Noncommutative supergeometry, duality and deformations.

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Abstract

We introduce a notion of $Q$-algebra that can be considered as a generalization of the notion of $Q$-manifold (a supermanifold equipped with an odd vector field obeying $\{Q,Q\} = 0$). We develop the theory of connections on modules over $Q$-algebras and prove a general duality theorem for gauge theories on such modules. This theorem containing as a simplest case $SO(d,d,\mathbb{Z})$-duality of gauge theories on noncommutative tori can be applied also in more complicated situations. We show that $Q$-algebras appear naturally in Fedosov construction of formal deformation of commutative algebras of functions and that similar $Q$-algebras can be constructed also in the case when the deformation parameter is not formal.

It was shown recently that noncommutative geometry is quite useful in the study of string theory/M-theory (see [4]-[10] and references therein). It was proved, in particular, that the gauge theory on noncommutative tori has $SO(d,d,\mathbb{Z})$ duality group, closely related to T-duality in string theory [5]. A very general duality theorem, containing $SO(d,d,\mathbb{Z})$-duality as a special case was derived in [11]. This theorem was formulated and proved in the framework of "noncommutative supergeometry". The main idea of noncommutative geometry is to consider every associative algebra as an algebra of functions on "noncommutative space". Of course, supergeometry...
fits very nicely in this approach: one of the most convenient definitions of a supermanifold is formulated in terms of the algebra of functions on it. One can say that supergeometry is "supercommutative $\mathbb{Z}_2$-graded noncommutative geometry".

One of important notions of supergeometry is the notion of $Q$-manifold (of a manifold equipped with an odd vector field $Q$ satisfying $\{Q, Q\} = 0$); see [13]. The first order differential operator $\hat{Q}$ corresponding to $Q$ obeys $\hat{Q}^2 = 0$; therefore the algebra of functions on a $Q$-manifold can be considered as a differential $\mathbb{Z}_2$-graded associative algebra and it is naturally to think that differential $\mathbb{Z}_2$-graded associative algebras are analogs of $Q$-manifolds. However, in [11] we introduced another notion, the notion of $Q$-algebra, that also can be considered as a natural generalization of $Q$-manifold and that can be used to develop the theory of connections and to prove a general duality theorem. Namely, one can define a $Q$-algebra as a $\mathbb{Z}_2$-graded associative algebra $A$ equipped with an odd derivation $Q$ obeying $Q^2a = [\omega, a]$; here $\omega \in A$ should satisfy $Q = 0$. (One says that a linear operator acting on graded algebra is a derivation, if it satisfies the graded Leibniz rule.) Of course, in the case when $A$ is supercommutative this definition coincides with the definition of differential algebra, but if we do not assume supercommutativity this definition is more general. The notion of $Q$-algebra is equivalent to the notion of CDGA-curved differential graded algebra-introduced in [14]. It is closely related to the notion of $A_{\infty}$-algebra.

One can define an $A_{\infty}$-algebra as a vector space $V$ equipped with multilinear operations $m_i$; these operations should satisfy some relations. (The operations $m_i$ determine a derivation of tensor algebra over $V$; the square of this derivation should be equal to zero.) In standard definition of $A_{\infty}$-algebra one considers operations $m_i$ where the number of arguments $i$ is $\geq 1$. However, one can modify the definition including an operation $m_0$ (if the number of arguments is equal to zero, then the operation is simply a fixed element of $V$). Using the modified definition one can say that $Q$-algebra is an $A_{\infty}$-algebra where all operations with the number of arguments $\geq 3$ vanish. (In standard definition this requirement leads to differential algebras.)

We define a connection on $A$-module $E$ as an odd linear map $\nabla : E \to E$ obeying the Leibniz rule $\nabla(ea) = \nabla e \cdot a + (-1)^{\deg e} e \cdot Qa$; the general duality theorem is formulated in terms of such connections.
We analyze the relation of the standard definition of connection in noncommutative geometry to our one. It seems that many well-known constructions and theorems become more transparent in the formalism $Q$-algebras. From the other side many considerations of [11] are similar to arguments employed previously, especially in [1], [2], [3].

Notice, that the theory of $Q$-algebras can be generalized in the following way. We can consider a $Q$-algebra as $\mathbb{Z}_2$-graded algebra $A$ equipped with an odd derivation $Q$; then $Q^2 = \rho$ is an even derivation that is not necessarily an inner derivation. In this case we should modify the definition of connection. Namely, to specify a connection on $A$-module $E$ we should consider along with an odd linear operator $\nabla$ obeying the Leibniz rule an even linear operator $\sigma$ that satisfies

$$\sigma(ea) = (\sigma e)a = e\rho(a)$$

Using the notation (5) we can represent this relation in the form

$$[\sigma, \hat{a}] = \hat{\rho}(a).$$

For the original definition of $Q$-algebras we should take $\sigma = \hat{\omega}$. It is easy to verify that this definition allows us to generalize the theory of connections presented below; the only essential modification is in definition of the curvature where we should replace $\hat{\omega}$ with $\sigma$.

The present paper contains a more detailed exposition of the results of the letter [11] as well as some applications of these results. In particular, we show that $Q$-algebras appear naturally in Fedosov construction of formal deformations of commutative algebras of functions and that similar $Q$-algebras can be constructed also in the case when the deformation parameter is not formal. We conjecture that these $Q$-algebras can be used to circumvent the problem of construction of non-formal deformation. (In formal case modules over deformed algebra corresponded to modules of appropriate $Q$-algebra equipped with zero curvature connection. The construction of deformed algebra is not known in non-formal case, but we hope that one can use the $Q$-algebra we constructed instead of this unknown algebra.)

1. Preliminaries.

When we talk about associative algebra $A$ we always have in mind graded ($\mathbb{Z}$-graded or $\mathbb{Z}_2$-graded) unital associative algebra over $\mathbb{C}$. 

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Graded commutator is defined by the formula

\[[a, b] = ab - (-1)^{\deg a \deg b} ba.\]

In what follows all commutators are understood as graded commutators.

A (right) module \(E\) over \(A\) is a graded vector space with operator of multiplication on elements of \(A\) from the right; this operation should have standard properties: \((ea) \cdot b = e \cdot (ab)\) \(e(a + b) = ea + eb\) etc. Grading on \(E\) should be compatible with grading on \(A\) (i.e. \(\deg(ea) = \deg e + \deg a\)). The definition of a left module is similar; by default our modules are right modules. For every module \(E\) we can construct a module \(\Pi E\) changing the grading: \(\tilde{\deg} e = \deg e = 1\) (for \(\mathbb{Z}_2\)-graded modules the operation \(\Pi\) is parity reversion).

A vector space \(E\) is called an \((A, B)\)-bimodule if it is a left \(A\)-module and a right \(B\)-module; we require that \((a_1 e) a_2 = a_1 (ea_2)\) where \(a_i \in A_i\), \(e \in E\).

If \(E_1, E_2\) are \(A\)-modules we define an \(A\)-homomorphism as a map \(\varphi : E_1 \rightarrow E_2\) obeying \(\varphi(xa) = \varphi(x)a\). The graded algebra of \(A\)-homomorphisms of the \(A\)-module \(E\) into itself (algebra of \(A\)-endomorphisms) is denoted by \(\text{End}_A E\). If \(E\) is an \((A, B)\)-bimodule there exist natural homomorphisms \(A \rightarrow \text{End}_B E\) and \(B \rightarrow \text{End}_A E\).

If \(E_1\) is a right \(A\)-module and \(E_2\) is a left \(A\)-module we define \(E_1 \otimes_A E_2\) as a vector space obtained from the standard tensor product \(E_1 \otimes_C E_2\) by means of identification \(e_1 a \otimes e_2 \sim e_1 \otimes ae_2\), where \(e_i \in E_i\), \(a \in A\).

A linear map \(E_1 \otimes_C E_2 \rightarrow F\) can be considered as \(F\)-valued bilinear pairing \(< e_1, e_2 >\); this map descends to \(E_1 \otimes_A E_2\) iff \(< e_1 a, e_2 > = < e_1, ae_2 >\).

A (finitely generated) free module \(A^n\) over \(A\) can be defined as the space of column vector with entries from \(A\) and with componentwise multiplication on elements of \(A\). We regard \(A^n\) as a right module, but it can be considered also as \((A, A)\)-bimodule. (We already used the structure of \((A, A)\)-bimodule on \(A^1 = A\).) The algebra \(\text{End}_A A^n\) of endomorphisms of \(A^n\) can be identified the algebra \(\text{Mat}_n A\) of \(n \times n\) matrices with entries from \(A\); these matrices act on \(A^n\) by means of multiplication from the left. A projective \(A\)-module can be defined as a direct summand \(E\) in a free module \(A^n\). The decomposition \(A^n = E + E'\) into a direct sum determines an endomorphism \(e : A^n \rightarrow A^n\) projecting \(A^n\) onto \(E\); in other words \(e^2 = e, \ ex = x\).
for $x \in E$, $ex' = 0$ for $x' \in E'$. Notice that in our terminology projective modules are always finitely generated.

Projective $A$-modules form a semigroup with respect to direct summation. Applying the Grothendieck construction to this semigroup we obtain the K-theory group $K_0(A)$. More precisely, we say that a projective module $E$ specifies an element $[E] \in K_0(A)$ and impose the relations $[E_1 + E_2] = [E_1] + [E_2]$, $[E + \Pi E] = 0$. If we work with $\mathbb{Z}_2$-graded modules there is no necessity to consider formal differences $E_1 - E_2$ (virtual modules); the relation $[E + \Pi E] = 0$ permits us to replace virtual module $E_1 - E_2$ with $\mathbb{Z}_2$-graded module $E_1 + \Pi E_2$. A $\mathbb{C}$-linear map $\tau : A \to \mathbb{C}$ is called a (graded) trace if it vanishes on all (graded) commutators: $\tau([a, b]) = 0$ for all $a, b \in A$. We always consider graded traces; therefore we almost always omit the word "graded” in our formulations.

A trace $\tau$ on $A$ generates a trace on $\text{End}_A A^n = \text{Mat}_n A$; this trace will be denoted by the same symbol $\tau$. (To calculate the trace of a matrix $(a_{ij}) \in \text{Mat}_n A$ one should take the supertrace of the matrix $(\tau(a_{ij}))$).

If $E \subset A^n$ is a projective module then the algebra $\text{End}_A E$ of endomorphisms of $E$ can be identified with the subalgebra of $\text{End}_A A^n = \text{Mat}_n \Omega$ consisting of elements of the form $eae$. (Here $e : A^n \to A^n$ is a projection of $A^n$ onto $E$, $a \in \text{End}_A A^n$). We define a graded trace $\bar{\tau}$ on $\text{End}_A E$ as a restriction of $\tau$ to this subalgebra.

If $E$ is an $A$-module, then starting with an element $g \in E$ and $A$-homomorphism $f : E \to A$ we can construct an endomorphism $g \otimes f : E \to E$ transforming $x \in E$ into $gf(x) \in E$. (The endomorphism $g \otimes f$ can be considered as a generalization of linear operator of rank 1.)

For any algebra $A$ we construct a vector space $\bar{A} = A/[A, A]$ factorizing the vector space $A$ with respect to the subspace $[A, A]$ spanned by all (graded) commutators $[a, b]$. This construction is closely related with the notion of trace: traces on $A$ correspond to linear functionals on $\bar{A}$.

If $E$ is a projective $A$-module, one can construct a $\mathbb{C}$-linear map $\text{Tr} : \text{End}_A E \to \bar{A}$ transforming an endomorphism of the form $g \otimes f$ into the class $f(g) \in \bar{A}$ of $f(g) \in A$. (Such a $\mathbb{C}$-linear map is unique because in the case of projective module every endomorphism can be represented as a finite sum of endomorphisms of the form $g \otimes f$.)
The map Tr has the main property of trace

$$\text{Tr} [\varphi, \psi] = 0$$

(trace of graded commutator of two $A$-endomorphisms $\varphi, \psi \in \text{End}_A E$ vanishes). In some sense the map $\text{Tr}: \text{End}_A E \to A$ can be considered as universal trace on $\text{End}_A E$. (As we mentioned every trace $\tau$ on $A$ determines a trace $\tilde{\tau}$ on $\text{End}_A E$. It is easy to verify that $\tilde{\tau}(\varphi) = \tau(\text{Tr}\varphi)$.)

2. $Q$-algebras.

**Definition.** Let $A$ be a graded associative algebra. We say the $A$ is a $Q$-algebra if it is equipped with derivation $Q$ of degree 1 and there exists an element $\omega \in A^2$ satisfying

$$Q^2 x = [\omega, x]$$

for all $x \in A$.

Calculating $Q^3 x$ in two ways we obtain

$$Q^3 x = Q([\omega, x]) = Q\omega \cdot x + \omega \cdot Qx - Qx \cdot \omega - x \cdot Q\omega \cdot (-1)^{\deg x}$$

$$Q^3 x = Q^2 \cdot Qx = [\omega, Qx] = \omega \cdot Qx - Qx \cdot \omega.$$ We see that $Q\omega \cdot x = x \cdot Q\omega \cdot (-1)^{\deg x}$, i.e.

$$[Q\omega, x] = 0$$

(2)

We proved that $Q\omega \in A^3$ commutes with all elements of $A$ (in the sense of superalgebra). In almost all interesting cases it follows from this condition that $Q\omega$ vanishes.

We will include the additional condition

$$Q\omega = 0$$

(3)

in the definition of $Q$-algebra.

We almost always consider unital algebras. It is easy to to check that applying $Q$ to the unit we get 0. (This follows from the Leibniz rule.)

Let us consider a (graded) $A$-module $E$. We define a connection on $E$ as a $C$-linear operator $\nabla : E \to E$ having degree 1 and obeying the Leibniz rule:

$$\nabla(xa) = (\nabla x) \cdot a + (-1)^{\deg x} \cdot x \cdot Qa.$$  

(4)
for all $x \in E, \ a \in A$.

Let us introduce the notation

$$\hat{a}x = (-1)^{\deg x \deg a}xa$$  \hspace{1cm} (5)

The formula (4) can be rewritten in the form

$$\left[ \nabla, \hat{a} \right] = \hat{Q}a$$  \hspace{1cm} (6)

It is easy to check that some standard statements about connections remain true in our case. However, the definition of curvature should be modified.

1) If $\nabla$ is a fixed connection on $E$, then every other connection has the form

$$\nabla' = \nabla + A$$

where $A \in \text{End}^1_A E$ is an arbitrary endomorphism of degree 1.

2) If $\varphi \in \text{End}_A E$ is an endomorphism then $\left[ \nabla, \varphi \right]$ is also endomorphism.

3) The operator $\nabla^2 + \hat{\omega}$ is an endomorphism: $\nabla^2 + \hat{\omega} \in \text{End}^2_A E$. This endomorphism is called the curvature of connection $\nabla$; it is denoted by $F(\nabla)$ (or simply by $F$). It obeys $\left[ \nabla, F \right] = 0$.

To check this statement we represent $\nabla^2$ as $\frac{1}{2}[\nabla, \nabla]$ and calculate $[[\nabla, \nabla], \hat{a}]$ by means of (4) and Jacobi identify.

4) Let us define the operator $\tilde{Q} : \text{End}_A E \to \text{End}_A E$ by the formula

$$\tilde{Q}\varphi = [\nabla, \varphi].$$

It is easy to verify that

$$\tilde{Q}^2 \varphi = \left[ F, \varphi \right].$$  \hspace{1cm} (7)

where $F$ is the curvature of $\nabla$ It follows from this statement and from $\tilde{Q}F = [\nabla, F] = 0$ that the algebra $\text{End}_A E$ equipped with the operator $\tilde{Q}$ is a $Q$-algebra with $\tilde{\omega} = F$. (One should notice, however, that we can also take $\tilde{\omega} = F + c$, where $c$ is a central element obeying $\nabla c = 0$.)

5) If $E$ is considered as a module over $Q$-algebra $\text{End}_A E$, then $\nabla$ is a connection on this module.

As we mentioned, the notion of $Q$-algebra is a generalization of the notion of differential algebra. It is important to notice that the endomorphism algebra $\text{End}_A E$ is not necessarily a differential
algebra: we have $\tilde{Q}^2 = 0$ only in the case when $F$ is a central element. In particular, a structure of differential algebra on $\text{End}_AE$ arises if $\tilde{Q}$ is defined by means of constant curvature connection $\nabla$.

3. Equivalent $Q$-algebras.

Let us consider a graded associative algebra $A$. Let us suppose that a structure of $Q$-algebra on $A$ is specified by means of operator $Q$ obeying (1). If $\gamma$ is element of $A$ we denote by $\tilde{\gamma}$ a derivation of $A$ defined by the formula

$$\tilde{\gamma}(a) = [\gamma, a].$$

(8)

Taking $\gamma \in A^1$ we can construct a derivation of degree 1:

$$Q' = Q + \tilde{\gamma}.$$  

(9)

It is easy to check that

$$Q'^2 x = [\omega', x]$$

(10)

where $\omega' = \omega + (Q\gamma + \gamma^2)$ and $Q'\omega' = 0$.

This means that $Q'$ specifies another structure of $Q$-algebra on $A$. We will show that this new structure is in some sense equivalent to the original one. More precisely, we fix an $A$-module $E$ and consider connections on $E$ with respect to the original $Q$-structure ($Q$-connections) and with respect to new $Q$-structure ($Q'$-connections). We will prove that there exists one-to-correspondence between $Q$-connections and $Q'$-connections. Namely, for every $Q$-connection $\nabla : E \to E$ we can construct a $Q'$-connection $\nabla' = \nabla - \tilde{\gamma}$, where $\tilde{\gamma}$ is defined by (5). (To check that $\nabla'$ is a $Q'$-connection we use (6).) The curvature $F'$ of $\nabla'$ is equal to the curvature $F$ of $\nabla$.

As we mentioned in Sec.2 a $Q$-connection $\nabla : E \to E$ induces a structure of $Q$-algebra on the algebra of endomorphisms $\text{End}_AE$; the operator $\tilde{Q}$ on $\text{End}_AE$ is defined by the formula $\tilde{Q}_\varphi = [\nabla, \varphi]$. It is obvious that replacing a connection $\nabla$ with another connection $\nabla + \alpha$ we obtain an equivalent $Q$-algebra structure specified by the operator transforming $\varphi \in \text{End}_AE$ into $\tilde{Q}_\varphi + [\alpha, \varphi]$. Identifying equivalent $Q$-structures we can say that $Q$-structure on $\text{End}_AE$ does not depend on the choice of connection $\nabla$.

Let us define a dg-module over a $Q$-algebra as a module equipped with a connection with $F = 0$ (zero curvature connection). This terminology agrees with standard terminology in the case when a
Q-algebra is a $dg$-algebra ($Q^2 = 0$), because in this $F = \nabla^2$ and $\nabla$ can be regarded as a differential. It follows from the above statements that there exists a one-to-correspondence between $dg$-modules over $Q$-algebra $(A, Q)$ and $dg$-modules over equivalent $Q$-algebra $(A, Q') = (A, Q + \gamma)$. If $Q'^2 = 0$ (a $Q$-algebra is equivalent to a differential algebra), we can reduce the study of $dg$-modules over $Q$-algebra to the study of $dg$-modules over equivalent differential algebra (or to go in opposite direction if the differential algebra is more complicated.)

4. Connections on projective modules.

First of all it is easy to construct a connection on an arbitrary projective $A$-module $E$ where $A$ is a $Q$-algebra. Namely, if $E$ is specified by means of projection $e : A^n \to A^n$ (i.e. $e\Omega^n = E$) we can construct a connection on $E$ (so called Levi-Civita connection) by means of the formula $D = eQe$ where $Q$ acts on $A^n$ componentwise. (The Leibniz rule for $D$ follows from $e^2 = e$ and from the Leibniz rule for $Q$.) The curvature of the Levi-Civita connection is given by the formula:

$$F = e((Qe)^2 + \omega \cdot 1).$$

For any algebra $A$ we defined a vector space $\tilde{A} = A/[A, A]$. If $A$ is a $Q$-algebra we have $Q([A, A]) \subset [A, A]$. This means that the operator $Q : A \to A$ descends to an operator $\tilde{Q} : \tilde{A} \to \tilde{A}$. It is easy to check that $\tilde{Q}$ is a differential: $\tilde{Q}^2 = 0$.

Now we will define the Chern character of a connection $D$ on a projective $A$-module $E$ as an element of $\tilde{A}$:

$$\text{ch}D = \sum_{q=0}^{1} q! \text{Tr} F^q$$

(Recall that we defined a map $\text{Tr} : \text{End}_A E \to \tilde{A}$ using the formula $\text{Tr}(g \otimes f) = f(g)$. Here $f : E \to A$ is an $A$-homomorphism, $g \in A$ and $g \otimes f$ transforms $x \in E$ into $gf(x) \in E$. The map $a \to \tilde{a}$ transforms $a \in A$ into its class $\tilde{a} \in \tilde{A}$.)

One can prove the following statements:

1) $\text{ch}D$ is closed with respect to the differential $\tilde{Q}$ in $\tilde{A}$:

$$\tilde{Q}\text{ch}D = 0 \quad (11)$$

2) If $D', D$ are two connections on $A$-module $E$ then $\text{ch}D' - \text{ch}D$ is exact with respect to the differential $\tilde{Q}$:

$$\text{ch}D' - \text{ch}D = \tilde{Q}(\text{something}). \quad (12)$$
The proof is based on the following lemma:

For every endomorphism $\psi \in \text{End}_A E$ we have

$$\text{Tr}[D, \psi] = \bar{Q}\text{Tr}\psi$$  \hspace{1cm} (13)

It is sufficient to verify (14) for Levi-Civita connection $D = cQe$ (because $\text{Tr}[D' - D, \psi] = 0$) and for endomorphisms of the form $\psi = g \otimes f$ (because these endomorphisms span $\text{End}_A E$).

Using (14) we deduce (12) from the relation $[D, F^q] = 0$ that follows immediately from $[D, F] = 0$.

To derive (13) we will consider a smooth family $D(t) = D + t(D' - D)$ of connections on $E$ and prove that

$$\frac{d}{dt}\text{ch}D(t) = \bar{Q}(\text{something}).$$

First of all we notice that the curvature $F(t)$ of connection $D(t)$ obeys

$$\frac{dF(t)}{dt} = [\Gamma, D(t)]$$

where $\Gamma = D' - D \in \text{End}_A E$. We see that

$$\frac{dF}{dt} = [\Gamma, D] \mod \bar{A}, \bar{A}.$$

and therefore

$$\frac{dF^q}{dt} = q[\Gamma, D]F^{q-1} = q[D, \Gamma F^{q-1}] \mod \bar{A}, \bar{A},$$

$$\frac{d\text{Tr}F^q}{dt} = q\text{Tr}[D, \Gamma F^{q-1}] \in \bar{Q}(\bar{A}).$$

Integrating over $t$ we obtain (13).

In the proof of (13) we assumed that $A$ is equipped with topology having some properties that permit us to justify the calculations above. These assumptions are not necessary; it is easy to modify our consideration to obtain completely algebraic proof (as in [2] for example).

Sometimes it is convenient to reformulate (13) using the notion of closed trace. We say that a linear functional on $A$ is a closed trace if it vanishes on (graded) commutators and on elements of the form $Qa$. It follows from (13) that for a closed trace $\tau$ the number
\( \tau(\text{ch}(D)) \) does not depend on the choice of the connection \( D \) on the module \( E \); it depends only on the \( K \)-theory class of the module \( E \).

Using the differential \( \bar{Q} \) we can define the homology \( H(A) \) in the standard way: \( H(A) = \ker \bar{Q}/\text{Im} \bar{Q} \). It follows from (12), (13) that the Chern character specifies a homomorphism \( \text{ch}: K_0(A) \to H_{\text{even}}(A) \).

5. Morita equivalence of \( Q \)-algebras

Let us consider an \((A,B)\)-bimodule \( P \) where \( A \) is a \( Q \)-algebra with respect to the operator \( Q_1 \) and \( B \) is a \( Q \)-algebra with respect to the operator \( Q_2 \). We say that an operator \( \nabla_P : P \to P \) is a connection on bimodule \( P \) if

\[
\nabla_P(ax) = (-1)^{\deg a} \cdot a \nabla_p(x) + Q_1 a \cdot x \\
\nabla_P(xb) = \nabla_p x \cdot b + (-1)^{\deg x} \cdot xQ_2 b
\]

for all \( x \in P, \ a \in A_1, \ b \in B_2 \).

In other words, \( \nabla_P \) should be a connection with respect to \( A \) and with respect to \( B \) at the same time.

It follows from the above statements that every \( A \)-module \( E \) equipped with a connection \( \nabla \) can be considered as \((\text{End}_AE, A)\)-bimodule and \( \nabla \) is a connection on this bimodule.

Using an \((A,B)\)-bimodule \( P \) we can assign to every (right ) \( A \)-module \( E \) a (right) \( B \)-module \( \tilde{E} \) taking the tensor product with \( P \):

\[
\tilde{E} = E \otimes_A P \quad (14)
\]

(To take the tensor product over \( A \) we identify \( ea \otimes p \) with \( e \otimes ap \) in the standard tensor product \( E \otimes_C P \). Here \( e \in E, \ p \in P, \ a \in A \).)

If we have a connection \( \nabla_P \) in the bimodule \( P \) we can transfer a connection on \( E \) to a connection on \( \tilde{E} \). Namely, for every connection \( \nabla \) on \( E \) we define an operator \( \nabla \otimes 1 + 1 \otimes \nabla_P \) on \( E \otimes_C P \). It is easy to check that this operator is compatible with identification \( ea \otimes p \sim e \otimes ap \) and therefore descends to an operator \( \nabla : \tilde{E} \to \tilde{E} \). The operator \( \nabla \) can be considered as a connection on \( B \)-module \( \tilde{E} \).

It is easy to relate the curvatures of the connections \( \nabla \) and \( \nabla \). We should take into account that correspondence between \( E \) and \( \tilde{E} \) is natural, i.e. to every endomorphism \( \sigma \in \text{End}_AE \) we can assign an endomorphism \( \tilde{\sigma} \in \text{End}_B\tilde{E} \) (the map \( \sigma \otimes 1 : E \otimes_C P \to E \otimes_C P \) descends to an endomorphism \( \tilde{\sigma} : \tilde{E} \to \tilde{E} \)). In particular, the curvature \( F(\nabla) \in \text{End}_AE \) determines an endomorphism \( F(\nabla) \in \).
One can verify that the curvature $F(\tilde{\nabla})$ of the connection $\tilde{\nabla}$ on $\tilde{E}$ can be represented in the form:

$$F(\tilde{\nabla}) = \tilde{F}(\nabla) + \tilde{\varphi},$$  \hspace{1cm} (15)

where $\tilde{\varphi}$ is a fixed element of $\text{End}_B \tilde{E}$.

To verify (7) we notice that $\nabla^2 \otimes 1 + \hat{\omega}_1 \otimes 1 + \hat{\omega}_2$ descends to the endomorphism $\tilde{F}(\nabla) : \tilde{E} \to \tilde{E}$ and $\nabla^2 \otimes 1 + 1 \otimes \nabla^2 + 1 \otimes \hat{\omega}_2$ descends to $\tilde{F}(\nabla) : \tilde{E} \to \tilde{E}$. Using the relation $\hat{\omega}_1 \otimes 1 = -1 \otimes \hat{\omega}_1$ we obtain that the map $\varphi = F(\tilde{\nabla}) - \tilde{F}(\nabla)$ is induced by the map $\psi : E \otimes CP \to E \otimes CP$ where the map $\psi : P \to P$ is given by the formula

$$\psi = \nabla^2 + \hat{\omega}_1 + \hat{\omega}_2.$$

It is easy to check that $\psi \in \text{End}_A P \cap \text{End}_B P$  \hspace{1cm} (16)

(i.e. $\psi(ax) = a\psi(x)$, $\psi(xb) = \psi(x)b$ for $x \in P$, $a \in A$, $b \in B$).

To check that $\psi$ commutes with $a \in A$ we represent it in the form $\psi = F_1(\nabla_P) + \hat{\omega}_2$, where $F_1(\nabla_P)$ stands for the curvature of $\nabla_P$ considered as $A$-connection; the representation $\psi = F_2(\nabla_P) + \hat{\omega}_1$ should be used to prove that $\psi \in \text{End}_B P$.

It follows from (8) that $\varphi = 1 \otimes \psi$ descends to $\tilde{E}$ and gives an $B$-endomorphism $\tilde{\varphi}$. One should notice that these facts are clear also from the representation $\tilde{\varphi} = F(\tilde{\nabla}) - \tilde{F}(\nabla)$.

To illustrate the above statements we can start with an arbitrary $Q$-algebra $A$ and arbitrary $A$-module $P$ with connection $\nabla_P$. We consider $P$ as $(A,B)$-bimodule, where $A = \text{End}_A P$, $B = A$. (We have seen that $A = \text{End}_A P$ is a $Q$-algebra with respect to the operator $Q \varphi = [\nabla_P, \varphi]$ and that $\nabla_P$ is a connection also with respect to this $Q$-algebra.) It follows from our calculations that $F = F_2(\nabla_P) = \nabla^2 + \hat{\omega}_2$, $\hat{\omega}_1 = -F$ and therefore $\psi = F + \hat{\omega}_1 = 0$. (We can obtain the same result noticing that $F_1(\nabla_2) = \nabla^2 + \hat{\omega}_1 = (F - \hat{\omega}_1 = -\hat{\omega}_2)$.

We see that in our situation $\varphi = 0$; hence, $F(\tilde{\nabla}) = F(\nabla)$. (However, as we noticed above one can modify the definition of $Q$-algebra $\text{End}_A P$ adding central element $c$ with $\nabla c = 0$ to $\omega_1$; then $\varphi \neq 0$.)
We would like to give conditions when gauge theories on $A$-module $E$ and in $B$-module $\tilde{E}$ are equivalent. To establish such an equivalence we need $(A,B)$-bimodule $P'$ equipped with connection $\nabla_{P'}$. Such a bimodule permits us to transfer modules and connections in opposite direction. If the constructions obtained by means of $P'$ are inverse to constructions specified by $P$ we say that bimodules $P, P'$ give Morita equivalence of $Q$-algebras $A$ and $B$ (or that they are Morita equivalence bimodules). Of course, this notion generalizes the standard notion of Morita equivalence of associative algebras, when we do not use the operator $Q$ and connections. The definition of Morita equivalence bimodules can be reformulated in the following more constructive way. Let us suppose that there exist two bilinear scalar products between $P$ and $P'$ taking values in $A$ and in $B$ respectively. We assume that scalar products are $A$-invariant and $B$-invariant correspondingly. In other words, we assume that for $p \in P, \ p' \in P'$ we have $<p, p'_1>_1 \in A, \ <p', p>_2 \in B$ and $<p\omega, p'_1>_1 = <p, \omega p'_1>_1$ for $\omega \in B, \ <p'\sigma_1, p>_2 = <p'_1, \sigma_1 p>_2$ for $\sigma_1 \in A$. We require also that

\begin{align}
\sigma_1 <p, p'_1>_1 \sigma_2 &= <\sigma_1 p, p' \sigma_2>_1, \quad \omega_1 <p', p>_2 \omega_2 = <\omega_1 p', p \omega_2>_2 \\
p_1 <p, p'_1>_1 &= <p_1, p>_2 p'_1, \quad <p', p>_2 p'_1 = p' <p, p'_1>_1
\end{align}

(17)

(18)

Here $p, p_1 \in P, \ p', p'_1 \in P'$, $\sigma_i \in A, \ \omega_i \in B$. The scalar products determine maps

$$\alpha : P \otimes_B P' \rightarrow A, \ \beta : P' \otimes_A P \rightarrow B.$$ We can consider $P \otimes_B P'$ and $A$ as $(A,A)$-bimodules; then it follows from (9), that $\alpha$ is a homomorphism of bimodules; similarly $\beta$ is a homomorphism of $(B,B)$-bimodules. We require that $\alpha$ and $\beta$ be isomorphisms. Then

$$(E \otimes_A P) \otimes_B P' = E \otimes_A (P \otimes_B P') = E \otimes_A A = E$$

for every $A$-module $E$. This statement together with similar statement for $B$-modules gives us one-to-one correspondence between $A$-modules and $B$-modules (more precisely it gives us equivalence of categories of $A$-modules and $B$-modules). To obtain one-to-one correspondence between connections we should impose additional requirements

$$<\nabla_{Pp}, p'_1>_1 + <p, \nabla_{P'p'}>_1 = Q <p, p'_1>_1,$$

(19)
\[
<\nabla_P p', p>_2 + <p', \nabla_P p>_2 = Q <p', p>_2
\]

It follows from our assumptions that the operator
\[
\nabla_P \otimes 1 + 1 \otimes \nabla_{P'}
\]
on \[P \otimes_C P'\] descends to operator \[Q\] on \[P \otimes_B P'\]. Using that \[Q \cdot 1 = 0\] we obtain that the operator \[\nabla \otimes 1 + 1 \otimes \hat{Q}\] on \[E \otimes_C A\] descends to \[\nabla\] on \[E \otimes_A A = E\]. This means that going from \(A\)-connection to \(B\)-connection and back we obtain the original \(A\)-connection. This fact together with similar statement about \(B\)-connections gives one-to-one correspondence between \(A\)-connections and \(B\)-connections.

We see that under our conditions we have equivalence between gauge theories on \(A\)-module \(E\) and on \(B\)-module \(\tilde{E}\) (duality). We will describe later how the duality of gauge theories on noncommutative tori can be obtained this way.

Using well known results about Morita equivalence associative algebras [12] one can describe \(Q\)-algebras that are equivalent to a given \(Q\)-algebra \(A\) in the following way. Let us consider a projective \(A\)-module \(P\) that is equipped with a connection \(\nabla\). Let us assume that \(P\) is a generator (i.e. \(A^1\) is a direct summand in \(P^n\)). Then \(\hat{A} = \text{End}_A P\) is Morita equivalent to \(A\) as a \(Q\)-algebra. (The structure of \(Q\)-algebra on \(\hat{A}\) is specified by an operator \(\hat{Q}\) defined by the formula \(\hat{Q}\varphi = [\nabla, \varphi]\).) All \(Q\)-algebras that are Morita equivalent to \(A\) can be obtained by means of this construction. (This follows easily from the remark that two Morita equivalent \(Q\)-algebras are Morita equivalent as associative algebras.)

6. Connections on modules over associative algebras.

The theory of connections on modules over \(Q\)-algebras can be considered as a generalization of the theory of connections on associative algebras. If \(A\) is an associative algebra one can construct a differential \(\mathbb{Z}\)-graded algebra \(\Omega(A) = \sum_{n \geq 0} \Omega^n(A)\) (universal differential graded algebra) in the following way. The vector space \(\Omega^n(A)\) is is spanned by formal expressions \(a_0 da_1 ... da_n\) and \(\lambda da_1 ... da_m\) where \(a_0, ... a_n \in A, \ n \geq 0, \ m \geq 1, \ \lambda \in \mathbb{C}\). The multiplication and the differential on \(\Omega(A)\) are defined by means of Leibniz rule and relation \(d^2 = 0\). If \(E\) is an \(A\)-module we define an \(\Omega(A)\)-module \(\mathcal{E}\) as a tensor product: \(\mathcal{E} = E \otimes_A \Omega(A)\) where \(\Omega(A)\) is considered as \((A, \Omega(A))\)-bimodule. We can define a connection on \(A\)-module \(E\) as a connection of \(\Omega(A)\)-module \(\mathcal{E}\); this definition is equivalent to the definition given by Connes (see[1]).
Closed traces on $\Omega(A)$ can be identified with cyclic cocycles of algebra; taking into account that for closed trace $\tau$ the number $\tau(\text{ch} D)$ does not depend on the choice of connection $D$ we obtain a pairing of $K_0(E)$ with cyclic cohomology.

In the definition of connection on $A$-module $E$ the algebra $\Omega(A)$ can be replaced with any differential extension of the algebra $A$ (with any differential graded algebra $\Omega$ that contains $A$ as a subalgebra of $\Omega^0$). Moreover, one can consider any $Q$-extension of $A$ (any $Q$-algebra $\Omega$ obeying $A \subset \Omega^0$) and define a connection on $A$-module $E$ as a connection on $\Omega$-module $E \otimes_A \Omega$. It is interesting to notice that under certain conditions on algebra $\Omega$ any projective $\Omega$-module $E$ can be represented in the form $E$ \otimes_A \Omega$ where $E$ is projective $A$-module, $A = \Omega^0$ (see[12]). In particular, this statement is correct if $\Omega = \sum_{0 \leq k \leq n} \Omega^k$, i.e. the degree of an element of $\Omega$ is non-negative and bounded from above. (In this case elements of $\Omega$ having positive degree form a nilpotent ideal $I$ and $A$ can be identified with $\Omega/I$.)

If a Lie algebra $L$ acts on $A$ by means of infinitesimal automorphisms (derivations) we can construct a differential graded algebra $\Omega = \Omega(L, A)$ of cochains of Lie algebra $L$ with values in $A$. The elements of $\Omega$ can be considered as $A$-valued functions of anticommuting variables $c^1, ..., c^n$ corresponding to the elements of the basis $\delta_1, ..., \delta_n \in L$; the differential $d$ has the form

$$d\omega = (\delta_\alpha \omega) c^\alpha + \frac{1}{2} f^\alpha_{\beta \gamma} c^\beta c^\gamma \frac{\partial}{\partial c^\alpha},$$

where $f^\alpha_{\beta \gamma}$ are the structure constants of $L$ in the basis $\delta_1, ..., \delta_n$.

In other words we can describe the vector space $\Omega(L, A)$ as a tensor product $\Lambda(L^*) \otimes A$ where $L^*$ stands the vector space dual to $L$ and $\Lambda(M)$ denotes the Grassmann algebra generated by vector space $M$ (as vector space $\Lambda(M)$ is a direct sum of antisymmetric tensor powers of $M$). The grading on $\Omega(L, A)$ is defined by means of the natural grading on $\Lambda L^*$; if $A$ is a graded algebra one should take into account also the grading on $A$.

Let us consider in more detail connections on $A$-module $E$ with respect to differential extension $\Omega = \Omega(L, A)$, i.e. connections on graded $\Omega$-module $\mathcal{E} = E \otimes_A \Omega$. In this case $\mathcal{E}^0 = E \otimes_A \Omega^0 = E$, $\mathcal{E}^1 = E \otimes_A \Omega^1 = E \otimes C L^*$. The elements $e \otimes \omega, \ e \in E, \ \omega \in \Omega$ span $E$, therefore, the connection $\nabla : \mathcal{E}^r \to \mathcal{E}^{r+1}$ is completely determined by the map $\nabla : \mathcal{E}^0 \to \mathcal{E}^1$ that can be considered as a map $\nabla : E \to E \otimes L^*$ or as a family of maps $\nabla_x : E \to E$ that
depend linearly on $x \in L$. Instead of the family $\nabla_x$ we can consider $n$ maps $\nabla_1, ..., \nabla_n$ corresponding to the elements of the basis $f_1, ..., f_n$ of the Lie algebra $L$. These maps obey the Leibniz rule

$$\nabla_{\alpha}(ea) = \nabla_{\alpha}e \cdot a + e\delta_{\alpha}a$$

where $\delta_{\alpha}$ stands for the derivation of the algebra $A$ that corresponds to $f_{\alpha} \in L$.

Notice that the Grassmann algebra $\Lambda(L^*)$ is supercommutative, therefore the space $\tilde{\Omega} = \Omega/[[\Omega, \Omega]]$ that corresponds to $\Omega = \Omega(L, A) = \Lambda L^* \otimes A$ can be identified with $\Lambda(L^*) \otimes \hat{A}$. This means that the Chern character takes values in $H^{\text{even}}(\tilde{\Omega}) = H^{\text{even}}(L, \hat{A})$ (in the cohomology of the Lie algebra $L$ with coefficients in $\hat{A}$).

Let us consider an $(A, \hat{A})$-bimodule $P$ assuming that Lie algebra $L$ acts on $A$ and $\hat{A}$ by means of infinitesimal automorphisms. We would like to transfer connections from $A$-module $E$ to $\hat{A}$-module $\hat{E} = E \otimes_A P$. It is easy to see that we can do this using the formula $\hat{\nabla}_x = \nabla_x \otimes 1 + 1 \otimes \nabla_x^P$ if the bimodule $P$ equipped with constant curvature connection $\nabla_x^P$ (i.e. we have family of maps $\nabla_x^P$ satisfying the Leibniz rule with respect to $A$ and $\hat{A}$, the curvature $F_{xy} = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$ should be equal to $f_{xy} \cdot 1$). If $P$ generates Morita equivalence between $A$ and $\hat{A}$, the curvature $\hat{F}_{xy} = [\hat{\nabla}_x, \hat{\nabla}_y] - \hat{\nabla}_{[x,y]}$ should be equal to $\hat{f}_{xy} \cdot 1$). If $P$ generates Morita equivalence between $A$ and $\hat{A}$ and the map $\nabla \to \hat{\nabla}$ is one-to-one correspondence between connections on $E$ and $\hat{E}$ we say that $A$ and $\hat{A}$ are gauge Morita equivalent (in [5] I used the term "complete Morita equivalence" for this notion). It is easy to check that endomorphism algebra $\text{End}_{\Omega(L, A)}P$ of the $\Omega(L, A)$-module $P = P \otimes_A \Omega(L, A)$ is isomorphic to $\Omega(L, \hat{A})$. (If we consider elements of $\mathcal{P}$ as $P$-valued functions of anticommuting variables $c^1, ..., c^n$ then multiplying an element of $\mathcal{P}$ from the left by an $\hat{A}$-valued function of $c^1, ..., c^n$ we obtain an endomorphism of $\mathcal{P}$. The homomorphism from $\Omega(L, \hat{A})$ into $\text{End}_{\Omega(L, A)}\mathcal{P}$ obtained this way is an isomorphism.) It follows from this remark that $\Omega(L, \hat{A})$ is Morita equivalent to $\Omega(L, A)$. A connection on $P$ is by definition a connection on $\mathcal{P}$; if the connection has a constant curvature than corresponding operator $\hat{Q}$ on $\Omega(L, \hat{A}) = \text{End}_{\Omega(L, A)}\mathcal{P}$ is a differential. It is easy to check that under the conditions above $\hat{Q}$ coincides with the differential on $\Omega(L, \hat{A})$ considered as algebra of cochains of Lie algebra $L$ with values in $\hat{A}$. We obtain that differential algebras $\Omega(L, A)$ and $\Omega(L, \hat{A})$ are Morita equivalent as $Q$-algebras if the algebras $A$ and
A are gauge Morita equivalent.

Let $A$ be an algebra $A_\theta$ of smooth functions on $d$-dimensional noncommutative torus (i.e. an algebra of expressions of the form $\sum c_n U_n$, where $c_n$ is a $\mathbb{C}$-valued function on a $d$-dimensional lattice that vanishes at infinity faster than any power and the multiplication is defined by the formula $U_n U_m = \exp(\pi i \theta_{nm}) U_{n+m}$, where $\theta_{nm}$ is a bilinear function on the lattice). Then it is natural to construct a differential extension $\Omega_\theta$ of $A_\theta$ taking as $L$ the Lie algebra of derivations $\delta_x$ where $\delta_x U = < x, l > U_l$. (We assume that the lattice is embedded into vector space $V$. The vector $x$ belongs to the dual space $V^*$ that can be identified with the Lie algebra $L$.) Connections corresponding to this differential extension of $A_\theta$ appear naturally in the study of toroidal compactifications of M(atrix) theory.

Let us suppose now that in addition to the action of Lie algebra on $A$ we have a finite group $G$ acting on $A$ and $L$ by means of automorphisms and that actions of $G$ on $A$ and $L$ are compatible. (If we denote automorphisms of $A$ and of $L$ corresponding to the element $\gamma \in G$ by the same letter $\gamma$ this means that $\gamma(T(a)) = (\gamma T) \cdot (\gamma a)$ for every $\gamma \in G$, $T \in L$, $a \in A$.) One can define in natural way the action of $G$ on the algebra $\Omega = \Omega(L, A)$; this action commutes with the differential. This means that we can regard the crossed product $\Omega \rtimes G$ as a differential algebra; we have $(\Omega \ltimes G)_0 = \Omega_0 \ltimes G = A \ltimes G$ and therefore the crossed product can be considered as differential extension of $A \ltimes G$.

An $A \ltimes G$-module $E$ can be considered as an $A$-module equipped with action of the group $G$ that is compatible with the action of $G$ on $A$ (more precisely we should have $\gamma(xa) = \gamma(x) \cdot \gamma(a)$). As always a connection on $E$ is defined as a connection $\nabla$ on $\Omega \rtimes G$-module

$$E = E \otimes_{A \ltimes G} (\Omega \rtimes G).$$

Again this connection is completely determined by the map $\nabla : E^0 \to E^1$ that can be considered as a map

$$\nabla : E \to E \otimes_{A \ltimes G} (\Omega^1(L, A) \rtimes G)$$

or as a map

$$\nabla : E \to E \otimes L^*$$

that determines a connection on $A$-module $E$ and is compatible with the action of the group $G$ on $E$ and on $E \otimes L^*$.  

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In the case when $A$ is an algebra of functions on noncommutative torus the connections we obtained are precisely the connections that arise by compactification of M(atrix) theory on toroidal orbifolds (see [9],[10]).

7. Deformations of commutative algebras

The problem of deformation of algebras of functions on smooth manifolds is closely related to the problem of quantization. It can be formulated in the following way. Let $C(M)$ denote an algebra of functions on a smooth manifold $M$. To quantize a manifold $M$ we should construct a family $A_\epsilon$ of associative algebras obeying $A_{\epsilon=0} = C(M)$. This definition should be made more precise. First of all one can use various algebras of functions on $M$ in this definition. If $M$ is compact it is natural to work with the algebra $C^\infty(M)$ of smooth functions on $M$, but in the case of noncompact $M$ there are various interesting versions of $C(M)$. One can impose various conditions on dependence of $A_\epsilon$ from $\epsilon$. In the most popular approach one considers $\epsilon$ as a formal parameter; this means that $A_\epsilon$ coincides with $A_{\epsilon=0} = C^\infty(M)$ as a vector space and the product in $A_\epsilon$ (star-product) is a formal power series with respect to $\epsilon$:

$$f \star g = f \cdot g + B_1(f, g)\epsilon + B_2(f, g)\epsilon^2 + ....$$

It is easy to check that the operation $\{f, g\} = B_1(f, g) - B_1(g, f)$ (Poisson bracket corresponding to the family $A_\epsilon$) specifies a structure of Poisson manifold on $M$, therefore usually one speaks about quantization of Poisson manifolds. The problem of formal quantization was solved in the most important case of symplectic manifolds in [15] and for general Poisson manifolds in [16]. However, the situation in the case when $A_\epsilon$ is assumed to be a smooth family depending on parameter $\epsilon \in \mathbb{R}$ remains unclear. It is difficult to construct such families even for simple manifolds $M$. The most important constructions of this kind are based on the formula

$$f \star g = \int \alpha_{\theta u}(f)\alpha_{\nu}(g)e^{iuv}du\,dv \quad (20)$$

where $\alpha_u$ stands for a strongly continuous action of an abelian Lie group $L = \mathbb{R}^d$ on associative algebra $A_0$, $\theta$ is an antisymmetric $d \times d$ matrix (see [17] for details). The new product (star-product) determines an associative algebra $A_\theta$ that depends continuously on $\theta$. 

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Notice that in (20) we assumed that \( u, v \in L = \mathbb{R}^d \). It is more convenient to think that \( v \in L, u \in L^* \) and \( \theta \) is a linear operator acting from \( L^* \) into \( L \) and obeying \( \theta^* = -\theta \). This interpretation of (20) permits us to say that we don’t need inner product on \( L \) to apply (20). Applying (20) to the algebra of smooth functions on torus \( T^d = \mathbb{R}^d/\mathbb{Z}^d \) with natural action of \( L = \mathbb{R}^d \) we obtain a family \( \mathcal{A}_\theta \) as a continuous deformation of the algebra \( \mathcal{A}_0 = C^\infty(T^d) \). This algebra is by definition an algebra of smooth functions on noncommutative torus \( T^d_{\theta} \). Analogously, we obtain various classes of functions on noncommutative euclidean space \( \mathbb{R}^d_{\theta} \). For example we can fix \( \rho \in (0, 1] \) and denote by \( \Gamma^m_{\rho} \) a class of smooth complex functions \( a(x) \) on \( \mathbb{R}^d \) obeying

\[
|\partial_\alpha a(z)| \leq C_\alpha < z >^m - \rho|\alpha|
\]

where \( \alpha = (\alpha_1, ..., \alpha_d) \), \( |\alpha| = \alpha_1 + ... + \alpha_d \), \( m \in \mathbb{R} \), \( < z > = (1 + |z|^2)^{1/2} \). Then one can prove that the star-product of \( a' \in \Gamma^{m_1}_{\rho} \) and \( a'' \in \Gamma^{m_2}_{\rho} \) belongs to \( \Gamma^{m_1+m_2}_{\rho} \). In particular, \( \Gamma^m_{\rho} \) is an algebra for \( m \leq 0 \) and the union \( \Gamma_{\rho} \) of all \( \Gamma^m_{\rho} \) is also an algebra with respect to star-product. One more algebra can be obtained if we assume that \( \theta^{ij} = \epsilon \omega^{ij} \) where \( \epsilon \) is a formal parameter and consider star-product on the space \( A^\epsilon \) of formal series

\[
a(x, \epsilon) = \sum_{k,l} \epsilon^k P_l(x)
\]

where \( k \) and \( l \) are nonnegative integers and \( P_l(x) \) stands for a polynomial of degree \( l \) on \( \mathbb{R}^d \). In all these cases we obtain an algebra \( \mathcal{A}_\theta \) that represents a class of functions on noncommutative \( \mathbb{R}^d \). In particular, we can assign an algebra \( \mathcal{A}(T) \) to every symplectic linear space \( T \). Now we can consider an arbitrary symplectic manifold \( M \) and a bundle of algebras \( \mathcal{A}(T_x) \) corresponding to tangent spaces \( T_x \) where \( x \in M \). The set of sections of this bundle also constitutes an associative algebra with respect to fibrewise multiplication. We shall denote this algebra by \( W \) following Fedosov. It is necessary to emphasize that Fedosov considered only the case of the algebra of formal power series, but we use the same notation \( W \) in all cases. A part of Fedosov’s constructions can be generalized immediately to other algebras. In particular, for every symplectic connection on \( M \) we can construct an operator \( \partial : W \otimes \Lambda \to W \otimes \Lambda \), acting on a tensor product of \( W \) and the algebra \( \Lambda \) of differential forms on \( M \).
In Darboux local coordinates $x^i$ this operator has the form

$$\partial a = da + i[\Gamma, a] \tag{22}$$

where $\Gamma = \frac{1}{2} \Gamma_{ijk}(x)y^i y^j dx^k$, $\Gamma_{ijk}$ is a symplectic tensor, and $y^i$ are coordinates on tangent space $T_x$. It is easy to check that

$$\partial^2 a = [R, a] \tag{23}$$

where

$$R = \frac{a}{4} R_{ijkl} y^i y^j dx^k \hat{dx}^l, \quad R_{ijkl} = \omega_{lm} R^m_{jkl} \tag{24}$$

stands for the curvature tensor of symplectic connection. (Our notations differ slightly from Fedosov’s notations; to convince ourselves that formulas (23) and (24) coincide with the formulas in [18] we should take into account that Fedosov’s $y^i$ contain an extra factor of $\epsilon^{1/2}$.) Let us suppose that the algebra $\mathcal{A}$ used in the construction of the algebra $W$ contains polynomials (for example we can take the algebra $\Gamma_\rho$ as $\mathcal{A}$). Then $R$ belongs to the algebra $W \otimes \Lambda$ and we obtain the following statement.

The algebra $W \otimes \Lambda$ equipped with the operator $Q = \partial$ can be considered as a $Q$-algebra.

The condition (1) follows from (23) and the condition (3) from the Bianchi identity $\partial R = 0$.

Let us introduce an operator $Q_\gamma = \partial + \tilde{\gamma}$ where $\tilde{\gamma}x = [\gamma, x]$ and $\gamma \in W \otimes \Lambda^1$.

The algebra $W \otimes \Lambda$ equipped with the operator $Q_\gamma$ is a $Q$-algebra; this $Q$-algebra is equivalent in the sense of Section 3 to the algebra $W \otimes \Lambda$ equipped with operator $Q = \partial$.

It follows from calculation of Sec. 3 that

$$Q_\gamma^2 = iR + \partial \gamma + \gamma^2.$$  

We see that in the case when

$$iR + \partial \gamma + \gamma^2$$

is a central element we have $Q_\gamma^2 = 0$, i.e. the our $Q$-algebra is a differential algebra that is equivalent as a $Q$-algebra to $W \otimes \Lambda$ equipped with $\tilde{Q} = \partial$.

The Fedosov’s approach to quantization of symplectic manifolds can be described as follows. In the framework of formal power series
with respect to $\epsilon$ we can find such a $\gamma = \gamma_0$ that the operator $Q_\gamma$ is a differential.

The elements of $W \subset W \otimes \Lambda$ that are annihilated by the operator $Q_{\gamma_0}$ form an associative algebra $A_\epsilon$ that can be considered as a formal deformation of the algebra of functions on symplectic manifold $M$. The associative algebra at hand is quasiisomorphic to differential algebra $(W \otimes \Lambda, Q_{\gamma_0})$, i.e. to $W \otimes \Lambda$ equipped with differential $Q_{\gamma_0}$. (M.Kontsevich, private communication). This means that modules over $A_\epsilon$ are in one-to one correspondence with $dg$-modules over $(W \otimes \Lambda, Q_{\gamma_0})$ (for every $A_\epsilon$-module one can construct a $dg$-module as tensor product of $E$ with $W \otimes \Lambda$ over $A_\epsilon$; the differential $Q_{\gamma_0}$ descends to a differential on this tensor product). From the other side there exists a one-to-one correspondence between $dg$-modules over differential algebra $(W \otimes \Lambda, Q_{\gamma_0})$ and $dg$-modules over $Q$-algebra $(W \otimes \Lambda, Q_\gamma)$ for arbitrary $\gamma \in W \otimes \Lambda$. In particular, we can take $\gamma = 0$ and study $dg$-modules over the $Q$-algebra $(W \otimes \Lambda, \partial)$.

One interesting example of $dg$-modules over $(W \otimes \Lambda, \partial)$ can be constructed in the case when $M$ is a Kaehler manifold and the symplectic connection $\partial$ corresponds to the Kaehler metric on $M$. Then we can construct an $W$-module $\mathcal{F}$ as a module of sections of the bundle of Fock modules. (Using Kaehler structure on $M$ we can define a Fock representation of $\mathcal{A}(T_x)$ for every $x \in M$.) The symplectic connection $\partial$ acts on $\mathcal{F} \otimes \Lambda$; one can check that $\mathcal{F} \otimes \Lambda$ is a $dg$-module over $(W \otimes \Lambda, \partial)$ with respect to this action (this fact easily follows from the results of [21]).

Notice that one can generalize the above constructions, considering the case when polynomials don’t belong to the algebra $W$, but can be considered as (left and right) multipliers on $W$. That we are dealing with generalized $Q$-algebras in the sense defined in Introduction; the module $\mathcal{F} \otimes \Lambda$ can be equipped with zero curvature connection (is a $dg$-module) also in this more general case.

8. **Gauge theories on noncommutative tori.**

Let consider in more detail modules over the algebra $A_\theta$ of smooth functions on $d$-dimensional noncommutative torus. Recall, that $\theta$ stands for bilinear function on a lattice; it will be convenient for us to identify it with $d \times d$ matrix $\theta^{ij}$. Notice, that the algebra $A_\theta$ has a trace that is unique up to a constant factor ; it is given by the relations $\text{Tr} U_n = 0$ for $n \neq 0$, $\text{Tr} U_0 = 1$. This trace induces a trace on the algebra $\text{End}_{A_\theta} E$ for every projective module $E$; the trace of
unit endomorphism can be interpreted as (fractional) dimension of module $E$.

The algebra $A_\theta$ for $\theta = 0$ (and, more generally, for integral $\theta$) is isomorphic to the algebra of smooth functions on usual torus. It depends continuously on parameter $\theta$, therefore one should expect that $K$-groups of $A_\theta$ coincide with $K$-groups of torus. This fact was proved in [19]. We can consider Chern character $\text{ch}E$ as an even element of Grassmann algebra $\Lambda$ having $d$ generators $\alpha^1, ..., \alpha^d$. (By definition the Chern character is an element of $\Omega_\theta = \Omega_\theta/[\Omega_\theta, \Omega_\theta]$. We can represent $\Omega_\theta$ as $A_\theta \otimes \Lambda$. The algebra $\Lambda$ is supercommutative, therefore $\Lambda = \Lambda$. Uniqueness of trace on $A_\theta$ means that $\bar{A_\theta} = \mathbb{C}$. We obtain $\bar{\Omega_\theta} = \Lambda$.) Operators of multiplication by $\alpha^i$ and (left) derivatives $\partial/\partial \alpha^i$ satisfy canonical anticommutation relation (i.e. specify a representation of Clifford algebra); this means that $\Lambda$ can be considered as a fermionic Fock space. One can prove, that

$$\mu(E) = e^{\frac{i}{4} \frac{\partial}{\partial \alpha} \theta \frac{\partial}{\partial \alpha}} \text{ch}E$$

is an integral element of $\Lambda$ [19]. This element characterizes completely the $K$-theory class of projective module $E$. One can identify the group $K_0(E)$ with the lattice $\Lambda^{\text{even}}(\mathbb{Z})$ of integral even elements of $\Lambda$. The element $\mu(E)$ can be considered as a collection of topological numbers corresponding to the module $E$. The group $K_1(A_\theta)$ can be identified with the lattice $\Lambda^{\text{odd}}(\mathbb{Z})$ of odd integral elements of $\Lambda$. (Talking about even and odd elements of $\Lambda$ we have in mind Grassmann parity.)

If $E = E_0$ (i.e. graded module $E$ has only elements of degree 0) then obviously $\dim E = \text{Tr} 1 = \text{ch}_0 E > 0$. Expressing the zeroth component $\text{ch}_0 E$ of Chern character $\text{ch}E$ in terms of $\mu(E)$ we obtain necessary condition for existence of module obeying $E = E_0$ and having $\mu(E) = \mu$; such a module will be denoted by $E_\mu^\theta$ or simply by $E_\mu$ if $\theta$ is fixed.

If $\theta$ is irrational (i.e. has at least one irrational entry) then this condition is also sufficient [20]. For irrational $\theta$ two projective modules having only elements of degree 0 are isomorphic iff they belong to the same $K$-theory class [20]. This means that the module $E_\mu$ is unique (up to isomorphism).

For simplicity we’ll assume that $\theta$ is irrational. Then every $\mathbb{Z}_2$-graded projective module has a unique representation in the form $E_{\mu_1} + \Pi E_{\mu_2}$.
Let $G$ be an abelian group that can be represented as a direct sum of $\mathbb{R}^p$ and finite group. If $(\gamma, \tilde{\gamma}) \in G \times G^*$ where $G^*$ is the group of characters of $G$ one defines an operator $U_{\gamma, \tilde{\gamma}}$ acting on functions on $G$ by the formula

$$(U_{\gamma, \tilde{\gamma}} f)(x) = \tilde{\gamma}(x) f(x + \gamma).$$

More precisely, we should consider $U_{\gamma, \tilde{\gamma}}$ as operators on the Schwartz space $S(G)$ (or the space of smooth functions on $G$ that tend to zero at infinity faster than any power.)

If $\Gamma$ is a lattice in $G \times G^*$ (i.e. $\Gamma$ is a discrete subgroup of $G \times G^*$ and $G \times G^*/\Gamma$ is compact ) the operators $U_{\gamma, \tilde{\gamma}}$ with $(\gamma, \tilde{\gamma}) \in \Gamma$ specify a projective module over noncommutative torus ; modules of such a kind are called Heisenberg modules [20]. Every Heisenberg module $E$ has a constant curvature connection; this means that $\text{ch} E$ is a quadratic exponent and $\mu(E)$ is a generalized quadratic exponent i.e. a limit of quadratic exponents (see [5] for details). We'll say that a module admitting constant curvature connections is a basic module if it cannot be represented as a direct sum of isomorphic modules. In other words a basic module is a module $E_\mu$ where $\mu$ is a generalized quadratic exponent and $\mu$ cannot be represented in the form $k\mu_0$ where $\mu_0 \in \Lambda^{\text{even}}(\mathbb{Z})$, $k > 1$. One can check that endomorphism algebra $\text{End}_{A_\theta} E_\theta^\mu$ of basic module $E_\mu$ is an algebra of functions on another noncommutative torus. This algebra $A_\hat{\theta}$ is Morita equivalent to the original algebra $A_\theta$. One can consider a basic module $E_\theta^\mu$ as an $(A_\theta, A_\theta)$-bimodule that establishes Morita equivalence between $A_\theta$ and $A_\theta$. We'll denote this bimodule by $E_{\theta, \theta}$. In particular, $E_{\theta, \theta} = A_\theta$ where $A_\theta$ is considered as $(A_\theta, A_\theta)$-bimodule.

One can check that $\hat{\theta} = g(\theta)$ where $g \in SO(d, d, \mathbb{Z})$ [5]. (The group $SO(d, d, \mathbb{R})$ acts in the space of antisymmetric matrices by means of fractional linear transformations $g(\theta) = (A\theta + B)(C\theta + D)^{-1}$. ) More precise notation for Morita equivalence bimodules $E_{\theta, \theta}$ should include the element $g \in SO(d, d, \mathbb{Z})$ connecting $\theta$ and $\hat{\theta}$. The bimodules $E_{\theta, \theta}$ can be equipped with constant curvature connection. Using this connection we can establish gauge Morita equivalence of $A_\theta$ and $A_\hat{\theta}$. One can prove that all gauge Morita equivalences on noncommutative tori are of this kind. In other words, tori $A_\theta$ and $A_\hat{\theta}$ are gauge Morita equivalent iff $\hat{\theta} = g(\theta)$ where $g \in SO(d, d, \mathbb{Z})$ [5].
It is obvious that the tensor product of $E_{\hat{\theta}, \theta}$ and $E_{\theta, \theta'}$ over $A_{\theta}$ is a bimodule that establishes Morita equivalence between $A_{\theta}$ and $A_{\theta'}$; we will write

$$E_{\hat{\theta}, \theta} \otimes_{A_{\theta}} E_{\theta, \theta'} = E_{\hat{\theta}, \theta'}$$

This means, in particular, that one can define natural $E_{\hat{\theta}, \theta'}$-valued pairing between $E_{\hat{\theta}, \theta}$ and $E_{\theta, \theta'}$.

Let us describe the space $\{E_{\mu_1} \to E_{\mu_2}\}$ of $A_{\theta}$-linear maps of basic $A_{\theta}$-module $E_{\mu_1}$ into basic $A_{\theta}$-module $E_{\mu_2}$. We can consider $E_{\mu_1}$ as $(A_{\theta_1}, A_{\theta})$-bimodule $E_{\theta_1, \theta}$ and $E_{\mu_2}$ as $(A_{\theta_2}, A_{\theta})$-bimodule $E_{\theta_2, \theta}$. The space $\{E_{\mu_1} \to E_{\mu_2}\}$ can be considered as $(A_{\theta_2}, A_{\theta_1})$-bimodule. One can check this bimodule establishes Morita equivalence between $A_{\theta_2}$ and $A_{\theta_1}$; this permits us to identify it with $E_{\theta_2, \theta_1}$.

Let us consider now matrices $\theta_1, \ldots, \theta_k$ belonging to the same orbit of $\text{SO}(d,d,\mathbb{Z})$ in the space of antisymmetric $d \times d$ matrices. Corresponding noncommutative tori are Morita equivalent, therefore we can consider bimodules $E_{\theta_i, \theta_j}$. Let us consider a space $A_{\theta_1, \ldots, \theta_k}$ consisting of $k \times k$ matrices where the entry in the $i$-th row and $j$-th column belongs to $E_{\theta_i, \theta_j}$. It follows from (25) that there exists natural pairing between $E_{\theta_i, \theta_j}$ and $E_{\theta_j, \theta_i}$ with values in $E_{\theta_i, \theta_j}$. Using this pairing we define an algebra structure on $A_{\theta_1, \ldots, \theta_k}$.

It follows from the results mentioned above that every algebra $A_{\theta_1, \ldots, \theta_k}$ is Morita equivalent to noncommutative torus and, conversely, every algebra that is Morita equivalent to noncommutative torus is isomorphic to one of algebras $A_{\theta_1, \ldots, \theta_k}$. (Recall, that we assume that the parameter of noncommutativity $\theta$ is an irrational matrix.)

The proof is based on identification of an algebra Morita equivalent to $A_{\theta}$ with $\text{End}_{A_{\theta}} E$ where $E$ is a projective $A_{\theta}$-module and on remark that $E$ can be represented as a direct sum of basic modules [20]. (We used the fact that every projective $A_{\theta}$-module is a generator and therefore can be used to construct Morita equivalence between $A_{\theta}$ and $\text{End}_{A_{\theta}} E$.) It follows from the above statements that the endomorphism algebra of a direct sum of basic modules can be considered as algebra $A_{\theta_1, \ldots, \theta_k}$ and that every algebra $A_{\theta_1, \ldots, \theta_k}$ can be obtained this way.

Notice, that there exist numerous non-trivial isomorphisms between algebras $A_{\theta_1, \ldots, \theta_k}$. This follows from the fact that the relation $\mu_1 + \ldots + \mu_k = \mu'_1 + \ldots + \mu'_l$ implies an isomorphism of $A_{\theta}$-modules $E_{\mu_1} + \ldots + E_{\mu_k}$ and $E_{\mu'_1} + \ldots + E_{\mu'_l}$ (here $E_{\mu_i}$ and $E_{\mu'_j}$ are basic
\(A_\theta\)-modules obeying \(\mu(E_{\mu_i}) = \mu_i, \ \mu(E_{\mu'_j}) = \mu'_j\). Isomorphisms of corresponding endomorphism algebras leads to a conclusion that \(A_{\theta_1,\ldots,\theta_k}\) is isomorphic to \(A_{\theta'_1,\ldots,\theta'_l}\) where \(\theta_i\) and \(\theta'_j\) are defined by the formula

\[
A_{\theta_i} = \text{End}_{A_\theta} E_{\mu_i}, \quad A_{\theta'_j} = \text{End}_{A_\theta} E_{\mu'_j}.
\]

This result can be reformulated in the following way. Let us suppose that \(\theta_i = g_i(\theta), \ \theta'_j = g'_j(\theta)\) and \(\sum g_i(1) = \sum g'_j(1)\) where \(g_i, g'_j\) are elements of the group \(SO(d, d; \mathbb{Z})\) acting on \(\theta\) by means of fractional linear transformations and on elements of \(\Lambda\) (on Fock space) by means of linear canonical transformations (=spinor representation). Then \(A_{\theta_1,\ldots,\theta_k}\) is isomorphic to \(A_{\theta'_1,\ldots,\theta'_l}\).

In the dimensions \(d = 2, 3\) every projective module is isomorphic to the direct sum of identical basic modules. (This follows from the fact that in this dimensions \(\mu(E)\) is always a generalized quadratic exponent, hence every module admits a constant curvature connection). We obtain that in the dimensions 2, 3 every algebra that is Morita equivalent to \(A_\theta\) is isomorphic to matrix algebra \(\text{Mat}_n(A_{\hat{\theta}})\) where \(\hat{\theta} = g(\theta), \ g \in SO(d, d; \mathbb{Z})\).

Notice, that bimodules \(E_{\hat{\theta},\theta}\) and algebras \(A_{\theta_1,\ldots,\theta_k}\) have a nice physical interpretation. It was shown in [8] that \(E_{\hat{\theta},\theta}\) can be interpreted as a state space of strings connecting two \(D\)-branes carrying gauge theories corresponding to noncommutative tori \(A_{\hat{\theta}}, A_\theta\). One can say that \(A_{\theta_1,\ldots,\theta_k}\) is an algebra of string states in presence of \(k\) \(D\)-branes. (It was shown in [8] that different boundary conditions for given theory in the bulk correspond to Morita equivalent algebras. In our case we have \(k\) Morita equivalent tori.)

A connection on \(A_\theta\)-module \(E\) can be identified with a connection on \(\Omega_\theta\)-module \(E \otimes_{A_\theta} \Omega_\theta\) where \(\Omega_\theta = \Omega(L, A_\theta)\) denotes a differential extension of \(A_\theta\) corresponding abelian Lie algebra \(L\) of derivations of \(A_\theta\). Gauge Morita equivalence of \(A_\theta\) and \(A_{\hat{\theta}}\) implies Morita equivalence of \(Q\)-algebras \(\Omega_\theta, \Omega_{\hat{\theta}}\) and physical equivalence of corresponding gauge theories. We can generalize this statement considering \(Q\)-extension \(\Omega_{\theta_1,\ldots,\theta_k}\) of the algebra \(A_{\theta_1,\ldots,\theta_k}\). The definition of the space \(\Omega_{\theta_1,\ldots,\theta_k}\) repeats the definition of \(A_{\theta_1,\ldots,\theta_k}\), but instead of bimodules \(E_{\theta_i,\theta_j}\) we should use bimodules \(\mathcal{E}_{\theta_i,\theta_j}\) that establish Morita equivalence between \(\Omega_{\theta_i}\) and \(\Omega_{\theta_j}\). Using the natural \(\mathcal{E}_{\theta_i,\theta_j}\)-valued pairing between \(\mathcal{E}_{\theta_i,\theta_j}\) and \(\mathcal{E}_{\theta_j,\theta_i}\) we introduce an algebra structure on \(\Omega_{\theta_1,\ldots,\theta_k}\). One can consider \(\Omega_{\theta_1,\ldots,\theta_k}\) as a \(Q\)-algebra introducing
Q as an operator acting on every bimodule \( E_{\theta_i, \theta_j} \), as constant curvature connection. (Such a connection is determined uniquely up to an additive constant.) It is easy to check that \( Q^2 x = [\omega, x] \), where \( \omega = \text{diag}(\omega_1, \ldots, \omega_k) \), where \( \omega_i \in E_{\theta_i, \theta_i} = \Omega_{\theta_i} \) is a scalar (an element of the form const·1). It follows from the above consideration that \( \Omega_{\theta_1, \ldots, \theta_k} \) is Morita equivalent to \( \Omega_{\theta_i} \) as a \( Q \)-algebra. This means that the gauge theory based on the \( Q \)-algebra \( \Omega_{\theta_1, \ldots, \theta_k} \) is physically equivalent to the gauge theory on noncommutative torus.

In the case of two-dimensional noncommutative tori we can make the consideration above more explicit using the results of [22] (see also [23] for more complete formulas and some applications). In this case we can work with the group \( SL(2, \mathbb{Z}) \) instead of \( SO(2, 2, \mathbb{Z}) \).

Representing a \( 2 \times 2 \) antisymmetric matrix as

\[
\begin{pmatrix}
0 & \theta \\
-\theta & 0
\end{pmatrix}
\]

we consider the noncommutativity parameter as a real number \( \theta \); the group \( SL(2, \mathbb{Z}) \) acts on \( \theta \) by means of fractional linear transformations. A basic bimodule \( E_{\hat{\theta}, \theta} \) can be realized in the space \( S(\mathbb{R} \times \mathbb{Z}_m) \) by means of operators

\[
(U_1 f)(x, \mu) = f(x - r, \mu - \rho),
\]

\[
(U_2 f)(x, \mu) = \exp[2\pi i (tx - s\mu)] f(x, \mu),
\]

\[
(Z_1 f)(x, \mu) = f(x - r', \mu - \rho'),
\]

\[
(Z_2 f)(x, \mu) = \exp[2\pi i (t'x - s'\mu)] f(x, \mu).
\]

Here \( \hat{\theta} = \frac{b + a \theta}{m + n \theta}, \ x \in \mathbb{R}, \ \mu \in \mathbb{Z}_m \), the numbers \( r, t, s, r', t', s' \in \mathbb{R} \) and \( \rho, \rho' \in \mathbb{Z}_m \) are chosen in such a way that the operators \( U_1, U_2 \) obey \( U_1 U_2 = e^{-2\pi i \hat{\theta}} U_2 U_1 \) and therefore specify a right \( A_{\hat{\theta}} \)-module \( E_{n, m}(\theta) \), the operators \( Z_1, Z_2 \), commuting with \( U_1, U_2 \), obey \( Z_1 Z_2 = e^{2\pi i \hat{\theta}} Z_2 Z_1 \) and specify a left \( A_{\hat{\theta}} \)-module.

We take \( r = \frac{1}{m}, \rho = -n, \ t = n + m \theta, \ s = -\frac{1}{m}, \ r' = \frac{1}{m(n + m \theta)}, \rho' = -1, \ t' = 1, \ s' = -\frac{a}{m} \).

The natural bilinear \( E_{\hat{\theta}, \theta} \)-valued pairing between \( E_{\hat{\theta}, \theta} \) and \( E_{\hat{\theta}, \theta'} \) sends a pair \((f, g)\) into

\[
h(x, \Delta) = \sum_{q \in \mathbb{Z}} f(\tilde{A}x + Bq + C\Delta, Dq + E) g(\tilde{A}x + \tilde{B}q + \tilde{C}\Delta, \tilde{D}q + \tilde{E}).
\]

(30)
Here \( \hat{\theta} = (b + a\theta)(n + m\theta)^{-1} \), \( \theta = (\beta + \alpha\theta')(k + l\theta')^{-1} \), the bimodule \( E_{\hat{\theta},\theta'} \) consists of functions of \( x \in \mathbb{R} \), \( \Delta \in \mathbb{Z}_{nl+m\alpha} \) (it is isomorphic to \( E_{bk+aa,nl+m\alpha}(\theta') \) as \( A_{\theta'} \)-module). The coefficients in (30) are given by the formulas:

\[
A = 1, \quad B = \frac{1}{m}, \quad C = -\frac{l}{m(nl + m\alpha)}, \quad D = -n, \quad E = 1,
\]

\[
\tilde{A} = n + m\theta, \quad \tilde{B} = \frac{l\theta - \alpha}{l}, \quad \tilde{C} = \frac{\alpha - l\theta}{nl + m\alpha}, \quad \tilde{D} = 1, \quad \tilde{E} = 0.
\]

Remark. Our considerations were not completely rigorous in the following relation. The bilinear \( E_{\hat{\theta},\theta'}\)-valued pairing, \( \rho_{\hat{\theta},\theta'} \) between \( E_{\hat{\theta},\theta'} \) and \( E_{\theta,\theta'} \) is not completely canonical; it is defined only up to a constant factor. (Multiplication by a constant can be considered as an automorphism of a bimodule.) This means that the identification (25) is possible, but not unique. In principle, associativity of tensor product can be violated:

\[
\rho_{\hat{\theta},\theta'}(\rho_{\hat{\theta},\theta'} \otimes 1_{\theta',\theta''}) = \text{const} \cdot \rho_{\hat{\theta},\theta'}(1_{\hat{\theta},\theta} \otimes \rho_{\theta,\theta'}), \quad (31)
\]

where the constant is not necessarily equal to 1. It is easy to see that this problem does not appear in two-dimensional case: defining the pairing by (30) we obtain (31) with \( \text{const} = 1 \) (see [23] for detailed calculation). This statement answers a question asked by Yu. Manin in [24]. Moreover in any dimension the constant in (31) is equal to 1 for the appropriate choice of pairing. (We implicitly assume that such a choice is made; only with such a choice we can say that the algebra \( A_{\theta_1,...,\theta_k} \) is associative.) The absence of appropriate choice would lead to unremovable non-associativity of the algebra \( A_{\theta_1,...,\theta_k} \).

From the other side we know that an associative algebra \( \text{End}_{A_{\theta}} E \) of endomorphisms of direct sum \( E \) of basic modules can be interpreted as \( A_{\theta_1,...,\theta_k} \).

**Conclusion.**

In present paper we generalized the theory of connections on modules over associative algebra. We embedded this theory into the theory of connections on modules over \( Q \)-algebras and proved a general duality theorem in this framework. Namely, we proved that
under certain conditions there exists one-to-one correspondence between connections on modules over one $Q$-algebra and connections on modules over another $Q$-algebra and found relation between corresponding curvatures. More precisely, it follows from our results that under certain conditions gauge theory constructed by means of $Q$-algebra $\Omega$ is equivalent to gauge theory corresponding to the $Q$-algebra $\text{End}_\Omega \mathcal{E}$ where $\mathcal{E}$ is an $\Omega$-module equipped with a connection. (We use the fact that every connection determines a structure of $Q$-algebra on the algebra of endomorphisms.) This theorem can be applied to many concrete situations; we gave an example in Sec. 8.

We have shown that $Q$-algebras appear naturally in Fedosov’s quantization of symplectic manifolds and conjectured that they can be used to circumvent the problems arising in the attempts to quantize symplectic manifolds beyond the framework of perturbation theory.

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