we require the above amplitude be changed into

$$T_n = A_L[F_{Ln} + V_{ij}(\frac{\delta g_n}{g} + \frac{1}{2} \delta Z_{Wn} + \frac{1}{2} \delta Z u_{Ln}^{n+1}|m_{i,d}, m_{j,d} - m_{u,d} - m_{u,n} + \frac{1}{2} \delta Z d_{Ln}^{n+1}|m_{u,n} - m_{u,d})] + AR_F + B_L G_{Ln} + B_R G_{Ln}$$

(9)

here the footnote "[l]" represents the new meaning: changing the quantity according to the two steps we have listed. From these two equations the CKM counterterm $\delta V_n$ is determined as

$$\delta V_n = F_{Ln} + \frac{1}{2} V_{ij}(\frac{\delta Z u_{Ln}^{n+1}|m_{i,d} - m_{j,d} - m_{u,n} - m_{u,d})} - F_{Ln} - \frac{1}{2} \delta Z d_{Ln}^{n+1}|V - \frac{1}{2} \delta Z d_{Ln}^{n+1}$$

(10)

Obviously our prescription complies with the first and second criterions. If it complies with the unitary criterion we will get an eligible CKM renormalization prescription. To test this thought we use the Taylor’s series to expand the CKM counterterm at one loop level. Using the definition of quark WRC in Ref. [22], dimension regularization [23] and $R_L$-gauge [24], the one order result of $\delta V_1$ is

$$\delta V_{1j} = \frac{\alpha(\Delta - 11)}{128 \pi M_W^2 s_W} \left[ V_{ij}(12 \ln m_W^2 m_{d,j}^2 + 6 m_{d,j}^4 + 12 \ln m_W^2 m_{d,j}^2 + 6 m_{d,j}^4 + 12 m_{d,j}^2 m_{u,i}^2 - \sum_k V_{ik} V_{kj} (6 \ln m_W^2 m_{d,j}^2 + 3 m_{d,j}^4 + 12 m_{d,j}^2 m_{u,i}^2) - \sum_k V_{ik} V_{kj} (6 \ln m_W^2 m_{d,j}^2 + 3 m_{d,j}^4 + 12 m_{d,j}^2 m_{u,i}^2)) + \frac{2}{m_{d,i} - m_{d,j}} \sum_{k,l \neq j} m_{d,j} m_{u,k} (4 m_{d,j}^2 + 6 \ln m_W^2 m_{d,j}^2 + 3 m_{d,j}^2) V_{ik} V_{kj} - \frac{2}{m_{d,j} - m_{d,k}} \sum_{k,l \neq i} m_{d,k} m_{u,i} (4 m_{d,k}^2 + 6 \ln m_W^2 m_{d,k}^2 + 3 m_{d,k}^2) V_{ik} V_{kj} - \frac{2}{m_{u,i} - m_{u,k}} \sum_{k,l \neq i} m_{u,k} m_{d,i} (4 m_{u,k}^2 + 6 \ln m_W^2 m_{d,i}^2 + 3 m_{d,i}^2) V_{ik} V_{kj} \right]$$

(11)

the subscript "1" in $\delta V_1$ is omitted, the superscript ",(1)" denotes the one order result of the quantity, $\alpha$ is the fine structure constant, $\Delta = 2/(D - 4) + \gamma_E - \ln(4\pi) + \ln(M_W^2/\mu^2)$, $D$ is the space-time dimensionality, $\gamma_E$ is the Euler’s constant, $\mu$ is an arbitrary mass parameter, and $s_W$ is the sine of Weak mixing angle $\theta_W$. Substituting this result for $\delta V_{ij}$ in Eq. (2) we have verified that it satisfies the unitary criterion. Next, the two order result of $\delta V_1$ is

$$\delta V_{1j} = \frac{\alpha(\Delta - 11)}{128 \pi M_W^2 s_W} \left[ V_{ij}(12 \ln m_W^2 m_{d,j}^2 + 6 m_{d,j}^4 + 12 \ln m_W^2 m_{d,j}^2 + 6 m_{d,j}^4 + 12 m_{d,j}^2 m_{u,i}^2 - \sum_k V_{ik} V_{kj} (6 \ln m_W^2 m_{d,j}^2 + 3 m_{d,j}^4 + 12 m_{d,j}^2 m_{u,i}^2) - \sum_k V_{ik} V_{kj} (6 \ln m_W^2 m_{d,j}^2 + 3 m_{d,j}^4 + 12 m_{d,j}^2 m_{u,i}^2)) + \frac{2}{m_{d,i} - m_{d,j}} \sum_{k,l \neq j} m_{d,j} m_{u,k} (4 m_{d,j}^2 + 6 \ln m_W^2 m_{d,j}^2 + 3 m_{d,j}^2) V_{ik} V_{kj} - \frac{2}{m_{d,j} - m_{d,k}} \sum_{k,l \neq i} m_{d,k} m_{u,i} (4 m_{d,k}^2 + 6 \ln m_W^2 m_{d,k}^2 + 3 m_{d,k}^2) V_{ik} V_{kj} - \frac{2}{m_{u,i} - m_{u,k}} \sum_{k,l \neq i} m_{u,k} m_{d,i} (4 m_{u,k}^2 + 6 \ln m_W^2 m_{d,i}^2 + 3 m_{d,i}^2) V_{ik} V_{kj} \right]$$

(12)

We also find this result satisfies the unitary criterion.

But it isn’t true at three order level. Substituting the three order result $\delta V(3)$ for $\delta V_{1j}$ in Eq. (2), we obtain

$$\sum_k (\delta V_{kj}^3 V_{kj} + V_{kj}^3 V_{kj}^3) = \frac{\alpha(\Delta - 11)}{128 \pi M_W^2 s_W} \left[ \sum_{k,i} m_{u,k}^2 m_{d,i} - 2 m_{d,j}^2 m_{u,i}^2 + m_{d,j}^4 + m_{d,ij}^2 m_{d,ij}^2 m_{d,ij}^2 m_{d,ij}^2 V_{kj} - \sum_k l \sum_{n=1}^{n} m_{d,ui}^2 m_{d,ij}^2 m_{d,ij}^2 m_{d,ij}^2 V_{kj}^3 \right]$$

(13)

which shows that $\delta V_1$ doesn’t comply with the unitary criterion. Although this prescription breaks the unitarity of the bare CKM matrix, it only brings little effect to actual calculations. Since the breaking effect comes from the terms proportional to $m_{quark}^6/M_W^6$, the deviation of $(V + \delta V_1)$ from unitary matrix is very small. The only important
terms in CKM counterterm that contribute to the non-unitarity are the series of $m_{\text{quark}}^2/M_W^2$, which approximates to 2. Calculating to fifth order of $m_{\text{quark}}^2/M_W^2$, we find the largest deviation of $(\delta V^\dagger V + V^\dagger \delta V)$ from 0 is proportional to $\alpha |V_{3\ell}| m_b^2/M_W^2$, which approximates to $10^{-7}$. Comparing with the present measurement precision of the CKM matrix elements this deviation can be neglected. So we can use this CKM renormalization prescription in actual calculations.

Of course we need a CKM renormalization prescription with satisfying the three criterions in the academic point. For this purpose we firstly use our prescription to construct CKM counterterms, then shift them to satisfy the unitary criterion. This prescription should be carried out order by order. Now we need to introduce a set of denotations: δVn, the shifted deltaVn which satisfies the unitary criterion. Here we emphasize that deltaVn is calculated with using δVn−1, · · ·, δV1 as the lower loop level CKM counterterms. Using these denotations the unitary criterion Eq. (2) is

\[
\begin{align*}
\delta \bar{V}_1 V^\dagger + V \delta \bar{V}_1^\dagger &= 0, \\
\delta \bar{V}_2 V^\dagger + V \delta \bar{V}_2^\dagger &= -\delta \bar{V}_1 V_{\text{DG}}^\dagger, \\
\delta \bar{V}_3 V^\dagger + V \delta \bar{V}_3^\dagger &= -\delta \bar{V}_1 V_{\text{DG}}^\dagger - \delta \bar{V}_2 V_{\text{DG}}^\dagger, \\
&\cdots \\
\delta \bar{V}_n V^\dagger + V \delta \bar{V}_n^\dagger &= -\delta \bar{V}_1 \delta \bar{V}_{n-1} - \delta \bar{V}_2 \delta \bar{V}_{n-2} \cdots - \delta \bar{V}_{n-2} \delta \bar{V}_2 - \delta \bar{V}_{n-1} V^\dagger,
\end{align*}
\]

(14)

Using the prescription of Ref. [12] the one loop level CKM counterterm up to the academic standard has been obtained, as shown in Eq. (6). When dealing with the higher loop level case, we should identify the CKM counterterms at different loop levels as different variables when solving Eqs. (14) for they are different from each other very much. We introduce a symbol $B_n$ to denote

\[
B_0 = 0, \\
B_n = \sum_{i=1}^{n-1} -\delta \bar{V}_i V_{\text{DG}}^\dagger.
\]

(15)

Obviously $B_n$ satisfies

\[
B_n = B_n^\dagger
\]

(16)

Assuming that we have obtained the counterterms $\delta \bar{V}_1, \delta \bar{V}_2, \cdots, \delta \bar{V}_{n-1}$ and $\delta \bar{V}_n$, the n-loop level shifted counterterm $\delta \bar{V}_n$ can be obtained by this way

\[
\delta \bar{V}_n = \frac{1}{2}(\delta \bar{V}_n - V \delta \bar{V}_n^\dagger V + B_n V)
\]

(17)

Inserting Eq. (17) and Eq. (15) into Eqs. (14), we find, using induction, the CKM counterterm $\delta \bar{V}_1 + \delta \bar{V}_2 + \cdots + \delta \bar{V}_n$ satisfies the unitary criterion to n-loop level.

The next aiming is to test if the shifted CKM counterterm satisfies the first and second criterions. For this purpose we only need to prove the divergent and gauge-dependent part of $\delta \bar{V}_1 + \delta \bar{V}_2 + \cdots + \delta \bar{V}_n$ equal to the same one of $\delta \bar{V}_1 + \delta \bar{V}_2 + \cdots + \delta \bar{V}_n$, since the latter contains the right divergent and gauge dependent terms. Based on the renormalizability of SM, we predict that since $\delta \bar{V}_1 + \delta \bar{V}_2 + \cdots + \delta \bar{V}_n$ makes the physical amplitude convergent the divergent part of $\delta \bar{V}_1 + \delta \bar{V}_2 + \cdots + \delta \bar{V}_n$ must satisfy the unitary criterion, otherwise $\delta \bar{V}_1 + \delta \bar{V}_2 + \cdots + \delta \bar{V}_n$ must be changed according to the unitarity of bare CKM matrix and reduce the prediction of the physical process involving quark mixing divergent. The same conclusion holds true for the gauge dependent part of $\delta \bar{V}_1 + \delta \bar{V}_2 + \cdots + \delta \bar{V}_n$, if there are gauge dependent terms in it. That’s to say, at n-loop level

\[
\delta \bar{V}_n^{DG} V^\dagger + V \delta \bar{V}_n^{DG} = B_n^{DG}
\]

(18)

the superscript "DG" denotes the divergent or gauge dependent part of the quantity. Using this relationship and Eq. (17), we obtain

\[
(\delta \bar{V}_n^{DG} - \delta \bar{V}_n^{DG}) V^\dagger = \frac{1}{2}(B_n^{DG} - \delta \bar{V}_n^{DG} V^\dagger - V \delta \bar{V}_n^{DG} V^\dagger) = 0.
\]

(19)

This identity manifests that $\delta \bar{V}_n^{DG} = \delta \bar{V}_n^{DG}$, i.e. $\delta \bar{V}_n$ contains the same divergent and gauge dependent terms as $\delta \bar{V}_n$. By induction we can prove $\delta \bar{V}_1 + \delta \bar{V}_2 + \cdots + \delta \bar{V}_n$ contains the same divergent and gauge dependent terms as $\delta \bar{V}_1 + \delta \bar{V}_2 + \cdots + \delta \bar{V}_n$. Thus our CKM counterterm complies with the first and second criterions.

Now we have obtained the eligible CKM counterterm $\delta \bar{V}_1 + \delta \bar{V}_2 + \delta \bar{V}_3 \cdots$ which complies with the three criterions. We guess this CKM renormalization prescription doesn’t break the present symmetries of SM, e.g. Ward-Takahashi identity, since it only changes the value of CKM matrix from $V^0$ to $V + \delta V$. 

4
\{\delta V_1, \delta V_2, \ldots, \delta V_n, \ldots\} constructs a series. Here we list the results of \(\delta V_1, \delta V_2, \delta V_3\) and \(\delta V_4\)

\[
\delta V_1 = \frac{1}{2} (\delta V_1 - V \delta V_1^T V),
\]

\[
\delta V_2 = \frac{1}{2} (\delta V_2 - V \delta V_2^T V) + \frac{1}{8} (\delta V_1 V^T \delta V_1 + V \delta V_1 V \delta V_1^T V - V \delta V_1 \delta V_1 - \delta V_1 \delta V_1^T V),
\]

\[
\delta V_3 = \frac{3}{4} (\delta V_3 - V \delta V_3^T V) + \frac{3}{8} (\delta V_1 V^T \delta V_1 + \delta V_2 V^T \delta V_1 + V \delta V_2 V \delta V_2^T V + V \delta V_2 \delta V_2^T V - V \delta V_2 \delta V_1 V \delta V_2^T V + V \delta V_1 \delta V_2^T V + V \delta V_2 \delta V_1^T V - \delta V_1 \delta V_1^T V),
\]

\[
\delta V_4 = \frac{5}{4} (\delta V_4 - V \delta V_4^T V) + \frac{5}{8} (\delta V_1 V^T \delta V_1 + \delta V_2 V^T \delta V_2 + V \delta V_1 V \delta V_1^T V + V \delta V_1 V \delta V_2 + V \delta V_1 \delta V_2 V + V \delta V_1 \delta V_2^T V + V \delta V_1 \delta V_1^T V + \delta V_2 \delta V_2 V + V \delta V_2 \delta V_1 V - \delta V_1 \delta V_2 V),
\]

\[
\delta V_5 = \frac{7}{4} (\delta V_5 - V \delta V_5^T V) + \frac{7}{8} (\delta V_1 V^T \delta V_1 + \delta V_2 V^T \delta V_2 + V \delta V_1 V \delta V_1^T V + V \delta V_1 V \delta V_2 + V \delta V_1 \delta V_2 V + V \delta V_1 \delta V_2^T V + V \delta V_1 \delta V_1^T V + \delta V_2 \delta V_2 V + V \delta V_2 \delta V_1 V - \delta V_1 \delta V_2 V) + \delta V_5 \delta V_5^T V.
\]

We have known the CKM counterterm should be gauge independent at one loop level \([17]\). Is it also true at higher loop level? Using Nielsen identities \([25]\) it has been proven that any physical parameter’s counterterm, especially the effect of the choice of \(\delta V_1\) can affect the gauge dependence of Eq. (3) can affect the gauge dependence. Hence, we cannot guarantee the choice of \(\delta V_1\) is right. In order to check this problem we express the amplitude of \(W \rightarrow u_i d_j\) as

\[
T(V^0) = T(V + \delta V) = T(V) + T'(V) \delta V + \frac{1}{2} T''(V) (\delta V)^2 + \cdots
\]

the superscript ”\(r\)” denotes the partial derivative with respect to CKM matrix of the quantity. To two loop level, the above equation is

\[
T_2(V^0) = T_2(V) + T'_2(V) \delta V_1 + \delta V_2 A_L
\]

\(T_2(V)\) denotes the 2-loop amplitudes of \(W \rightarrow u_i d_j\) without including CKM counterterms. In order to identify the effect of the choice of \(\delta V_1\) on the gauge dependence of \(\delta V_2\) we need to calculate \(T'_2(V)\) analytically. From Eq. (3), since \(F_R\) and \(G_L\) are gauge independent and don’t contain CKM matrix element, only the terms in the first bracket of Eq. (3) can affect the gauge dependence of \(\delta V_2\). Based on the fact that the terms in the first bracket of Eq. (3) is gauge independent \([17]\), we have

\[
T'_1(\delta V_1) |_\xi = -\frac{\delta V_1}{\delta V_1^T V} \left[ \sum_{k \neq i} \delta Z_{k i}^\ell V_{k j} + \sum_{k \neq i} V_{k i} \delta Z_{k j}^\ell \right] + \frac{1}{2} \sum_{k \neq i} \left( \frac{1}{2} \frac{\delta Z_{k i}^{\ell \ell}}{\delta V_{k j}} + \frac{1}{2} \frac{\delta Z_{k j}^{\ell \ell}}{\delta V_{i k}} \right) \delta V_{k j} + \frac{1}{2} \left( \delta Z_{k i}^{\ell \ell} \delta V_{k j} + \sum_{k \neq j} \delta V_{k i} \delta Z_{k j}^{\ell \ell} \delta V_{k j} \right)
\]

the factor \(A_L\) and the subscript ”1” of \(\delta V_1\) have been omitted, and the subscript ”\(\xi\)” denotes the gauge dependent part of the quantity. Now we still use the Taylor’s series \(m^2_{\text{quark}}/M_W^2\) to expand this result. To one order of \(m^2_{\text{quark}}/M_W^2\) we have

\[
T^{(1)}_1(\delta V) |_\xi = \frac{\alpha \ln \xi_W}{8 \pi M_W^2} \sum_{k, l} [m^2_{d, k} V_{ik} (V_{ik} \delta V_{ij} + \delta V_{ik}^* V_{ij}) + (\delta V_{ik} V_{ik} + V_{ik} \delta V_{ik}^*) V_{ij} m^2_{u, k}].
\]

\(T^{(1)}_1\) denotes the 1-order result of \(T_1\), and \(\xi_W\) is the W boson gauge parameter. It can be seen that if \(\delta V_1\) satisfies the unitary criterion Eq. (2) at one-loop level, the gauge dependent part of \(T^{(1)}_1(\delta V) V_{ij}\) will be equal to zero. This conclusion holds true to 5-order of \(m^2_{\text{quark}}/M_W^2\), as shown below
The 1-loop, 2-loop, 3-loop, 4-loop and n-loop level renormalization prescription preserves the basic structure of SM, i.e. the unitarity of the bare CKM matrix. On the other hand, the process involving quark mixing convergent and gauge independent. The 1-loop, 2-loop, 3-loop, 4-loop and n-loop level for one loop level. In this paper, we have generalized the prescription of Ref. [21,12] to make it suitable for any loop level. In addition, we find that how to choose the one loop level CKM counterterm doesn’t affect the gauge dependence of the two loop level CKM counterterm only if the former preserve the unitarity of the bare CKM matrix at one loop level. This may be a signal that CKM counterterms are gauge parameter independent.

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