The Line Elements in the Hubble Expansion

Moshe Carmeli

Department of Physics, Ben Gurion University, Beer Sheva 84105, Israel
(Email: carmelim.bgumail.bgu.ac.il)

In this lecture I present the line elements that express the Hubble expansion. The coordinates here are the spatial coordinates $x$, $y$, $z$, and the velocity coordinate $v$, which are actually what astronomers use in their measurements. Two such line elements are presented: the first is the empty space (no matter exists), and the second with matter filling up the Universe. These line elements are the comparable to the standard ones, the Minkowski and the FRW line elements in ordinary general relativity theory.

1 Introduction

In the standard cosmological theory one uses the Einstein concepts of space and time as were originally introduced for the special theory of relativity and later in the general relativity theory. According to this approach all physical quantities are described in terms of the continuum spatial coordinates and the time. Using the general relativity theory a great progress has been made in understanding the evolution of the Universe.

Accordingly in the standard cosmological model one has the Friedmann-Robertson-Walker (FRW) line element

$$ds^2 = dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\},$$

where $k$ is a constant, $k < 0$, open Universe, $k > 0$, closed Universe, $k = 0$, flat Universe. Here $a(t)$ is a scale function, and by the Einstein field equations, it satisfies

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} G \rho - \frac{k}{a^2},$$

$$\ddot{a} = -\frac{4\pi}{3} G (\rho + 3p) a,$$

$$\frac{d}{dt} (a^3 \rho) = -\rho \frac{d}{dt} (a^3),$$
where $\rho$ is the mass density, $p$ is the pressure, $G$ is Newton’s constant, and the overdot denotes derivative with respect to $t$. Notice that the above equations are not independent.

It is well known that both Einstein’s theories are based on the fact that light propagates at a constant velocity. However, the Universe also expands at a constant rate when gravity is negligible, but this fact is not taken into account in Einstein’s theories. Moreover, cosmologists usually measure spatial distances and redshifts of faraway galaxies as expressed by the Hubble expansion. In recent years this fact was undertaken to develop new theories in terms of distances and velocities (redshift). While in Einstein’s special relativity the propagation of light plays the major role, in the new theory it is the expansion of the Universe that takes that role. It is the concept of cosmic time that becomes crucial in these recent theories. In the standard theory the cosmic time is considered to be absolute. Thus we talk about the Big Bang time with respect to us here on Earth as an absolute quantity. Consider, for example, another galaxy that has, let us say, a relative cosmic time with respect to us of 1 billion year. Now one may ask what will be the Big Bang time with respect to this galaxy. Will it be the Big Bang time with respect to us minus 1 billion year? A person who lives in that galaxy will look at our galaxy and say that ours is far away from him by also 1 billion year. Will that mean, with respect to him, our galaxy is closer to the Big Bang time by 1 billion year? Or will we seem to him to be farther by 1 billion year? All this leads to the conclusion that there is no absolute cosmic time. Rather, it is a relative concept.

Based on this assumption, and using the Hubble expansion $R = \tau v$, where $R$ is the distance from us to a galaxy, $v$ is the receding velocity of the galaxy and $\tau$ is the Hubble time (a universal constant equal to 12.486 Gyr - it is the standard Hubble time in the limit of zero distance and zero gravity - and thus it is a constant in this epoch of time, see Section 8). The Hubble expansion can now be written as

$$\tau^2 v^2 - (x^2 + y^2 + z^2) = 0. \quad (5)$$

($\tau v$ can alternatively be written in terms of the redshift. For relativistic velocities the relationship between the velocity and the redshift parameter is given by $z = [(1 + v/c)/(1 - v/c)]^{1/2} - 1$, $v/c = z(z + 2)/(z^2 + 2z + 2)$.) Accordingly we have a line element

$$ds^2 = \tau^2 dv^2 - (dx^2 + dy^2 + dz^2). \quad (6)$$

It is equal to zero for the Hubble expansion, but is otherwise not vanishing. It is similar to the Minkowskian metric

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2), \quad (7)$$

which vanishes for light propagation, but is otherwise different from zero for particles of finite mass.
If we now assume that the laws of physics are valid at all cosmic times and \( \tau \) is a universal constant which has the same value at all cosmic times, then we can develop a new special relativity just like Einstein’s original special relativity theory: the validity of the laws of physics at all cosmic times replaces the special relativistic assumption in Einstein’s theory of in all inertial coordinate systems, whereas the constancy of \( \tau \) at all cosmic times replaces the constancy of the speed of light in all inertial systems in ordinary special relativity.

In this way one also obtains a cosmological transformation that relates distances and velocities (redshifts) at different cosmic times, just as in ordinary special relativity we have the Lorentz transformation that relates spatial coordinates and time at different velocities. We now will have

\[
x' = \frac{x - tv}{\sqrt{1 - t^2/\tau^2}} \quad v' = \frac{v - xt/\tau^2}{\sqrt{1 - t^2/\tau^2}},
\]

for the case \( y' = y, z' = z \). In the above transformation \( t \) is the relative cosmic time and is measured backward (\( t = 0 \) now and \( t = \tau \) at the Big Bang). As seen from the above transformation, \( \tau \) is the maximum cosmic time that is allowed in nature and as such can be considered as the age of the Universe.

For example, if we denote the temperature at a cosmic time \( t \) by \( T \), and the temperature at the present time by \( T_0 (=2.73\text{K}) \), we then have \( T = T_0/\sqrt{1 - t^2/\tau^2} \). For instance, at \( t/\tau = 1/2 \) we get \( T = 2 \times 2.73/\sqrt{3} = 3.15\text{K} \). (This result does not take into account gravity that needs a correction by a factor of 13, and thus the temperature at \( t = \tau/2 \) is 40.95K.)

Since we always use forward rather than backward times, we write the transformation (8) in terms of such a time \( t' = \tau - t \). The resulting transformation will have the form

\[
x' = \frac{x - (\tau - t)v}{\sqrt{(t/\tau)(2 - t/\tau)}}, \quad v' = \frac{v - x(\tau - t)/\tau^2}{\sqrt{(t/\tau)(2 - t/\tau)}},
\]

where primes have been dropped for brevity, \( 0 \leq t \leq \tau, t = 0 \) at the Big Bang, \( t = \tau \), now.

The above introduction gives a brief review of a new special relativity (cosmological special relativity, for more details see [1]). Obviously the Universe is filled up with gravity and therefore one has to go to a Riemannian space with the Einstein gravitation field equations in terms of space and redshift (velocity). This is done in the next section.

# 2 Extension to curved space: cosmological general relativity

The theory presented here, cosmological general relativity, uses a Riemannian four-dimensional presentation of gravitation in which the coordinates are those of
Moshe Carmeli

Hubble, i.e. distances and velocity rather than the traditional space and time. We solve the field equations and show that there are three possibilities for the Universe to expand. The theory describes the Universe as having a three-phase evolution with a decelerating expansion, followed by a constant and an accelerating expansion, and it predicts that the Universe is now in the latter phase. It is shown, assuming \( \Omega_m = 0.245 \), that the time at which the Universe goes over from a decelerating to an accelerating expansion, i.e., the constant-expansion phase, occurs at 8.5 Gyr ago. Also, at that time the cosmic radiation temperature was 146K. Recent observations of distant supernovae imply, in defiance of expectations, that the Universe’s growth is accelerating, contrary to what has always been assumed, that the expansion is slowing down due to gravity. Our theory confirms these recent experimental results by showing that the Universe now is definitely in a stage of accelerating expansion. The theory predicts also that now there is a positive pressure, \( p = 0.034 g/cm^2 \), in the Universe. Although the theory has no cosmological constant, we extract from it its equivalence and show that \( \Lambda = 1.934 \times 10^{-35} s^{-2} \). This value of \( \Lambda \) is in excellent agreement with the measurements obtained by the High-Z Supernova Team and the Supernova Cosmology Project. It is also shown that the three-dimensional space of the Universe is Euclidean, as the Boomerang experiment shows. Comparison with general relativity theory is finally made and it is pointed out that the classical experiments as well as the gravitational radiation prediction follow from the present theory, too.

3 Cosmology in spacevelocity

In the framework of cosmological general relativity (CGR) gravitation is described by a curved four-dimensional Riemannian spacevelocity [2, 3, 4, 5]. CGR incorporates the Hubble constant \( \tau \) at the outset. The Hubble law is assumed in CGR as a fundamental law. CGR, in essence, extends Hubble’s law so as to incorporate gravitation in it; it is actually a distribution theory that relates distances and velocities between galaxies. The theory involves only measured quantities and it takes a picture of the Universe as it is at any moment. The following is a brief review of CGR as was originally given by the author in 1996 in Ref. 2.

The foundations of any gravitational theory are based on the principle of equivalence and the principle of general covariance [6]. These two principles lead immediately to the realization that gravitation should be described by a four-dimensional curved spacetime, in our theory spacevelocity, and that the field equations and the equations of motion should be written in a generally covariant form. Hence these principles were adopted in CGR also. Use is made in a four-dimensional Riemannian manifold with a metric \( g_{\mu \nu} \) and a line element \( ds^2 = g_{\mu \nu} dx^\mu dx^\nu \). The difference from Einstein’s general relativity is that our coordinates are: \( x^0 \) is a velocitylike coordinate (rather than a timelike coordinate), thus \( x^0 = \tau v \) where \( \tau \) is the Hubble time in the
zero-gravity limit and \( v \) the velocity. The coordinate \( x^0 = \tau v \) is the comparable to \( x^0 = ct \) where \( c \) is the speed of light and \( t \) is the time in ordinary general relativity. The other three coordinates \( x^k, k = 1, 2, 3 \), are spacelike, just as in general relativity theory.

An immediate consequence of the above choice of coordinates is that the null condition \( ds = 0 \) describes the expansion of the Universe in the curved spacetimevelocity (generalized Hubble’s law with gravitation) as compared to the propagation of light in the curved spacetime in general relativity. This means one solves the field equations (to be given in the sequel) for the metric tensor, then from the null condition \( ds = 0 \) one obtains immediately the dependence of the relative distances between the galaxies on their relative velocities.

As usual in gravitational theories, one equates geometry to physics. The first is expressed by means of a combination of the Ricci tensor and the Ricci scalar, and follows to be naturally either the Ricci trace-free tensor or the Einstein tensor. The Ricci trace-free tensor does not fit gravitation in general, and the Einstein tensor is a natural candidate. The physical part is expressed by the energy-momentum tensor which now has a different physical meaning from that in Einstein’s theory. More important, the coupling constant that relates geometry to physics is now also different.

Accordingly the field equations are

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu},
\]

exactly as in Einstein’s theory, with \( \kappa \) given by \( \kappa = 8\pi k/\tau^4 \), (in general relativity it is given by \( 8\pi G/c^4 \)), where \( k \) is given by \( k = G\tau^2/c^2 \), with \( G \) being Newton’s gravitational constant, and \( \tau \) the Hubble constant time. When the equations of motion will be written in terms of velocity instead of time, the constant \( k \) will replace \( G \). Using the above equations one then has \( \kappa = 8\pi G/c^2\tau^2 \).

The energy-momentum tensor \( T^{\mu\nu} \) is constructed, along the lines of general relativity theory, with the speed of light being replaced by the Hubble constant time. If \( \rho \) is the average mass density of the Universe, then it will be assumed that \( T^{\mu\nu} = \rho u^\mu u^\nu \), where \( u^\mu = dx^\mu/ds \) is the four-velocity. In general relativity theory one takes \( T^0_0 = \rho \). In Newtonian gravity one has the Poisson equation \( \nabla^2 \phi = 4\pi G\rho \).

At points where \( \rho = 0 \) one solves the vacuum Einstein field equations in general relativity and the Laplace equation \( \nabla^2 \phi = 0 \) in Newtonian gravity. In both theories a null (zero) solution is allowed as a trivial case. In cosmology, however, there exists no situation at which \( \rho \) can be zero because the Universe is filled with matter. In order to be able to have zero on the right-hand side of (7) one takes \( T^0_0 \) not as equal to \( \rho \), but to \( \rho_{\text{eff}} = \rho - \rho_c \), where \( \rho_c \) is the critical mass density, a constant in CGR given by \( \rho_c = 3/(8\pi G\tau^2) \), whose value is \( \rho_c \approx 10^{-29}g/cm^3 \), a few hydrogen atoms per cubic meter. Accordingly one takes \( T^{\mu\nu} = \rho_{\text{eff}} u^\mu u^\nu \); \( \rho_{\text{eff}} = \rho - \rho_c \) for the
energy-momentum tensor.

In the next sections we apply CGR to obtain the accelerating expanding Universe and related subjects.

4 Gravitational field equations

In the four-dimensional spacevelocity the spherically symmetric metric is given by

\[ ds^2 = \tau^2 dv^2 - e^{\mu} dv^2 - R^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

(11)

where \( \mu \) and \( R \) are functions of \( v \) and \( r \) alone, and comoving coordinates \( x^\mu = (x^0, x^1, x^2, x^3) = (\tau v, r, \theta, \phi) \) have been used. With the above choice of coordinates, the zero-component of the geodesic equation becomes an identity, and since \( r, \theta \) and \( \phi \) are constants along the geodesics, one has \( dx^0 = ds \) and therefore \( u^\alpha = u_\alpha = (1, 0, 0, 0) \). The metric (11) shows that the area of the sphere \( r = \text{constant} \) is given by \( 4\pi R^2 \) and that \( R \) should satisfy \( R' = \partial R/\partial r > 0 \). The possibility that \( R' = 0 \) at a point \( r_0 \) is excluded since it would allow the lines \( r = \text{constants} \) at the neighboring points \( r_0 \) and \( r_0 + dr \) to coincide at \( r_0 \), thus creating a caustic surface at which the comoving coordinates break down.

As has been shown in the previous sections the Universe expands by the null condition \( ds = 0 \), and if the expansion is spherically symmetric one has \( d\theta = d\phi = 0 \). The metric (11) then yields \( \tau^2 dv^2 - e^{\mu} dr^2 = 0 \), thus

\[ \frac{dr}{dv} = \tau e^{-\mu/2}. \]

(12)

This is the differential equation that determines the Universe expansion. In the following we solve the gravitational field equations in order to find out the function \( \mu (r, v) \).

The gravitational field equations (10), written in the form

\[ R_{\mu\nu} = \kappa (T_{\mu\nu} - g_{\mu\nu} T/2), \]

(13)

where

\[ T_{\mu\nu} = \rho_{\text{eff}} u_\mu u_\nu + p (u_\mu u_\nu - g_{\mu\nu}), \]

(14)

with \( \rho_{\text{eff}} = \rho - \rho_c \) and \( T = T_{\mu\nu} g^{\mu\nu}, \) are now solved. One finds that the only nonvanishing components of \( T_{\mu\nu} \) are \( T_{00} = \tau^2 \rho_{\text{eff}}, T_{11} = c^{-1} \tau p e^\mu, T_{22} = c^{-1} \tau p R^2 \)

and \( T_{33} = c^{-1} \tau p R^2 \sin^2 \theta, \) and that \( T = \tau^2 \rho_{\text{eff}} - 3c^{-1} \tau p \).

The field equations obtained are given by

\[ -\ddot{\mu} - \frac{4}{R} \dot{R} - \frac{1}{2} \mu^2 = \kappa (\tau^2 \rho_{\text{eff}} + 3c^{-1} \tau p), \]

(15)
\[ 2\ddot{R} - R'\dot{\mu} = 0, \]  
\[ \ddot{\mu} + \frac{1}{2} \dot{\mu}^2 + \frac{2}{R} \dot{R} \dot{\mu} + e^{-\mu} \left( \frac{2}{R} R' \mu' - \frac{4}{R} R'' \right) = \kappa \left( \tau^2 \rho_{eff} - c^{-1} \tau p \right), \]  
\[ \frac{2}{R} \dddot{R} + 2 \left( \frac{\dot{R}}{R} \right)^2 + \frac{1}{R} \dddot{R} \dot{\mu} + \frac{2}{R^2} e^{-\mu} \left[ \frac{1}{R} R' \mu' - 2 \left( \frac{R'}{R} \right)^2 - \frac{2}{R} R'' \right] = \kappa \left( \tau^2 \rho_{eff} - c^{-1} \tau p \right). \]  

Combinations of Eqs. (15)–(18) then give three independent field equations:

\[ e^\mu \left( 2RR + R'^2 + 1 \right) - R'^2 = -\kappa c^{-1} e^\mu R^2 p, \]  
\[ 2\ddot{R} - R'\dot{\mu} = 0, \]  
\[ e^{-\mu} \left[ \frac{1}{R} R' \mu' - \left( \frac{R'}{R} \right)^2 - \frac{2}{R} R'' \right] + \frac{1}{R} \dddot{R} \dot{\mu} + \left( \frac{\dot{R}}{R} \right)^2 + \frac{1}{R^2} = \kappa \tau^2 \rho_{eff}, \]

other equations being trivial combinations of (19)–(21).

5 Solution of the field equations

The solution of (20) satisfying the condition \( R' > 0 \) is given by

\[ e^\mu = R'^2 / (1 + f(r)), \]

where \( f(r) \) is an arbitrary function of the coordinate \( r \) and satisfies the condition \( f(r) + 1 > 0 \). Substituting (22) in the other two field equations (19) and (21) then gives

\[ 2RR + R'^2 - f = -\kappa c^{-1} \tau R^2 p, \]  
\[ \frac{1}{RR'} \left( 2\ddot{R}R' - f' \right) + \frac{1}{R^2} \left( \dot{R}^2 - f \right) = \kappa \tau^2 \rho_{eff}, \]

respectively.

The simplest solution of the above two equations, which satisfies the condition \( R' = 1 > 0 \), is given by \( R = r \). Using this in Eqs. (23) and (24) gives \( f(r) = \kappa c^{-1} \tau pr^2 \), and \( f + f/r = -\kappa \tau^2 \rho_{eff} r \), respectively. Using the values of \( \kappa = 8\pi G/c^2 \tau^2 \) and \( \rho_c = 3/8\pi G \tau^2 \), we obtain

\[ f(r) = (1 - \Omega_m) r^2 / (c^2 \tau^2), \]

where \( \Omega_m = \rho/\rho_c \). We also obtain

\[ p = \frac{1 - \Omega_m}{\kappa \tau^3} = \frac{c}{8\pi G} = 4.544 (1 - \Omega_m) \times 10^{-2} g/cm^2, \]
\[ e^{-\mu} = 1 + f(r) = 1 + \tau e^{-1} \kappa p r^2 = 1 + (1 - \Omega_m) r^2 / (c^2 \tau^2). \]  

(27)

Accordingly, the line element of the Universe is given by

\[ ds^2 = \tau^2 dv^2 - \frac{dr^2}{1 + (1 - \Omega) r^2 / (c^2 \tau^2)} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]  

or,

\[ ds^2 = \tau^2 dv^2 - \frac{dr^2}{1 + (\kappa \tau / c)p r^2} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  

(28)

(29)

This line element is the comparable to the FRW line element in the standard theory.

It will be recalled that the Universe expansion is determined by Eq. (12),

\[ \frac{dr}{dv} = \tau e^{-\mu / 2}. \]

The only thing that is left to be determined is the signs of \((1 - \Omega_m)\) or the pressure \(p\). Thus we have

\[ \frac{dr}{dv} = \tau \sqrt{1 + \kappa e^{-1} pr^2} = \tau \sqrt{1 + \frac{1 - \Omega_m}{c^2 \tau^2}}. \]  

(30)

For simplicity we confine ourselves to the linear approximation, thus Eq. (30) yields

\[ \frac{dr}{dv} = \tau \left( 1 + \frac{\kappa}{2} e^{-1} pr^2 \right) = \tau \left[ 1 + \frac{1 - \Omega_m}{2c^2 \tau^2} r^2 \right]. \]  

(31)

6 Physical meaning

To see the physical meaning of these solutions, one does not need the exact solutions. Rather, it is enough to write down the solutions in the lowest approximation in \(\tau^{-1}\). One obtains, by differentiating Eq. (31) with respect to \(v\), for \(\Omega_m > 1\),

\[ \frac{d^2 r}{dv^2} = -kr, \quad k = \frac{(\Omega_m - 1)}{2c^2}, \]  

(32)

the solution of which is

\[ r(v) = A \sin \frac{v}{c} + B \cos \frac{v}{c}, \]  

(33)

where \(\alpha^2 = (\Omega_m - 1)/2\) and \(A\) and \(B\) are constants. The latter can be determined by the initial condition \(r(0) = 0 = B\) and \(dr(0)/dv = \tau = A\alpha / c\), thus

\[ r(v) = \frac{c\tau}{\alpha} \sin \frac{v}{c}. \]  

(34)

This is obviously a closed Universe, and presents a decelerating expansion.

For \(\Omega_m < 1\) we have

\[ \frac{d^2 r}{dv^2} = (1 - \Omega_m) r / (2c^2), \]  

(35)
whose solution, using the same initial conditions, is
\[ r(v) = \frac{ct}{\beta \sinh \beta v/c}, \] (36)
where \( \beta^2 = (1 - \Omega_m)/2 \). This is now an open accelerating Universe.

For \( \Omega_m = 1 \) we have, of course, \( r = \tau v \).

7 The accelerating universe

We finally determine which of the three cases of expansion is the one at present epoch of time. To this end we have to write the solutions (34) and (36) in ordinary Hubble’s law form \( v = H_0 r \). Expanding Eqs. (34) and (36) into power series in \( v/c \) and keeping terms up to the second order, we obtain
\[ r = \tau v \left(1 - \alpha^2 v^2/6c^2\right) = \tau v \left(1 + \beta^2 v^2/6c^2\right), \] (37)
for \( \Omega_m > 1 \) and \( \Omega_m < 1 \). Using now the expressions for \( \alpha \) and \( \beta \), then Eqs. (37) reduce to the single equation
\[ r = \tau v \left[1 + (1 - \Omega_m) v^2/6c^2\right]. \] (38)

Inverting now this equation by writing it as \( v = H_0 r \), we obtain in the lowest approximation
\[ H_0 = h \left[1 - (1 - \Omega_m) v^2/6c^2\right], \] (39)
where \( h = \tau^{-1} \). To the same approximation one also obtains
\[ H_0 = h \left[1 - (1 - \Omega_m) z^2/6\right] = h \left[1 - (1 - \Omega_m) r^2/6c^2\tau^2\right], \] (40)
where \( z \) is the redshift parameter. As is seen, and it is confirmed by experiments, \( H_0 \) depends on the distance it is being measured; it has physical meaning only at the zero-distance limit, namely when measured locally, in which case it becomes \( h = 1/\tau \).

It is well known that the measured value of \( H_0 \) depends on the “short” and “long” distance scales [8]. The farther the distance \( H_0 \) is being measured, the lower the value for \( H_0 \) is obtained. By Eq. (40) this is possible only when \( \Omega_m < 1 \), namely when the Universe is accelerating. By Eq. (26) we also find that the pressure is positive.

The possibility that the Universe expansion is accelerating was first predicted using CGR by the author in 1996 [2] before the supernovae experiments results became known.

It will be noted that the constant expansion is just a transition stage between the decelerating and the accelerating expansions as the Universe evolves toward its present situation.
Table 1: The Cosmic Times with respect to the Big Bang, the Cosmic Temperature and the Cosmic Pressure for each of the Curves in Fig. 1.

<table>
<thead>
<tr>
<th>Curve No*</th>
<th>$\Omega_m$</th>
<th>Time in Units of $\tau$</th>
<th>Temperature</th>
<th>Pressure</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>$3.1 \times 10^{-6}$</td>
<td>$3.87 \times 10^{-5}$</td>
<td>1096</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td>$9.8 \times 10^{-5}$</td>
<td>$1.22 \times 10^{-3}$</td>
<td>195.0</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>$3.0 \times 10^{-4}$</td>
<td>$3.75 \times 10^{-3}$</td>
<td>111.5</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>$1.2 \times 10^{-3}$</td>
<td>$1.50 \times 10^{-2}$</td>
<td>58.20</td>
</tr>
<tr>
<td>5</td>
<td>1.5</td>
<td>$1.3 \times 10^{-2}$</td>
<td>$1.62 \times 10^{-1}$</td>
<td>16.43</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>$3.0 \times 10^{-2}$</td>
<td>$3.75 \times 10^{-1}$</td>
<td>11.15</td>
</tr>
<tr>
<td>7</td>
<td>0.5</td>
<td>$1.3 \times 10^{-1}$</td>
<td>$1.62 \times 10^{-1}$</td>
<td>5.538</td>
</tr>
<tr>
<td>8</td>
<td>0.245</td>
<td>1.0</td>
<td>12.50</td>
<td>2.730</td>
</tr>
</tbody>
</table>

Figure 1 describes the Hubble diagram of the above solutions for the three types of expansion for values of $\Omega_m$ from 100 to 0.245. The figure describes the three-phase evolution of the Universe. Curves (1)-(5) represent the stages of **decelerating expansion** according to Eq. (34). As the density of matter $\rho$ decreases, the Universe goes over from the lower curves to the upper ones, but it does not have enough time to close up to a big crunch. The Universe subsequently goes over to curve (6) with $\Omega_m = 1$, at which time it has a constant expansion for a fraction of a second. This then followed by going to the upper curves (7) and (8) with $\Omega_m < 1$, where the Universe expands with **acceleration** according to Eq. (36). Curve no. 8 fits the present situation of the Universe. For curves (1)-(4) in the diagram we use the cutoff when the curves were at their maximum. In Table 1 we present the cosmic times with respect to the big bang, the cosmic radiation temperature and the pressure for each of the curves in Fig. 1.

Figure 2 shows the Hubble diagrams for the distance-redshift relationship predicted by the theory for the accelerating expanding Universe at the present time, and Figure 3 shows the experimental results.

Our estimate for $h$, based on published data, is $h \approx 80$ km/sec-Mpc. Assuming $\tau^{-1} \approx 80$ km/sec-Mpc, Eq. (40) then gives

$$H_0 = h \left[ 1 - 1.3 \times 10^{-4} \left( 1 - \Omega_m \right) r^2 \right], \quad (41)$$
where \( r \) is in Mpc. A computer best-fit can then fix both \( h \) and \( \Omega_m \).

To summarize, a theory of cosmology has been presented in which the dynamical variables are those of Hubble, i.e. distances and velocities. The theory describes the Universe as having a three-phase evolution with a decelerating expansion, followed by a constant and an accelerating expansion, and it predicts that the Universe is now in the latter phase. As the density of matter decreases, while the Universe is at the decelerating phase, it does not have enough time to close up to a big crunch. Rather, it goes to the constant-expansion phase, and then to the accelerating stage. As we have seen, the equation obtained for the Universe expansion, Eq. (36), is very simple.

### 8 Theory versus experiment

The Einstein gravitational field equations with the added cosmological term are [9]:

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu},
\]

where \( \Lambda \) is the cosmological constant, the value of which is supposed to be determined by the theory of quantum gravity which nobody knows much about it. In Eq. (42) \( R_{\mu\nu} \) and \( R \) are the Ricci tensor and scalar, respectively, \( \kappa = 8\pi G \), where \( G \) is Newton’s constant and the speed of light is taken as unity.

Recently the two groups (the Supernovae Cosmology Project and the High-Z Supernova Team) concluded that the expansion of the Universe is accelerating [10, 11, 12, 13, 14, 15, 16]. The two groups had discovered and measured moderately high redshift (0.3 < \( z \) < 0.9) supernovae, and found that they were fainter than what one would expect them to be if the cosmos expansion were slowing down or constant. Both teams obtained \( \Omega_m \approx 0.3, \Omega_\Lambda \approx 0.7 \), and ruled out the traditional \( (\Omega_m, \Omega_\Lambda) = (1, 0) \) Universe. Their value of the density parameter \( \Omega_\Lambda \) corresponds to a cosmological constant that is small but, nevertheless, nonzero and positive, \( \Lambda \approx 10^{-52} m^{-2} \approx 10^{-35} s^{-2} \).

In previous sections a four-dimensional cosmological theory (CGR) was presented. Although the theory has no cosmological constant, it predicts that the Universe accelerates and hence it has the equivalence of a positive cosmological constant in Einstein’s general relativity. In the framework of this theory (see Section 3) the zero-zero component of the field equations (10) is written as

\[
R^0_0 - \frac{1}{2}g_{00}R = \kappa \rho_{eff} = \kappa (\rho - \rho_c),
\]

where \( \rho_c = 3/(\kappa \tau^2) \) is the critical mass density and \( \tau \) is Hubble’s time in the zero-gravity limit.
Comparing Eq. (43) with the zero-zero component of Eq. (42), one obtains the expression for the cosmological constant of general relativity, \( \Lambda = \kappa \rho_c = 3/\tau^2 \).

To find out the numerical value of \( \tau \) we use the relationship between \( h = \tau^{-1} \) and \( H_0 \) given by Eq. (40) (CR denote values according to Cosmological Relativity):

\[
H_0 = h \left[ 1 - \left( 1 - \Omega^{CR}_m \right) z^2 / 6 \right],
\]

where \( z = v/c \) is the redshift and \( \Omega^{CR}_m = \rho_m / \rho_c \) with \( \rho_c = 3h^2 / 8\pi G \). (Notice that our \( \rho_c = 1.194 \times 10^{-20} \text{g/cm}^3 \) is different from the standard \( \rho_c \) defined with \( H_0 \).) The redshift parameter \( z \) determines the distance at which \( H_0 \) is measured. We choose \( z = 1 \) and take for \( \Omega^{CR}_m = 0.245 \), its value at the present time (see Table 1) (corresponds to 0.32 in the standard theory), Eq. (44) then gives \( H_0 = 0.874h \). At the value \( z = 1 \) the corresponding Hubble parameter \( H_0 \) according to the latest results from HST can be taken \([17]\) as \( H_0 = 70 \text{km/s-Mpc} \), thus \( h = (70/0.874) \text{km/s-Mpc} \), or \( h = 80.092 \text{km/s-Mpc} \), and \( \tau = 12.486 \text{Gyr} = 3.938 \times 10^{17} \text{s} \).

What is left is to find the value of \( \Omega^{CR}_\Lambda \). We have \( \Omega^{CR}_\Lambda = \rho^{ST}_c / \rho_c \), where \( \rho^{ST}_c = 3H_0^2 / 8\pi G \) and \( \rho_c = 3h^2 / 8\pi G \). Thus \( \Omega^{CR}_\Lambda = (H_0/h)^2 = 0.874^2 \), or \( \Omega^{CR}_\Lambda = 0.764 \). As is seen from the above equations one has

\[
\Omega_T = \Omega^{CR}_m + \Omega^{CR}_\Lambda = 0.245 + 0.764 = 1.009 \approx 1,
\]

which means the Universe is Euclidean.

As a final result we calculate the cosmological constant. One obtains

\[
\Lambda = 3/\tau^2 = 1.934 \times 10^{-35} \text{s}^{-2}.
\]

Our results confirm those of the supernovae experiments and indicate on the existence of the dark energy as has recently received confirmation from the Boomerang cosmic microwave background experiment \([18, 19]\), which showed that the Universe is Euclidean.

9 Remarks

In this paper the cosmological general relativity, a relativistic theory in spacevelocity, has been presented and applied to the problem of the expansion of the Universe. The theory, which predicts a positive pressure for the Universe now, describes the Universe as having a three-phase evolution: decelerating, constant and accelerating expansion, but it is now in the latter stage. Furthermore, the cosmological constant that was extracted from the theory agrees with the experimental result. Finally, it has also been shown that the three-dimensional spatial space of the Universe is Euclidean, again in agreement with observations.

Recently \([20, 21]\), more confirmation to the Universe accelerating expansion came from the most distant supernova, SN 1997ff, that was recorded by the Hubble Space
Telescope. As has been pointed out before, if we look back far enough, we should find a decelerating expansion (curves 1-5 in Figure 1). Beyond $z = 1$ one should see an earlier time when the mass density was dominant. The measurements obtained from SN 1997ff’s redshift and brightness provide a direct proof for the transition from past decelerating to present accelerating expansion (see Figure 4). The measurements also exclude the possibility that the acceleration of the Universe is not real but is due to other astrophysical effects such as dust.

Table 2 gives some of the cosmological parameters obtained here and in the standard theory.

<table>
<thead>
<tr>
<th></th>
<th>COSMOLOGICAL RELATIVITY</th>
<th>STANDARD THEORY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theory type</td>
<td>Spacevelocity</td>
<td>Spacetime</td>
</tr>
<tr>
<td>Expansion type</td>
<td>Tri-phase: decelerating, constant, accelerating</td>
<td>One phase</td>
</tr>
<tr>
<td>Present expansion</td>
<td>Accelerating (predicted)</td>
<td>One of three possibilities</td>
</tr>
<tr>
<td>Pressure</td>
<td>0.034 g/cm$^2$</td>
<td>Negative</td>
</tr>
<tr>
<td>Cosmological constant</td>
<td>$1.934 \times 10^{-35}$ s$^{-2}$ (predicted)</td>
<td>Depends</td>
</tr>
<tr>
<td>$\Omega_T = \Omega_m + \Omega_\Lambda$</td>
<td>1.009</td>
<td>Depends</td>
</tr>
<tr>
<td>Constant-expansion occurs at</td>
<td>8.5 Gyr ago (Gravity is included)</td>
<td>No prediction</td>
</tr>
<tr>
<td>Constant-expansion duration</td>
<td>Fraction of second</td>
<td>Not known</td>
</tr>
<tr>
<td>Temperature at constant expansion</td>
<td>146K (Gravity is included)</td>
<td>No prediction</td>
</tr>
</tbody>
</table>

Table 2: Cosmological parameters in cosmological general relativity and in standard theory.

In order to compare the present theory with general relativity, we must add the time coordinate. We then have a time-space-velocity Universe with two time-like and three space-like coordinates, with signature (+ − − +). We are concerned with the classical experiments of general relativity and the gravitational waves predicted by that theory. It can be shown that all these results are also obtained from the present theory [22].

In the case of gravitational waves we get in this theory a more general formula
than that obtained in general relativity theory. Writing the metric $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a first-approximation term, and using the notation $h_{\mu\nu} = \gamma_{\mu\nu} - \eta_{\mu\nu}\gamma/2$, with $\gamma = \eta^{\alpha\beta}\gamma_{\alpha\beta}$, then the linearized Einstein field equations yield

$$\Box \gamma_{\mu\nu} = -2\kappa T_{\mu\nu},$$

(47)

along with the supplementary condition

$$\eta^{\rho\sigma}\gamma_{\mu\rho,\sigma} = 0,$$

(48)

which solutions $\gamma_{\mu\nu}$ of Eq. (47) should satisfy. In Eq. (47) we have used the notation for a generalized wave equation

$$\Box f = \eta^{\alpha\beta}f_{,\alpha\beta} = \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{1}{\tau^2} \frac{\partial^2}{\partial v^2}\right)f.$$

(49)

Finally we see from Eq. (47) that a necessary condition for Eq. (48) to be satisfied is that

$$\eta^{\alpha\beta}T_{\mu\alpha,\beta} = 0,$$

(50)

which is an expression for the conservation of the energy and momentum without including gravitation.

In vacuum, Eq. (47) reduces to

$$\Box \gamma_{\mu\nu} = 0,$$

(51)

or

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right)\gamma_{\mu\nu} = \frac{1}{\tau^2} \frac{\partial^2\gamma_{\mu\nu}}{\partial v^2}.$$

(52)

Thus the gravitational field, like the electromagnetic field, propagates in vacuum with the speed of light. The above analysis also shows the existence of gravitational waves.

References


Figure 1: Hubble’s diagram describing the three-phase evolution of the Universe according to cosmological general relativity theory. Curves (1) to (5) represent the stages of *decelerating* expansion. As the density of matter $\rho$ decreases, the Universe goes over from the lower curves to the upper ones, but it does not have enough time to close up to a big crunch. The Universe subsequently goes to curve (6) with $\Omega = 1$, at which time it has a *constant* expansion for a fraction of a second. This then followed by going to the upper curves (7), (8) with $\Omega < 1$ where the Universe expands with *acceleration*. Curve no. 8 fits the present situation of the Universe. (Source: S. Behar and M. Carmeli, Ref. 3)
Figure 2: Hubble’s diagram of the Universe at the present phase of evolution with accelerating expansion. (Source: S. Behar and M. Carmeli, Ref. 3)
Figure 3: Distance vs. redshift diagram showing the deviation from a constant toward an accelerating expansion. (Source: A. Riess et al., Ref. 12)
Figure 4: Hubble diagram of SNe Ia minus an empty (i.e., “empty” \( \Omega = 0 \)) Universe compared to cosmological and astrophysical models. The points are the redshift-binned data from the HZT (Riess et al. 1998) and the SCP (Perlmutter et al. 1999). The measurements of SN 1997ff are inconsistent with astrophysical effects which could mimic previous evidence for an accelerating Universe from SNe Ia at \( z \approx 0.5 \).
(Source: A. Riess et al., Ref. 21)