On a supersymmetric completion of the $R^4$ term in IIB supergravity

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Abstract

We analyze the possibility of constructing a supersymmetric invariant that contains the $R^4$ term among its components as a superpotential term in type IIB on-shell superspace. We consider a scalar superpotential, i.e. an arbitrary holomorphic function of a chiral scalar superfield. In general, IIB superspace does not allow for the existence of chiral superfields, but the obstruction vanishes for a specific superfield, the dilaton superfield. This superfield contains all fields of type IIB supergravity among its components, and its existence is implied by the solution of the Bianchi identities. The construction requires the existence of an appropriate chiral measure, and we find an obstruction to the existence of such a measure. The obstruction is closely related to the obstruction for the existence of chiral superfields and is non-linear in the fields. These results imply that the IIB superinvariant related to the $R^4$ term is not associated with a scalar chiral superpotential.
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1 Introduction and summary of results

The effective description of massless modes of string theories at low energy is given to leading order by supergravity theories. Worldsheet and string loops introduce higher derivative corrections to the leading order supergravity theory. These terms represent the leading quantum effects of string theory and as such are of particular importance. In particular, one may test duality symmetries beyond the leading order by considering these terms. For example in the AdS/CFT duality, the leading higher derivative corrections of IIB string theory are related to subleading terms in the $1/N$ and the 't Hooft coupling expansion in the boundary theory. Furthermore, it is of interest to compute stringy corrections to supergravity solutions. For instance, one can compute stringy corrections to black hole solutions and their properties, such as their mass and their entropy. This would be the leading quantum gravity effects to the semi-classical results. Another motivation for studying higher derivative corrections is that they may allow to circumvent no-go theorems about de Sitter compactifications. Furthermore, they may lead to stabilization of moduli in compactifications that to leading order yield no-scale supergravities. All of these applications require a detailed knowledge of the leading higher derivative corrections.

The higher derivative corrections can be computed systematically by either computing scattering amplitudes [1]-[9] or by using sigma model techniques [10]-[16]. Despite considerable work, however, the complete set of the leading higher derivative terms is still missing. One way to extend the results in the literature is to use the symmetries of the theory under consideration. One such symmetry is supersymmetry. Starting from a given term in the leading order quantum corrections one may consider its orbit under supersymmetry. This procedure, although straightforward, is rather challenging from the technical point of view, and it has been carried out only in a few cases. Explicit results have been obtained for the heterotic effective action in [17, 18, 19, 20] in a component formalism and in [21, 22, 23] in superspace. The extension of these results to the other string theories as well as M theory has been discussed in the literature, see [24] and references therein. A complete supersymmetry analysis, however, as well as the complete set of terms that appear at leading order is still lacking.

One of the most elegant ways to construct superinvariants is to use superspace techniques. In this approach one is aiming in providing a superspace formula for the higher derivative corrections. Upon evaluating the fermionic integration, the result should contain the terms obtained by sigma model and/or scattering amplitude computations. This approach automatically provides all terms that are related to each other by supersymmetry.

The leading higher derivative corrections in IIB string theory are eight derivative terms and contain the well-known $R^4$ term [1]. Dimensional analysis suggests that a superinvariant associated with eight derivative terms may be constructed by integrating a superpotential over half of superspace. Type IIB superspace was constructed by Howe and West in [25]. The Bianchi identities imply the equations of motion, so this is an on-shell superspace. This is not a disadvantage, however. The freedom to do field redefinitions implies that the higher derivative terms are ambiguous up to lower order field equations, and by using on-shell fields we precisely mod out by this ambiguity.\footnote{On-shell (and off-shell) superinvariants were also extensively discussed in the supergravity literature in the context of possible counterterms, see [26]-[35] for an (incomplete) list of references.}

We should mention here that another manifestly supersymmetric way to study the
higher derivative terms is to relax some of the supergravity constraints when solving the Bianchi identities. As mentioned above, the solutions of the Bianchi’s in IIB supergravity imply the field equations. The equations may admit more general solutions when some of the conventional constraints are relaxed, and one might hope that the relaxed constraints imply the $\alpha'$-corrected field equations. Such an approach for the case of M-theory is followed in [36, 37], see also [38] for a review and further references.

In [25] Howe and West presented a linearized superfield that satisfies a “chirality” constraint (see section 4 for a discussion of our terminology), has as its leading component the physical (complex) scalar of the IIB supergravity, and contains in its components all fields of type IIB supergravity. Chiral superfields do not exist in general curved superspaces. We show, however, that type IIB superspace allows for a non-linear version of the linearized superfield of Howe and West. We call this superfield the dilaton superfield $V$.

Utilizing the solution of the Bianchi identities given in [25], we present iterative formulas for all its components. We then investigate the construction of a superinvariant as an integral over half of superspace of a arbitrary scalar function of the dilaton superfield. This choice is motivated by previous work of Green and collaborators, see [39] and references therein, where a similar construction in terms of the linearized superfield of Howe and West was advocated. This is also the simplest choice for a would-be superinvariant. Integration in superspace is a non-trivial operation: in supergravity theories the metric transforms, so one needs to obtain an appropriate supersymmetric version of the measure. Moreover in our case the measure should be appropriate for chiral superfields. We are thus looking for a superfield $\Delta$ whose leading component is $e$, where $e$ is the determinant of the vielbein, and is such that when we integrate over half of superspace an arbitrary holomorphic function $W[V]$ of the dilaton superfield, the resulting action is supersymmetric.

The existence of the measure is analyzed by studying the constraints imposed by supersymmetry. We show that one can systematically study the cancellation of $D^n W$ in the supersymmetry variation of the action starting from the terms with the highest $n$ and moving to lower orders. ($D$ is a supercovariant derivatives and $A|$ denotes evaluation of $A$ at $\theta = 0$.) Since the supersymmetry parameter $\zeta$ is complex, each $n$ leads to two different conditions. We show that the cancellation of the terms proportional $\sim \zeta$ uniquely determines all components of the measure $\Delta$. The terms proportional to $\zeta^*$ should cancel automatically. These are non-trivial conditions, and it turns out that the cancellation does occur at leading order and at the linearized level at next-to-leading order, but there is an obstruction to the existence of the chiral measure at the non-linear level at next-to-leading order. The obstruction is very closely related to the obstruction to the existence of chiral superfields. This proves that there is no superinvariant that that can be expressed as a scalar superpotential of the dilaton superfield.

It turns out that the superpotential term constructed using the measure $\Delta$ determined by the cancellation of the $\zeta$ terms has rather intriguing properties, even though it is not supersymmetric. In the remainder we discuss these properties.

IIB supergravity is invariant under $SL(2, R)$. In string perturbation theory the $SL(2, R)$ symmetry is broken by the vev of the dilaton, but it is believed that quantum theory possesses a local $SL(2, Z)$ symmetry. It is thus of interest to analyze the $SL(2, R)$ transform-
mation properties of the superpotential. To this end, we show that the chirality condition commutes with $SL(2, R)$, so the superpotential is compatible with $SL(2, R)$.

Given that we know all components of the dilaton superfield, it is straightforward but tedious to obtain the component form of the superpotential. We discuss in detail a few selected terms. In particular, we show that the action (in the Einstein frame) contains the terms (schematically)

$$S_{3} = \alpha'^{3} \int d^{10} x \epsilon \left( t^{(12,-12)}(\tau, \tau^*) \lambda^{16} + t^{(11,-11)}(\tau, \tau^*) \psi^* \lambda^{15} + \cdots + t^{(0,0)}(\tau, \tau^*) R^{4} + \cdots \right),$$

(1.1)

where $\tau$ is the dilaton-axion, $\lambda$ is the dilatino and $\psi$ is the gravitino. The $R^{4}$ term is given in (4.20) and it contains the well-known $R^{4}$ term, i.e. the contraction of the indices is exactly that of the $R^{4}$ term that arises in string theory. The coefficients $t^{(12,-12)}(\tau, \tau^*)$, $t^{(11,-11)}(\tau, \tau^*)$ and $t^{(0,0)}(\tau, \tau^*)$ are functions of the superpotential $W$ and its derivatives. We show by direct computation that each $t^{(w,-w)}$ is an eigenfunction of the $SL(2, R)$ Laplacian acting on $(w, -w)$ forms$^{6}$. Moreover, they are related to each other by the application of modular covariant derivatives. The relations between the modular forms can be understood as simple relations arising from the fact that they are derived from the same superpotential.

The eigenvalue of $t^{(0,0)}(\tau, \tau^*)$ under the action of the $SL(2, R)$ Laplacian turns out to be equal to 20. This eigenvalue was computed both directly, i.e. by acting on $t^{(0,0)}(\tau, \tau^*)$ with the $SL(2, R)$ Laplacian, and also indirectly by its relation to the eigenvalues of $t^{(12,-12)}(\tau, \tau^*)$ and $t^{(11,-11)}(\tau, \tau^*)$. Imposing $SL(2, Z)$ symmetry uniquely fixes $t^{(0,0)}(\tau, \tau^*)$ to be the Eisenstein non-holomorphic modular function $E_{5}(\tau, \tau^*)$. The asymptotics of $E_{5}(\tau, \tau^*)$ as $\tau_{2} \rightarrow \infty$ is $\tau_{2}^{5}$ and $\tau_{2}^{-4}$ plus exponentially suppressed terms$^{7}$. This implies that the string frame effective action at “weak coupling” contains the terms,

$$S = \int d^{10} x \sqrt{|g|} e^{-2\phi} R + ... + \alpha'^{3} (c_{1} e^{-\frac{4\psi}{g_{s}}} + c_{2} e^{\frac{2\phi}{g_{s}}}) R^{4} + ...$$

(1.2)

where $c_{1}$ and $c_{2}$ are non-zero numerical constants. This asymptotic behavior is not consistent with IIB string theory (as we know it). In closed string perturbation theory the leading behavior is $g_{s}^{-2}$ and only even powers of $g_{s}$ appear. The leading behavior in our case is more singular than the string tree-level contribution and is half-integral. Moreover, the difference of the two “perturbative” contributions is an odd power of $g_{s}$, i.e. $g_{s}^{9}$, so even if one would normalize by hand the leading power to be one (which $SL(2, Z)$ does not allow), the resulting series would still be inconsistent with closed string perturbation theory. Open string loops can give odd powers of $g_{s}$ but in our computation there are no open strings.

One should contrast these results to other results reported in the literature. In [41] Green and Gutperle conjectured that the coefficient of the $R^{4}$ term that arises in string theory is the non-holomorphic Eisenstein series $E_{3/2}$. The asymptotics of this modular function is consistent with tree-level and one-loop contributions and implies that there are no further perturbative contributions. Supporting evidence for this conjecture as well as related works appeared in [42]-[56], see [39] for a review. In particular, Berkovits constructed in [48] an invariant that contains the $R^{4}$ term in $N = 2 d = 8$ linearized

$^{6}$A $(p, q)$ form $t^{(p,q)}$ transforms as $t^{(p,q)}(\tau, \tau^*) \rightarrow t^{(p,q)}(\tau, \tau^*) (\gamma \tau + \delta)^{p} (\gamma \tau^* + \delta)^{q}$ under $\tau \rightarrow (\alpha \tau + \beta)/(\gamma \tau + \delta)$. 

$^{7}$The powers in the asymptotic terms are directly linked to the eigenvalue of Laplacian. A short computation shows that $\nabla_{SL(2)} \tau_{2}^{p} = s(s-1) \tau_{2}^{p}$, so with eigenvalue 20, one has that each of $\tau_{2}^{5}$ and $\tau_{2}^{-4}$ are eigenfunctions.
superspace. This theory can be viewed as a $T^2$ reduction of type IIB supergravity. He also showed that it contains exactly tree-level and one-loop contributions. Pioline in [50] showed that the superinvariant constructed in the 8$d$ linearized superspace is an eigenmode of the Laplacian with eigenvalue that implies that the coefficient of the $R^4$ term is $E_{3/2}$. Finally, Green and Sethi [45] analyzed the supersymmetry constraints on the coefficients of $\lambda^{16}$ and $\psi^*\lambda^{15}$ terms. Their analysis involved specific $\alpha'$ corrections to the supersymmetry rules and with these corrections they showed that the coefficient of the $R^4$ term is the Eisenstein series $E_{3/2}$.

As discussed, the superpotential term in our case is not supersymmetric, so our results are not in conflict with existing results. Our results, however, imply that the tree- and one-loop $R^4$ terms are not associated with a scalar superpotential term, even at the linearized level. Notice that our results do not rule out that the superpotential term is invariant under linearized supersymmetry – the obstruction is non-linear in the fields. It is the fact that we get $E_5$ rather than $E_{3/2}$ as a coefficient of the $R^4$ term that implies that the superpotential term is not associated with the stringy $R^4$ term. Notice that these considerations are consistent with the discussion in [9] where it was argued that certain terms obtained by string amplitude computations cannot be part of the superinvariant based on the linearized superfield of Howe and West.

In this paper we investigate whether the superinvariant associated with the $R^4$ term can be constructed as an integral over half of superspace of a scalar superpotential. Even though the answer turned out to be negative, we believe that the techniques developed here will be useful for the construction of the actual superinvariant associated with the $R^4$ term. Our analysis was specific to IIB supergravity. The method, however, is general and we expect that similar constructions apply to the other string theories and to M-theory. The superspace for type IIA theory has been constructed in [57], of type I supergravity coupled to Yang-Mills in [58]-[63], and of eleven dimensional supergravity in [64]-[67]. A construction similar to ours for the superinvariant associated with the $R^4$ terms in type I theories has been discussed in [21]. Although the details in all these cases will be different, we expect that one has to go through the same steps we present here. Keeping this in mind, we shall present in some detail the development of the tools required in order to carry out the computations.

This paper is organized as follows. In the next section we discuss the set up of the computation. In particular, we discuss the issue of working with on-shell fields. In section 3 we review type IIB supergravity both in components and in superspace. Chiral superfields and in particular the construction of the dilaton superfield and its components are discussed in section 4. The (non-existence of the) chiral measure is discussed in section 5. Section 6 presents the computation of the components of the superpotential as well as the $SL(2, R)$ transformation properties. We end with the discussion of our results and of future directions.

An effort is made to make this paper self-contained. Several appendices summarize relevant results from the literature. In appendix A we discuss our conventions, in appendix B we summarize the solution of the Bianchi identities, and in appendix C we provide the supersymmetry rules of the type IIB supergravity. Appendices D and E present details of computations used in the main text. In particular, in appendix D we discuss the $F_5$ dependence of the dilaton superfield and in appendix E we show that the superpotential contains the well-known $R^4$ term.
2 Symmetries of the effective action

We discuss in this section the constraints imposed on low energy effective actions by symmetries. Recall that the low energy effective actions are expressed as an expansion in derivatives (with the dimension of various fields properly taken into account). In string theories the various terms are weighted by different powers of $\alpha'$. The effective action has thus the form

$$S = S_0 + \sum_{n \geq 3} \alpha^n S_n ,$$  \hspace{1cm} (2.1)

where we take the sum to start at $n = 3$ because in the type IIB string theory this is the leading correction.

Let us consider now a symmetry $\delta_0$ of $S_0$. The symmetry transformation may also receive corrections,

$$\delta = \delta_0 + \sum_{n \geq 3} \alpha^n \delta_n .$$  \hspace{1cm} (2.2)

Invariance of the effective action, $\delta S = 0$, then implies

$$\delta_0 S_0 = 0$$
$$\delta_0 S_3 + \delta_3 S_0 = 0 ,$$  \hspace{1cm} (2.3)

etc. This procedure constrains both the possible terms $S_3$ and also the possible deformations of the symmetry $\delta_3$.

The higher derivative terms, however, are ambiguous due to field redefinitions [68, 1, 2]. Let us call collectively $\phi^I$ all the fields. Then the field redefinition,

$$\phi^I \rightarrow \phi^I + \alpha^3 f^I(\phi^J)$$  \hspace{1cm} (2.4)

where $f^I(\phi^J)$ an arbi~tary non-singular function of $\phi^J$, implies that

$$S_3 \rightarrow S_3 + f^I \frac{\partial S_0}{\partial \phi^I} .$$  \hspace{1cm} (2.5)

Thus, $S_3$ is ambiguous up to lowest order field equations. To eliminate this ambiguity one may work with on-shell fields.

We now show that instead of solving (2.3), one may equivalently solve

$$\delta_0 S_3 \approx 0 ,$$  \hspace{1cm} (2.6)

where $\approx$ means that the equality is up to the lowest order field equations (i.e. the field equations that follow from $S_0$). Indeed, any solution of (2.3) is also a solution of (2.6). To see this we use the chain rule to rewrite (2.3) as

$$\delta_0 S_3 = - \frac{\delta S_0}{\delta \phi^I} \delta_3 \phi^I \Rightarrow \delta_0 S_3 \approx 0 .$$  \hspace{1cm} (2.7)

Conversely, for any solution of (2.6) there exists some $\delta_3$ such that (2.3) is satisfied. To see this we note that (2.6) means that

$$\delta_0 S_3 = g^I_3 \frac{\delta S_0}{\delta \phi^I}$$  \hspace{1cm} (2.8)
for some function $g^I$ of all fields. Now let us consider the following deformation of $\delta_0$

$$
\delta_3 \phi^I = -g^I. \tag{2.9}
$$

Then (2.8) implies that (2.6) is satisfied. We thus established that

$$(\delta_0 + \alpha^3 \delta_3)(S_0 + \alpha^3 S_3) = \mathcal{O}(\alpha^4) \tag{2.10}$$

with $\delta_3$ given by (2.9). This then implies

$$
[\delta, \delta] S = \mathcal{O}(\alpha^4). \tag{2.11}
$$

Thus $[\delta, \delta]$ is a symmetry of the action and the algebra of $\delta$ closes up to this order (in the absence of auxiliary fields the algebra will only close on-shell). This finishes the proof that one may consider either (2.3) or (2.6).

The above discussion also gives a prescription for calculating explicitly the deformation of the symmetry. One first solves the problem (2.6) in terms of on-shell fields. After $S_3$ is determined, one computes $\delta_0 S_3$ with the on-shell condition relaxed, and finally reads off $\delta_3 \phi^I$ from (2.8) and (2.9).

Notice also that there may be more than one (inequivalent) solutions of the problem (2.3) (or (2.6)). In other words, there may be different pairs $\delta_3, S_3$ that satisfy (2.3).

Because of the freedom of field redefinitions not all $\delta_3 \phi^I$ are non-trivial. Indeed some of them may be removed by a field redefinition. By this we mean that after the field redefinition (2.4) the action will be supersymmetric (i.e. (2.3) will hold) without the need for $\delta_3 \phi^I$. A condition for this to happen is that the variation is of the form

$$
\delta_3(\epsilon) \phi^I = \mathcal{L}_{\delta^I}(\delta_0(\epsilon) \phi^I) + \delta_0(\epsilon') \phi^I
= -f^I \partial^J (\delta_0(\epsilon) \phi^J) + \partial^J f^I (\delta_0(\epsilon) \phi^J) + \delta_0(\epsilon') \phi^I, \tag{2.12}
$$

where $\epsilon$ is the parameter of the variation, $\epsilon'$ is a possibly field dependent parameter, $\mathcal{L}_{\delta^I}$ is the Lie derivative in the space of fields, and $\partial_I = \delta/\delta \phi^I$.

### 3 Review of IIB supergravity

#### 3.1 Component formulation

Type IIB supergravity was constructed in [69, 25]. The bosonic field content consists of the metric $g_{mn}$, a complex antisymmetric two-form gauge field $a_{mn}$, a real four-form gauge field $a_{nrst}$ with a self-dual field strength (at the linearized level), and a complex scalar $a$. The fermions consist of a complex gravitino $\psi_m$ of negative chirality and a complex dilatino $\lambda$ of positive chirality (our conventions are given in appendix A)

$$
\Gamma_{11} \psi_m = -\psi_m, \quad \Gamma_{11} \lambda = \lambda. \tag{3.1}
$$

The covariant field equations are invariant under an $SU(1, 1)$ global symmetry which is realized non-linearly on the scalars. To realize the $SU(1, 1)$ linearly we add an auxiliary scalar and an extra $U(1)$ gauge invariance [70]. The extra scalar may be eliminated by fixing the $U(1)$ gauge invariance. We shall work with the gauge invariant formulation throughout this paper.
The scalars parametrize the coset space $SU(1,1)/U(1)$. We represent them as an $SU(1,1)$ group matrix,

$$\mathbf{V} = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix}$$  \hspace{1cm} (3.2)

where $uu^* - vv^* = 1$. The global $SU(1,1)$ acts by a left multiplication, and the local $U(1)$ by matrix multiplication from the right,

$$\mathbf{V}' = \begin{pmatrix} z & w \\ w^* & z^* \end{pmatrix} \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix} \begin{pmatrix} e^{-i\Sigma} & 0 \\ 0 & e^{i\Sigma} \end{pmatrix}.$$  \hspace{1cm} (3.3)

The components $v$ and $u^*$ have $U(1)$ charge $U = 1$ and $u$ and $v^*$ charge $U = -1$. It follows that the (complex) ratio

$$a = \frac{v}{u^*}$$  \hspace{1cm} (3.4)

is gauge invariant and represents the physical scalars of the theory. Under $SU(1,1)$ it transforms with a linear fractional transformation,

$$a' = \frac{za + w}{w^*a + z^*}.$$  \hspace{1cm} (3.5)

The complex scalar $a$ parametrizes the unit disc. The axion and dilaton of string theory are related to $a$ by a (non-linear) transformation that we describe below.

The metric and the four-form are inert under $SU(1,1)$ and neutral under $U(1)$. The antisymmetric tensor is neutral under $U(1)$, and transforms as a doublet under $SU(1,1)$, where the two components of the corresponding column vector are $a_{mn}^*$ and $a_{mn}$. The gravitino has charge $U = 1/2$ and the dilatino has $U = 3/2$. Both of them are inert under $SU(1,1)$. Finally the supersymmetry parameter $\zeta$ is neutral with respect to $SU(1,1)$ and has $U = 1/2$.

It will also be useful to discuss the transformation properties of the fields under the $U(1)$ subgroup of $SU(1,1)$. These transformations can be obtained from the $SU(1,1)$ transformations by setting $z = \exp i\sigma, w = 0$. In particular, $u$ and $v$ have charge 1 and $u^*$ and $v^*$ have charge -1. It follows that the physical scalar $a$ has charge 2. Notice that $a$ is invariant under the local $U(1)$ symmetry but transforms under the $U(1)$ subgroup of $SU(1,1)$. The dilatino and gravitino have charge $3/2$ and $1/2$, respectively. We summarize the results about the two $U(1)$ charges in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$g_{mn}$</th>
<th>$\psi_m$</th>
<th>$\lambda$</th>
<th>$a_{mn}$</th>
<th>$a_{mors}$</th>
<th>$u$</th>
<th>$v$</th>
<th>$u^*$</th>
<th>$v^*$</th>
<th>$a$</th>
<th>$\zeta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>local $U(1)$</td>
<td>0</td>
<td>1/2</td>
<td>3/2</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>1/2</td>
</tr>
<tr>
<td>global $U(1)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 1: The charges of the fields under the local $U(1)$ and the $U(1)$ subgroup of $SU(1,1)$.

The field equations are constructed using $SU(1,1)$ invariant combinations of the scalars and the 2-form $a_{mn}$. The 1-forms

$$p = u^*dv - vdu^*, \hspace{1cm} q = \frac{1}{2\ell}(u^*du - vdv^*)$$  \hspace{1cm} (3.6)
are $SU(1, 1)$ invariant. The composite field $q$ transforms as a connection under the local $U(1)$, and $p$ has charge 2. Let
\[ F = da_2 \]
be the field strength associated with $a_{mn}$. The $SU(1, 1)$ invariant field strengths can be constructed as
\[ (f_3^*, f_3) = (F^*, F) \]
We finally introduce
\[ f_5 = da_4 - 2i(a^*_2 \wedge F - a_2 \wedge F^*) \]
As mentioned above, we will work with the gauge invariant formulation throughout (except in section 6). We briefly discuss here gauge fixing. More details can be found in [69] (where a different realization is used). An explicit realization of $V$ is given by
\[ V = \frac{1}{\sqrt{1 - aa^*}} \begin{pmatrix} e^{-i\phi} & ae^{i\phi} \\ a^*e^{-i\phi} & e^{i\phi} \end{pmatrix}. \]
The local $U(1)$ symmetry leaves invariant $a_2$ and acts by shifting $\phi$. It follows that one may gauge fix the $U(1)$ symmetry by setting $\phi$ equal to some function of $a$, see (6.20) for the gauge fixing we will use later. $SU(1, 1)$ and supersymmetry transformations do not respect this gauge, so compensating $U(1)$ transformations are necessary. In particular, this implies that the fermions that were $SU(1, 1)$ invariant in the gauge invariant formulation, transform after gauge fixing. The compensating transformations complicate the computations. We therefore choose to work throughout in the gauge invariant formulation. Only in section 6, where we will compare our results with results in the literature, we will gauge fix the $U(1)$.

In our discussion so far we took the scalars to parametrize the $SU(1, 1)/U(1)$ coset space. From the string theory point of view, it is more appropriate to consider the (equivalent) description where the scalars $\tau$ parametrize the $SL(2, \mathbb{R})/SO(2)$ coset. To go from one description to another we note that the $a$ parametrizes the unit disc, whereas $\tau$ the Poincare upper half plane. The transformation from one to another is given by
\[ \tau = i\frac{1-a}{1+a}. \]
The fact that $\tau$ rather than $a$ is related to the axion $C_0$ and dilaton $\varphi$ of type IIb string theory by $\tau = \tau_1 + i\tau_2 = C_0 + i \exp(-\varphi)$ is explained, for instance, in [71]. The $SU(1, 1)$ transformation given in (3.3) is related to an $SL(2, \mathbb{R})$ transformation by
\[ \alpha = z_1 - w_1, \quad \beta = z_2 + w_2, \]
\[ \gamma = -z_2 + w_2, \quad \delta = z_1 + w_1, \]
where $z = z_1 + iz_2, w = w_1 + iw_2$. Using these results one can go from one formulation to another. For example, $\tau$ transforms as
\[ \tau' = \frac{\alpha \tau + \beta}{\gamma \tau + \delta}. \]
Notice also that $\tau$ does not transform linearly under the $SO(2)$ subgroup of $SL(2, \mathbb{R})$ (the $SO(2)$ subgroup is generated by $\alpha = \delta = \cos \sigma, \gamma = -\beta = \sin \sigma$). The infinitesimal transformation is given by
\[ \delta \tau = \sigma(1 - \tau^2). \]
In contrast $a$ transforms linearly under the same $SO(2)$ (it has charge 2, as we explained).
3.2 Superspace formulation

3.2.1 General set-up

IIB superspace was constructed in [25]. Our superspace conventions are the ones of [25] and are summarized in appendix A. We denote the superspace coordinates by \( z^M = (x^m, \theta^\mu, \bar{\theta}^{\bar{\mu}}) \), where \( x^m \) are the spacetime coordinates and \( \theta^\mu \) is a complex 16-component Weyl spinor of \( SO(9,1) \) and \( \bar{\theta}^{\bar{\mu}} \) is its complex conjugate. For every \( x \)-space field we introduce a superfield whose leading component is the \( x \)-space field. We will denote the superfield with the same letter as the \( x \)-field but capitalized. The \( SU(1,1) \) group matrix \( V \) becomes a superfield, and the scalar fields \( u \) and \( v \) are the lowest components of the superfields \( U \) and \( V \), respectively. They satisfy

\[
UU^* - VV^* = 1. \tag{3.15}
\]

We also define the superspace version of (3.6),

\[
P = U^*dV - VdU, \quad Q = \frac{1}{2i}(U^*dU - VdV^*), \tag{3.16}
\]

where we use a form notation. It will be useful to give these in components as well. For the composite connection we get

\[
Q_A = \frac{1}{2i}(U^*\partial_A U - V\partial_A V^*) = -\frac{1}{2i}(U\partial_A U^* - V^*\partial_A V), \tag{3.17}
\]

where in the second equality we used (3.15). Using the definition of the covariant derivative,

\[
D_A V = \partial_A V + 2iQ_A V, \quad D_A U = \partial_A U - 2iQ_A U, \tag{3.18}
\]

we derive the identities,

\[
U^*D_A U = VD_A V^*, \quad UD_A U^* = V^*D_A V^*. \tag{3.19}
\]

Using these identities we then obtain

\[
P_A = U^*D_A V - VD_A U^* = \frac{1}{U}D_A V, \quad \bar{P}_A = UD_A V^* - V^*D_A U = \frac{1}{U^*}D_A V^*. \tag{3.20}
\]

The theory is invariant under local \( SO(1,9) \times U(1) \) transformations that rotate the frame fields \( E^A = dx^M E^A_M \). The \( SO(1,9) \) is the 10d Lorentz group and the \( U(1) \) is identified with the local \( U(1) \) of \( SU(1,1)/U(1) \). There is a corresponding 1-form connection \( \Omega_A^B \). The \( SO(1,9) \) part is a superfield with lowest component (a supercovariant generalization of) the \( x \)-space spin-connection (see (3.43)) and the \( U(1) \) part is a superfield \( Q_A \).

The superspace geometry is encoded in the algebra of supercovariant derivatives,

\[
[D_A, D_B] = -T_{AB}^C D_C + \frac{1}{2}R_{ABC}^D L_D^C + 2iM_{AB} \kappa \tag{3.21}
\]

where \( T_{AB}^C \) is the torsion, \( L_A^B \) are the \( SO(9,1) \) generators, \( \kappa \) is the \( U(1) \) generator and \( R_{ABC}^D \) and \( M_{AB} \) are the spacetime and \( U(1) \) curvature tensors, respectively. The super-Jacobi identity

\[
[D_A, [D_B, D_C]] + \text{graded cyclic} = 0 \tag{3.22}
\]
implies the Bianchi identities

\[ I^{(1)}_{ABC}^D = \sum_{(ABC)} (D_A T_{BC}^D + T_{AB}^E T_{EC}^D - \hat{R}_{ABC}^D) = 0 \]  
\[ I^{(2)}_{ABCD}^E = \sum_{(ABC)} (D_A \hat{R}_{BCD}^E + T_{AB}^F \hat{R}_{FCD}^E) = 0 \]

where \( \sum_{(ABC)} \) denotes the graded cyclic sum, and the hat in \( \hat{R} \) means that it contains the contribution of the \( U(1) \) connection as well. There are also additional identities that stem from the fact that field strengths are closed forms. In form notation [25],

\[ I^{(3)} = DF_3 - F_3^* \wedge P, \]  
\[ I^{(4)} = dF_5 + 2iF_3 \wedge F_3^* \]  
\[ I^{(5)} = DP \]  
\[ I^{(6)} = M + \frac{1}{2} iP \wedge P^* . \]

The superfields introduced so far have a large number of components. In order to reduce the independent fields to the ones of the IIB supergravity multiplet described in the previous section one needs to impose constraints. Once the constraints are imposed the equations (3.23)-(3.28) are not identities any more and should be solved. Solving the Bianchi identities determines the corresponding superspace.

The Bianchi identities for IIB supergravity were solved in [25]. In superspace it is the torsion rather the curvature that is more important. The curvature is determined once the torsion coefficients are supplied. The non-zero torsion components are specified by the field content of the IIB supergravity. For each field of IIB supergravity there is a torsion coefficient whose leading component at the linearized level is the corresponding field strength. At the non-linear level the torsion coefficients contain fermion bilinears as well. The exact expressions are collected in appendix B. Furthermore, the Bianchi identities imply the IIB field equations. Notice that we do not need to consider \( \alpha' \) corrections to the torsion constraints, since our method makes use of the lowest order supersymmetry transformations only. However, as discussed in section 2, a specific superinvariant does imply specific corrections to the supersymmetry rules, and the latter induce \( \alpha' \) corrections to the torsion coefficients.

### 3.2.2 Solution of the Bianchi identities

To completely determine the theory we need the fermionic derivatives of all fields. This information can be obtained from Bianchi identities. We summarize these results here. The derivation of most of the formulas below can be found in [25].

The identities \( I^{(5)} \) and \( I^{(6)} \) imply,

\[ P_\bar{\alpha} = 0, \quad P_\alpha = -2\Lambda_\alpha, \quad \bar{P}_\alpha = 0, \quad \bar{P}_\bar{\alpha} = -2\Lambda^{*}_\alpha . \]  

Using (3.20), these results imply

\[ D^*_\alpha V = 0, \quad D_\alpha V = -2U\Lambda_\alpha \quad D^*_\alpha U^* = 0, \quad D_\alpha U^* = -2V^*\Lambda_\alpha \]  
\[ D_\alpha V^* = 0, \quad D^*_\alpha V^* = -2U^*\Lambda^{*}_\alpha \quad D_\alpha U = 0, \quad D^*_\alpha U = -2V\Lambda^{*}_\alpha . \]  

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A direct computation starting from the definition (3.20) gives the fermionic derivatives of $P_a$,

\[
D_a P_a = -2D_a \Lambda - 2T_{\alpha\beta} \Lambda_{\beta},
\]

\[
D_a \bar{P}_a = -2T_{\alpha\beta} \Lambda^*_{\beta}.
\]

(3.31)

Notice that $P^*_a = -\bar{P}_a$.

The dimension one Bianchi identities $I^{(1)}_{\alpha\beta\gamma}$, $I^{(1)}_{\alpha\beta\gamma}$, and $I^{(1)}_{\alpha\beta\gamma}$, determine the fermionic derivatives of $\Lambda$, 

\[
D_\alpha \Lambda_{\beta} = \frac{i}{24} \gamma_{\alpha\beta} F_{abc},
\]

\[
D_\alpha \Lambda^{*}_{\beta} = -\frac{1}{2} i (\gamma^\alpha)_{\alpha\beta} \bar{P}_a.
\]

(3.32)

(3.33)

The dimension 3/2 identities $I^{(1)}_{\alpha\beta\gamma}$, $I^{(1)}_{\alpha\beta\gamma}$, $I^{(1)}_{\alpha\beta\gamma}$ yield the fermionic derivatives of $F_5$ and $F_5$, but the fermionic derivative of $F_5$ is easier to obtain from $I^{(4)}$. The results are

\[
D_\alpha F_{abc} = \frac{1}{32} \left( \gamma_{abc} \right) f^{abc} + 3F^{*}_{[a\beta} \gamma_{bc]d} + 52F^{*}_{[\alpha\beta} \gamma_{d]c} + 28F^{*}_{abc} \right) \gamma_{\alpha\beta},
\]

\[
D_\alpha F^{*}_{abc} = -3(\gamma_{\alpha\beta}) \gamma_{\alpha\beta} \gamma_{\alpha\beta} + 3T_{\alpha\beta\gamma} \gamma_{\alpha\beta},
\]

(3.34)

(3.35)

D_\alpha F^{*}_{abcde} = -10(\gamma_{[ab} \gamma_{cde]} \alpha \beta + 20(\gamma_{[ab} \alpha \beta \gamma_{cde]}),
\]

(3.36)

where $\Psi_{ab\gamma}$ is a superfield whose leading component is the covariantized field strength of the gravitino.

The dimension two Bianchi’s $I^{(1)}_{\alpha\beta\gamma}$ and $I^{(1)}_{\alpha\beta\gamma}$ give

\[
D_\alpha \Psi_{bc}^\epsilon = \frac{1}{4} (\gamma_{bc}) \epsilon_{R_{bc}\epsilon} - (D_\beta T_{c\epsilon} + T_{\epsilon k} T_{ck} + T_{\epsilon k} T_{ck} - T_{\epsilon k} T_{ck} - (b \leftrightarrow c)) + i(\delta^\epsilon M_{bc},
\]

\[
D_\alpha \Psi_{ab} = -2D_\alpha T_{[b/\alpha} \beta - \Psi_{ab} \gamma T_{\alpha \beta} + 2T_{\alpha [b} \gamma T_{\beta]} \beta - 2T_{\alpha [b} \gamma T_{\beta]} \beta.
\]

(3.37)

Finally, the fermionic derivative of the Riemann tensor can be determined from $I^{(2)}_{ab\epsilon}$,

\[
D_\alpha R_{b\epsilon} = -2D_\alpha R_{\epsilon} - 2T_{\alpha\beta} R_{\epsilon} + 2T_{\alpha\beta} R_{\epsilon} - T_{\epsilon} R_{\alpha\beta} + T_{\epsilon} R_{\alpha\beta}.
\]

(3.38)

3.2.3 From superspace to components

The components of a covariant superfield $S$ may be obtained by the method of covariant projections, i.e. the components of $S$ are obtained by evaluating successive spinorial covariant derivatives at $\theta = 0$:

\[
s \equiv S |_{\theta = 0}, \quad s_\alpha \equiv D_\alpha S |_{\theta = 0}, \quad s_\alpha \equiv D_\alpha S |_{\theta = 0}, \quad \text{etc.}
\]

(3.39)

For ease in notation we will denote the projection by a vertical line without the subscript $\theta = 0$.

---

The dimension of the Bianchi identity $I_{ABC} \equiv 0$ is equal to $A + B + C - D$, where a bosonic index counts as 1 and a fermionic one as 1/2.
The vielbein and the gravitino are given by

\[ E_m^a = e_m^a \quad E_m^a = \psi_m^a. \tag{3.40} \]

To compute the components of a tensor whose indices have been converted to target space indices one uses manipulations of the form

\[ X_a = E_a^M X_M = e_a^m X_m - \bar{\psi}_a^\alpha X_\alpha + \psi_a^{*\alpha} X_\alpha. \tag{3.41} \]

Using such manipulations one can compute the leading components of the superfields \( P_a, F_{abc}, F_{abcde} \). These are the supercovariant field strengths which we denote by hats,

\[ P_a = \hat{p}_a = p_a + 2(\psi_a \lambda) \]
\[ F_{abc} = \hat{f}_{abc} = f_{abc} - 3(\psi^{*}_{[a} \gamma_{bc]} \lambda) - 3i(\psi_{[a} \gamma_b \psi_{c]}) \]
\[ F_{abcde} = \hat{f}_{abcde} = f_{abcde} + 20(\psi^{*}_{[a} \gamma_{bcd} \psi_{c]}). \tag{3.42} \]

Similarly, one obtains the supercovariant version of the spin-connection, gravitino field strength and Riemann tensor,

\[ \Omega_{mab} = \hat{\omega}_{mab} = \omega_{mab}(e) + \kappa_{mab}, \quad \Psi_{ab} = \hat{\psi}_{ab} = 2e_a^m e_b^n D_m \psi_n^a - 2\psi_{[a} \beta T_{b]b}^\alpha + 2\psi^{*}_{[a} \bar{\gamma}_{bc}^\alpha + \psi_{a} \gamma^c \psi_b (\gamma^c \lambda^*)^\alpha - 2\psi_{[a} \psi_{b]} \bar{\psi}^\beta \lambda^\beta \]
\[ R_{abc} = \hat{r}_{abc} = r_{abc} - 2\psi_{[a} \lambda R_{b]ac}^\beta \]
\[ + \psi^{*}_{a} \psi_{b} \lambda R_{b\beta ac} \]
\[ + \psi_{a} \psi_{b} \psi_{c} \lambda R_{b\beta ac} \]
\[ + \psi^{*}_{a} \psi_{b} \psi_{c} \lambda R_{b\beta ac}. \tag{3.43} \]

where \( \omega_{amn}(e) \) is the standard spin-connection associated with the vielbein, and

\[ \kappa_{a,bc} = \frac{1}{2} (t_{ab,c} + t_{ca,b} - t_{bc,a}), \quad t_{mn}^a = -2i \bar{\psi}_m^\gamma \gamma^a \psi_n. \tag{3.44} \]

General coordinate transformations in superspace with parameter \( \zeta^a \equiv \xi^a(z) \) yield the local supersymmetry rules for the components fields. For a covariant superfield \( S \), this yields

\[ \delta S = \zeta^a (D_a S) - \zeta^{*a} (D_a^* S). \tag{3.45} \]

The supersymmetry rule for the vielbein and gravitino is obtained using

\[ \delta E_M^A = D_M \xi^A - E_M^C \xi^B T_{BC}^A. \tag{3.46} \]

The supersymmetry rules for the component fields are collected in appendix C.

4 The dilaton superfield

4.1 General construction

Chiral superfields in four dimensions satisfy the linear constraint, \( D_4 \Phi = 0 \). In ten dimensions one may attempt to impose the linear constraint,

\[ D_4^a \Phi = 0. \tag{4.1} \]

A superfield satisfying such a constraint may be called an “analytic superfield” since in the chiral representation the superfield depends only on \( \theta \) but not \( \theta^* \). Because of the
similarity with the 4d chiral superfields, however, we will still use the terminology “chiral superfield” even though it is not appropriate in 10d where $D_\alpha^*$ and $D_\alpha$ have the same chirality.

In flat superspace one can always impose the constraint (4.1). In curved spacetime, however, there is an integrability condition: the anti-commutator of two $D_\alpha^*$ acting on the superfield should also vanish. If the torsion $T_{\gamma\alpha\beta}$ is non-zero then the chirality constraint (4.1) cannot be in general imposed. Indeed,

$$0 = \{D_\alpha^*, D_\beta^*\} \Phi = -T_{\alpha\beta\gamma} D_\gamma \Phi .$$

In IIB supergravity,

$$T_{\alpha\beta\gamma} = (\gamma^a)_{\alpha\beta}(\gamma_a)_{\gamma\delta} \Lambda^a_{\delta} - 2\delta^a_{(\alpha} \Lambda_{\beta)} .$$

So this equation yields

$$\gamma^a_{\alpha\beta} \Lambda_\gamma (\gamma_a)_{\gamma\delta} D_\delta \Phi - 2\Lambda_{(\alpha} D_{\beta)} \Phi = 0 ,$$

which implies

$$\Lambda_{(\alpha} D_{\beta)} \Phi = 0 .$$

The most general solution of this equation is

$$D_\alpha \Phi = g \Lambda_{\alpha} ,$$

where $g$ is any function of the fields. Notice also that if $\Phi$ is a chiral superfield then its covariant derivatives $D_a \Phi$ and $D_\alpha \Phi$ are not. This follows from the explicit form of the (anti)commutator of $D_\alpha^*$ with $D_a$ and $D_\alpha$.

We have seen in the previous section that the Bianchi identities imply that $V$ and $U^*$ are chiral superfields (and $V^*$ and $U$ antichiral ones), see (3.30). Indeed, in this case (4.6) is satisfied with $g = -2U$ and $g = -2V^*$, respectively. These two superfields are not independent. From (3.19) we get

$$D_\alpha U^* = \frac{V^*}{U} D_\alpha V .$$

Clearly, any function of $V$ and $U^*$ (but not of their covariant derivatives) is a chiral superfield as well. The gauge invariant superfield,

$$A = \frac{V}{U^*}$$

is chiral and has as its lowest component the physical scalar fields $a$. This superfield has $U(1)$ charge 2 with respect to the $U(1)$ subgroup of $SU(1,1)$. The linearized version of this superfield was constructed in [25]. It was shown there that it contains the entire supergravity multiplet in its components. Our considerations lead to the construction of all components, including all non-linear terms. Another related superfield is

$$T = i \frac{1 - A}{1 + A} .$$

The leading component of this superfield is $\tau$. 

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As discussed in the previous section, one can obtain all components of a superfield by a successive fermionic differentiation and then evaluation at $\theta = 0$. To obtain the components of all superfields discussed in this paper, it is sufficient to compute the fermionic derivatives of $V$ and this is the subject of the next subsection.

Before we proceed, however, we show that the chirality condition commutes with the $SU(1,1)$ (or equivalently $SL(2,R)$) action. From (3.3) we get that $SU(1,1)$ acts on the chiral superfields, $V$ and $U^*$ as

$$V' = zV + wU^*, \quad U'' = w^*V + z^*U^*$$

(4.10)

We thus see that the $SU(1,1)$ transformation rotates the two chiral superfields among themselves, and therefore the transformed superfields are still chiral. Furthermore, one may show that the fermionic derivatives are invariant under $SU(1,1)$ transformations,

$$D'_\alpha = D_\alpha,$$

(4.11)

when acting on functions of the dilaton superfield. This can be shown by starting from (3.30) and acting with an $SU(1,1)$ (or $SL(2,R)$) transformation. Since we know how all superfields transform, it is straightforward to determine how the fermionic derivatives transform.

### 4.2 Projections

All fermionic derivatives of the chiral superfield $V$ as well as their evaluation at $\theta = 0$ can be computed using the information in the previous section. The first derivative of $V$, $D_\alpha V$, is given in (3.30). To compute $D_\alpha D_\beta V$, we need $D_\alpha \Lambda_\beta$ and $D_\alpha U$. These are given in (3.32) and (3.30). At the next level, $D_\alpha D_\beta D_\gamma V$, we need in addition the derivative of $F_{abc}$. This is given in (3.34). In order to compute the fourth derivative of the superfield $V$, we now need the derivatives of $\Lambda_\alpha$, $F_{abc}$, $P_a$ and $\Psi_{ab\alpha}$, all of which are given in the previous section (see equations (3.32)-(3.33)-(3.35)-(3.37)). Notice that up to this order, new component fields were involved at every order: the dilatino appears in $DV$, $F_{abc}$ appears at $D^2V$, the gravitino field strength appears at $D^3V$ and the Riemann tensor and $F_5$ appear at order $D^4V$.

In order to proceed from here, we need the derivatives of all quantities appearing above. The only new derivatives we have to evaluate are the derivative of the curvature $R_{abcd}$, $P_a$ and of the torsions in (3.37). The components of the torsion involved in the above expression are $T_{aa}^{\beta}$, $T_{aa}^{\beta}$ and $T_{aa}^{\beta}$. These are given entirely in terms of the superfields $\Lambda$, $\tilde{\Lambda}$, $Z_{abcde}$, $F_{abc}$ and $F_{abc}^*$, where $Z_{abcde}$ is defined in (B.2). So to differentiate (3.37), we only need to determine $D_\alpha \Lambda_\alpha^*$, $D_\alpha Z_{abcde}$, $D_\alpha R_{abcd}$ and $D_\alpha P_a$. All of these are given in the previous section. At the next level the only new derivative that we encounter is $D_\alpha \Psi_{ab\alpha}$ and this is given in (3.37).

In the previous section we gave the $\theta = 0$ components and first covariant derivative of all fields. We can therefore determine all fermionic derivatives of the superfield $V$. Furthermore, the evaluation of the derivatives of $V$ at $\theta = 0$ is straightforward but tedious given the results in the previous section. One should remember that the evaluation of the field strengths at $\theta = 0$ gives rise to supercovariant objects.

To illustrate the procedure we discuss here the computation of the first four projections of $V$. This will be of use in section 6 where we compute the components of the
superpotential. Following the procedure discussed above we get

\[ V = u \nu \]  
\[ D \alpha V = -2u \lambda_\alpha \]  
\[ D_{[\alpha}^{} D_{\beta]}^{} V = \frac{i}{12} u^{\gamma}_{\alpha \beta} f_{abc} \]  
\[ D_{[\gamma}^{} D_{\beta]}^{} D_{\alpha]}^{} V = \frac{i}{12} u^{\gamma}_{\alpha \beta} \left\{ -\frac{1}{32} \left( \gamma_{abcdef} f_{abcdef}^{*} + 3f_{[a}^{} \gamma_{bc]de}^{*} + 52f_{[ab}^{} \gamma_{c]d}^{*} + 28f_{abc}^{*} \right) \gamma^{\epsilon} \lambda_\epsilon + 3\hat{p}_{[\alpha}^{} (\gamma_{bc]}^{*} \gamma_\epsilon^* \lambda_\epsilon + 3i(\gamma_\alpha)_{\gamma_\epsilon}^* \Psi_{bc} \right\} \]  

(4.12)  
(4.13)  
(4.14)  
(4.15)

These formulas are exact in that they contain all bosonic and fermionic terms. As we discussed, this procedure can be continued till all projections are obtained. The number of terms involved in the computation, however, grows as we go up in level. Since the computation is algorithmic, it can presumably be computerized. In the remaining of this section we compute the bosonic part of the fourth projection. The computation proceeds by taking the fermionic derivative of \( D^3 V \). Keeping only terms that contribute purely bosonic terms we get

\[ D_{[\beta}^{} D_{\gamma}^{} D_{\beta]}^{} D_{\alpha]}^{} V = \frac{i}{12} U^{\gamma}_{\beta \alpha} \left\{ -\frac{1}{32} \left( \gamma_{abcdef} f_{abcdef}^{*} + 3f_{[a}^{} \gamma_{bc]de}^{*} + 52f_{[ab}^{} \gamma_{c]d}^{*} + 28f_{abc}^{*} \right) \gamma^{\epsilon} \lambda_\epsilon + 3P_{[\alpha}^{} (\gamma_{bc]}^{*} \gamma_\epsilon^* \lambda_\epsilon + 3i(\gamma_\alpha)_{\gamma_\epsilon}^* \Psi_{bc} \right\} \]  

(4.16)

Using the expression for \( D_{\delta}^{} \Lambda_\epsilon, D_{\delta}^{} \Lambda_\epsilon^* \) and \( D_{\delta}^{} \Psi_{bc} \) given in the previous section we obtain a number of bosonic terms. In particular, the \( D_{\delta}^{} \Psi_{bc} \) term gives terms linear \( R \) and \( DF_{5} \). These are the terms that are present in the linearized superfield of Howe and West. This part is given by

\[ D_{[\delta}^{} D_{\gamma}^{} D_{\beta]}^{} D_{\alpha]}^{} V|_{linear} = \frac{1}{16} u(\gamma_{abc})_{\beta\alpha}^* (\gamma_{d}^* \gamma_{\delta}^* \gamma_{bcde} - \frac{i}{96} u(\gamma_{abc})_{\beta\alpha}^* (\gamma_{d}^* \gamma_{\delta}^* D_b f_{acdef} \]  

\[ \frac{1}{16} u(\gamma_{abc})_{\beta\alpha}^* (\gamma_{d}^* \gamma_{\delta}^* (g_{ad}^* c_{bcde} - \frac{i}{6} D_b f_{acdef} \right) \]  

(4.17)

Here \( \gamma_{bcde} \) is the Weyl tensor, and in passing to the second equality we used the Fierz identity (A.14) to show that only the Weyl tensor contributes.

There are further contributions, however, that are not captured by the linearized superfields. They are proportional to \( f_3^* f_3 \) and \( f_5^* f_5 \). The latter give

\[ D_{[\delta}^{} D_{\gamma}^{} D_{\beta]}^{} D_{\alpha]}^{} V|_{f_5^* f_5} = -\frac{u}{1536} \gamma_{[\delta}^{} (\gamma_{bcde}^{*} (3f_{bo} f_{mn} f_{ced}^{mn} - f_{abcd} f_{def}^{mn}) \]  

(4.18)

The computation leading to this term is elaborate and is given in appendix D. The \( f_3^* f_3 \) terms can also be computed straightforwardly, but we shall not present them here.

To summarize, we obtained

\[ D_{[\delta}^{} D_{\gamma}^{} D_{\beta]}^{} D_{\alpha]}^{} V = u^{\gamma}_{[\beta\alpha}^* (\gamma_{de}^{*} R_{acdef} \]  

(4.19)

where

\[ R_{acdef} = \frac{1}{16} (g_{ad} c_{bcde} - \frac{i}{6} D_b f_{acdef} - \frac{1}{1536} (3f_{bo} f_{mn} f_{ced}^{mn} - f_{abcd} f_{def}^{mn}) + f_3^* f_3 \]  

(4.20)
5 Obstruction to a supersymmetric action

In supergravity theories the measure \( e = \det e^a_m \) transforms under supersymmetry. To construct actions one needs an appropriate supersymmetric measure. For unconstrained superfields the supersymmetric measure is given by the superdeterminant of the super-vielbein, \( \text{sdet } E \). Chiral superfields, however, are only integrated over half of superspace, and \( \text{sdet } E \) is not the correct density. In four dimensions the chiral measure is a chiral superfield whose lowest component is \( e \). In our case, however, the superfield \( \Delta \) whose lowest component is \( e \) cannot be a chiral scalar superfield. As discussed in the previous section, chiral superfields should satisfy (4.6). From the supersymmetry rules one obtains \( D_\alpha \Delta | \sim (\gamma^b \psi_b)_\alpha \), so (4.6) is not satisfied. We will proceed by systematically analyzing the constraints imposed on the measure by supersymmetry.

We consider the following action

\[
S = \int d^{10}x \ d^{16} \Theta \Delta W[V, U^*] + \text{c.c.} \tag{5.1}
\]

where the superpotential \( W[V, U^*] \) is an arbitrary function of the chiral superfields \( V, U^* \) but not of their derivatives or complex conjugate superfields, i.e. \( W[V, U^*] \) is itself a chiral superfield. By definition,

\[
d^{16} \Theta = D^{16} \equiv \frac{1}{16!} \epsilon^{a_1..a_{16}} D_{a_1} \ldots D_{a_{16}}. \tag{5.2}
\]

The reason for considering (5.1) is that an action of this form was argued to capture the interactions of the effective action at the linearized level, see [39] and references therein. Furthermore, a scalar superpotential term is one of the simplest interaction terms one may consider. There are other possibilities one may consider such as considering a non-scalar superpotential.

The action should be gauge invariant. Each \( D_\alpha \) has charge \(-1/2\) under the local \( U(1) \) symmetry. This implies that \( W \) must have charge +8 in order for the action to be \( U(1) \) invariant. This can be achieved by setting

\[
W = (U^*)^8 \tilde{W}(A) \tag{5.3}
\]

where \( A \) is the gauge invariant dilaton superfield (since both \( A \) and \( T \) are gauge invariant one may consider \( \tilde{W} \) as either a function of \( A \) or \( T \)). The factor \( U^* \) may be considered as a \( U(1) \) compensator. Gauge fixing the \( U(1) \) symmetry amounts to setting \( U^* \) equal to some function of \( A \).

We shall determine the constraints imposed by supersymmetry on \( \Delta \) and analyze whether there is a \( \Delta \) such that the action is supersymmetric. By definition, the chiral measure \( \Delta \) is a superfield whose lowest component is the determinant of the vielbein

\[
\Delta | = \det e^a_m = e. \tag{5.4}
\]

The idea now is to determine the remaining projections by systematically arranging that the supersymmetry variation of the action vanishes. We shall see that IIB superspace allows for a precise formulation of the problem along these lines.

Integrating out the \( \Theta \) one obtains the following component action

\[
S = \int d^{10}x \epsilon^{a_1..a_{16}} \sum_{n=0}^{16} \frac{1}{n!(16-n)!} D_{a_1} \ldots D_{a_n} \Delta | D_{a_{n+1}} \ldots D_{a_{16}} W | \]

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\[ \int d^{10}x \sum_{n=0}^{16} \frac{1}{n!} D_{\alpha_1} \ldots D_{\alpha_n} \Delta \left| D^{16-n,\alpha_1 \ldots \alpha_n} W \right| \]  

(5.5)

where we have introduced the notation

\[ D^{16-n,\alpha_1 \ldots \alpha_n} W = \frac{1}{(16-n)!} \epsilon^{\alpha_1 \ldots \alpha_{16}} D_{\alpha_{n+1}} \ldots D_{\alpha_{16}} W . \]  

(5.6)

One would like to fix the higher projections of \( \Delta \) such that this action is supersymmetric for any superpotential \( W \). Notice that because of the \( \epsilon \)-symbol only fully antisymmetrized projections of \( \Delta \) enter in the action.

Invariance of the action under supersymmetry requires

\[ \delta S = \int d^{10}x \sum_{n=0}^{16} \frac{1}{n!} \left( \delta D_{\alpha_1} \ldots D_{\alpha_n} \Delta \left| D^{16-n,\alpha_1 \ldots \alpha_n} W \right| + D_{\alpha_1} \ldots D_{\alpha_n} \Delta \left| \delta D^{16-n,\alpha_1 \ldots \alpha_n} W \right| \right) = 0. \]  

(5.7)

Since we consider arbitrary superpotential \( W \) the terms \( D^n W \) are linearly independent and one can investigate their cancellation separately. Furthermore, this can be done systematically by starting from the term with the highest number of derivatives, \( D^{18} W \), and then moving to \( D^{15} W \) terms, etc. For each \( D^n W \), \( n = 1, \ldots, 16 \), supersymmetry implies two conditions: one for the terms that are proportional to \( \zeta \) and another for the ones proportional to \( \zeta^* \). We show below that the conditions proportional to \( \zeta \) uniquely determine all projections of the chiral measure \( \Delta \). This leaves 16 more conditions to be checked: the ones proportional to \( \zeta^* D^n W \). These conditions should be satisfied automatically for (5.1) to be supersymmetric. Furthermore, since the coefficients of \( \zeta^* D^n W \) are field dependent, these conditions further split into a number of independent conditions: one for each independent structure.

The supersymmetry variation of \( D^{16-n} W \) is given by

\[ \delta D^{16-n,\alpha_1 \ldots \alpha_n} W = \frac{1}{(16-n)!} \epsilon^{\alpha_1 \ldots \alpha_{16}} (\zeta^a D_a D_{\alpha_{n+1}} \ldots D_{\alpha_{16}} W - \zeta^{a*} D^*_a D_{\alpha_{n+1}} \ldots D_{\alpha_{16}} W) \]  

(5.8)

Inspection of the supersymmetry algebra reveals that (anti) commutators of covariant derivatives cannot increase the number of \( D_a \) derivatives acting on \( W \). It follows that the schematic form of the variation is

\[ \delta D^{16-n} W \sim \zeta \left[ D^{17-n} W + O(D^{15-n} W) \right] + \zeta^* O(\psi D^{16-n} W). \]  

(5.9)

Let us show this in some detail. Consider first the terms proportional to \( \zeta^{a*} \). Anticommuting \( D^*_a \) to the right where it annihilates \( W \) yields \( D_a D^{15-n} W \) plus other terms of order \( D^{14-n} W \). The term \( D_a D^{15-n} W \) is actually of order \( D^{16-n} W \). To see this, use (3.41) to convert the flat index in \( D_a \) to a curved one. This yields a term \( \psi D^{16-n} W \) plus terms of lower order. Thus the \( \zeta^{a*} \) terms are of order \( D^{16-n} W \). Let us now discuss the \( \zeta^a \) terms. Antisymmetrizing all derivatives we get \( D^{17-n} W \). The antisymmetrization involves anticommutators \( \{D_{a_1}, D_{a_2}\} \), and the corresponding curvature terms are of order \( D^{15-n} W \). The torsion term yields \( D_a^* D^{15-n} W \), which can be analyzed as the term \( D_a^* D^{16-n} W \), and is also of order \( D^{15-n} W \). We conclude that the supersymmetry variation of \( D^{16-n} W \) is of the form (5.9).

It follows from (5.9) that one can iteratively determine all projections of \( \Delta \) by arranging that the \( \zeta \) terms in the variation of the action cancel. In particular, the cancellation of the
terms of order $D^{17-n}W|$ determine the projection $D_n\Delta|$. We now prove this inductively. The $n=1$ case will be shown to hold in the next subsection. Let us assume that the statement holds true for all $n<k$, i.e. we assume that we determined all $D_n\Delta|, n<k,$ by arranging for the cancellation of the terms proportional to $D^{17-n}W|, n<k.$ Let us now consider the $n=k$ case. From (5.7) we get

$$\delta \zeta S = \int d^{10}x \left( \delta \zeta D^{k-1}\Delta| + \zeta D^k\Delta| + \zeta f(D_n\Delta|) \right) D^{17-k}W| + O(D^{16-k}W|) \quad (5.10)$$

where we suppress indices and numerical factors, $\delta \zeta$ denotes a supersymmetry variation with only $\zeta$ terms taken into account, and $f(D_n\Delta|)$ denotes the terms that originate from manipulations of higher order terms. To be more explicit, recall that the supersymmetry variation $\delta D^{16-n}W$ in (5.9) involves terms proportional to $D^{16-l}W|, l>n$. Thus the terms $D_n\Delta(\delta D^{16-n}W|), n<k$ in the variation of the action can contribute term proportional to $\zeta D^{17-k}W|$. These terms depend on $D_n\Delta, n<k$, which are known by the induction hypothesis and are denoted by $f(D_n\Delta|)$. It follows that by setting

$$\zeta D^k\Delta| \sim (\delta \zeta D^{k-1}\Delta|) + \zeta f(D_n\Delta|) \quad (5.11)$$

the $\zeta D^{17-k}W|$ terms vanish, as advertised. This finishes the inductive proof that the cancellation of $\zeta$ terms uniquely determine all components of $\Delta$. In the next subsection we will determine the exact form, including coefficients, of $D_\alpha\Delta|$ and $D_{[\alpha}D_{\beta]}\Delta|$ (up to certain fermion bilinears for the latter).

We are now left with the $\zeta^*$ terms in the supersymmetry variation. For the action to be supersymmetric, these terms should vanish automatically. We shall see that these terms cancel at leading order and at a linear level at the subleading order, but there is an obstruction which is non-linear in the fields in subleading order.\(^9\)

### 5.1 Obstruction at subleading order

We now discuss in detail the determination of the first two projections of $\Delta$ and the obstruction to the existence of the chiral measure. Using (5.4) and keeping terms up to order $D^{15}W|$ in the supersymmetry variation of the action, we find

$$\delta S = \int d^{10}x \left[ \delta e D^{16}W| + e(\delta D^{16}W|) + e \left( -\delta D^{15,\alpha}W|D_\alpha\Delta| 
+ D^{15,\alpha}W|\delta D_\alpha\Delta|\right) \right] = 0 \quad (5.12)$$

The variation of $e$ can be computed from the supersymmetry variation of the vierbein

$$\delta e = -ie((\zeta \gamma \psi^*) + (\zeta^* \gamma \psi)) \quad (5.13)$$

The variation of the $D^{16}W|$ term is given by

$$\delta D^{16}W| = \frac{1}{16!} \epsilon^{\alpha_1...\alpha_{16}}(\zeta^\alpha D_\alpha - \zeta^{*\alpha} D^*_\alpha) D_{\alpha_1}...D_{\alpha_{16}}W| \quad (5.14)$$

\(^9\)In the original version of this paper, we only considered the linear subleading terms. We are grateful to Nathan Berkovits and Paul Howe for informing us that the $\lambda \lambda^* D^{17}W|$ terms present an obstruction.
Let us consider separately the $\zeta$ terms and the $\zeta^*$ terms. To manipulate the $\zeta^*$ terms we need to compute the commutator $[D^*_\alpha, D^{16} W]$. Using the supersymmetry algebra we obtain

$$[D^*_\alpha, D^{16} W] = \frac{1}{16!} \epsilon^{\alpha_1...\alpha_6} \left[ -16 T^{c}_{\bar{a}a_1} D_c D_{a_2}...D_{a_{16}} W - \frac{15 \cdot 16}{2} \left( T^{c}_{\bar{a}a_2} T^{\delta}_{a_1c} D_\delta D_{a_3}...D_{a_{16}} W \right) \right. \\
+ R^{\alpha_1a_2} \delta_{D} D_{a_3}...D_{a_{16}} W ] + i \frac{17 \cdot 16}{2} M_{\alpha_1a_2} D_{a_2}...D_{a_{16}} W ] + O[D^{14} W].$$

(5.15)

where the curvature terms originate in the anticommutators, $\{ D^*_\alpha, D_{\alpha_p} \}$, and the double torsion terms arise in the process of commuting $D_c$ to the left. We now convert the $D_c$ derivative to a curved space derivative, $D_m$, using $D_c = E^M_c D_M$,

$$[D^*_\alpha, D^{16} W] = T^{c}_{\bar{a}a_1} \psi_c D^{16} W - \left( \epsilon^{m} T^{c}_{\bar{a} \beta} D_m \right) \left( -i \frac{17 \cdot 16}{2} M_{\beta a} + \frac{1}{2} \left( T^{c}_{\bar{a} \gamma} T^{\gamma}_{\beta c} - T^{\gamma}_{\bar{a} \beta} T^{\gamma}_{\gamma c} - R^{\beta \gamma a}_c \right) + O([\psi^* \psi]) \right) D^{\beta, 15} W + O[D^{14} W].$$

In the $\zeta$ terms we proceed by fully antisymmetrizing the $D_\alpha$ derivative with the rest of the derivatives. Since the index $\alpha$ takes only 16 values, the fully antisymmetric product is identically equal to zero. We therefore obtain

$$\epsilon^{\alpha_1...\alpha_6} D_\alpha D_{a_1}...D_{a_{16}} W = \epsilon^{\alpha_1...\alpha_6} \sum_{p=0}^{15} (-1)^p (16 - p) D_{a_1}...D_{a_p} \{ D_{a_{p+1}}, D_\alpha \} D_{a_{p+2}}...D_{a_{16}} W .$$

(5.17)

We now use the superalgebra

$$\{ D_{\alpha_p}, D_\alpha \} = -T^{\bar{a}}_{\alpha \beta} D^*_\beta + \frac{1}{2} R^{\alpha \beta \gamma} L_{\gamma \beta} .$$

(5.18)

The terms with $D^*_\beta$ can be manipulated in a way similar to (5.16). They give rise to terms of order $\psi^* D^{15} W$. The rest yields,

$$D_\alpha D^{16} W = \frac{1}{3} R^{\alpha \beta \gamma} D^{15, \gamma} W + O(\psi^* D^{15} W).$$

(5.19)

Similar manipulations yield

$$\frac{1}{15!} \epsilon^{\alpha_1...\alpha_6} \delta D_{a_1}...D_{a_{15}} W |D_{a_{16}} \Delta| = -\zeta (D \Delta) |D^{16} W | + O(\psi^* D^{15} W) .$$

(5.20)

At this point we have all contributions to order $D^{16} W$. Keeping terms of leading order only we obtain

$$\delta S = \int d^{10} x \left[ \zeta^{\alpha} (i e^{\gamma^{c}_{\alpha \beta} \psi_{c}^{* \beta} - D_\alpha \Delta) |D^{16} W | \\
- \zeta^{* \alpha} (|D^*_\alpha, D^{16} W | + i e^{\gamma^{c}_{\alpha \beta} \psi_{c}^{* \beta} D^{16} W |) + O(D^{15} W) \right].$$

(5.21)

The cancellation of the $\zeta$-terms uniquely fixes the first projection of $\Delta$,

$$D_\alpha \Delta | = -i e^{\gamma^{c}_{\alpha \beta} \psi_{c}^{* \beta} .$$

(5.22)

The $\zeta^*$ terms at leading order vanish by themselves upon using the leading order term in (5.16).
We next move to terms of order \( D^{15}W \). We will keep terms that are linear in the fields and from the fermion bilinears only the terms proportional to \( \lambda \lambda^* \). We now need to compute

\[
\delta D_\beta \Delta |D^{\beta,15}W| = T^c_{\bar{\beta} \gamma} \delta(e\psi^*e\gamma) D^{\beta,15}W
= T^c_{\bar{\beta} \gamma} e(m^c D_m \psi^*e\gamma + \zeta^\alpha T^c_{\bar{\alpha} \gamma} - \zeta^{* \alpha} T^c_{\bar{\alpha} \gamma}) D^{\beta,15}W
+ O(\psi^*\psi D^{15}W) + O(D^{14}W)
\]

(5.23)

where we found it useful to use (3.46) for \( \delta \psi^*e\gamma \) rather than substituting the expression from appendix C. Furthermore,

\[
\frac{1}{4 \cdot 14!} \delta D_{\alpha_1 \ldots \alpha_{14}} W [D_{\alpha_15}, D_{\alpha_{16}}] \Delta | = -\frac{1}{2} \zeta^\alpha [D_\alpha, D_\beta] \Delta |D^{\beta,15}W| + O(D^{14}W)
\]

(5.24)

Summing up all contributions we obtain

\[
\delta S_{15} = \int d^{10}x e \left\{ D_m \left( e^c_{\bar{m}} T^c_{\bar{\beta} \gamma} \zeta^{* \alpha} D^{\beta,15}W \right) \right\}
- \zeta^\alpha \left[ i\gamma_{\beta \gamma} T^c_{\bar{\alpha} \gamma} - \frac{1}{3} R_{\alpha\beta\gamma} + \frac{1}{2} [D_\alpha, D_\beta] \Delta | \right] D^{\beta,15}W
\]
\[
+ \zeta^{* \alpha} \left[ -i \frac{17}{2} M_{\beta \alpha} + \frac{1}{2} (T^c_{\bar{\alpha} \gamma} T^c_{\bar{\beta} \gamma} - T^c_{\bar{\beta} \gamma} T^c_{\bar{\gamma} \gamma} - R_{\alpha\gamma\beta} - T^c_{\bar{\gamma} \gamma} T^c_{\bar{\gamma} \gamma}) \right] D^{\beta,15}W
\}
\]

(5.25)

The total derivative term originates from (5.16) and the \( D_m \zeta^* \) term in \( \delta \psi^*e\gamma \).

Requiring that the \( \zeta \)-terms cancel determines the projection \( [D_\alpha, D_\beta] \Delta | \),

\[
ie\gamma_{\beta \gamma} T^c_{\bar{\alpha} \gamma} - \frac{1}{3} R_{\alpha\beta\gamma} + \frac{1}{2} [D_\alpha, D_\beta] \Delta | = 0.
\]

(5.26)

Using the explicit formulas for the curvature and torsions we obtain

\[
ie\gamma_{\beta \gamma} T^c_{\bar{\alpha} \gamma} = \frac{1}{24} i e^{abc}_{\alpha \beta} f_{abc}^* \quad R_{\alpha\gamma\beta} = \frac{1}{4} i e^{abc}_{\alpha \beta} f_{abc}^*
\]

(5.27)

which leads to

\[
[D_\alpha, D_\beta] \Delta | = \frac{1}{12} i e^{abc}_{\alpha \beta} f_{abc}^* + O[\psi^* \psi, \lambda^* \psi].
\]

(5.28)

We process the \( \zeta^* \) terms by using the Bianchi identities. They imply

\[
I_{\alpha\gamma\beta}^{(1)} = 0 \quad \Rightarrow \quad R_{\alpha\gamma\beta} = -17 i M_{\beta \alpha} - T^c_{\bar{\alpha} \gamma} T^c_{\bar{\beta} \gamma} - T^c_{\bar{\gamma} \gamma} T^c_{\bar{\gamma} \gamma} - T^c_{\bar{\gamma} \gamma} T^c_{\bar{\gamma} \gamma}
\]
\[
I_{\alpha\beta\gamma}^{(1)} = 0 \quad \Rightarrow \quad T^c_{\bar{\alpha} \gamma} T^c_{\bar{\beta} \gamma} - T^c_{\bar{\gamma} \gamma} T^c_{\bar{\gamma} \gamma} = 0
\]

(5.29)

(5.30)

where we used the notation introduced in (3.23). Inserting these expression in (5.25) we find that all terms but one cancel out and we end up with

\[
\delta S_{15} = \frac{1}{2} \int d^{10}x e \zeta^{* \alpha} T^c_{\bar{\alpha} \gamma} T^c_{\bar{\gamma} \gamma} D^{\beta,15}W
\]

(5.31)

Recall that we showed in section 4 that \( T^c_{\bar{\gamma} \gamma} \neq 0 \) is the obstruction for the existence of chiral superfields. Here we find that it is also the obstruction for the existence of a chiral measure.
In type IIB superspace $T_{\alpha\bar{\beta}}^\gamma$ is non-zero. Nevertheless, the dilaton superfield $V$ exists because the torsion coefficient is such that $T_{\alpha\bar{\beta}}^\gamma D_{\gamma}V = 0$. The superpotential is a function of $V$, so one should check that the obstruction does not vanish because of special properties of the dilaton superfield. Using the notation we introduce in (6.2) one finds,

$$D_\beta^{\beta,15}W = \frac{1}{15!} \epsilon_{\beta\alpha_1...\alpha_{15}} D_{\alpha_1} V ... D_{\alpha_{15}} V F^{(15)}(V) | + ...$$

where the dots indicate terms with $F^{(n)}(v), n < 15$. Since (5.31) should be valid for any superpotential, the terms proportional to different $F^{(n)}(v)$ should vanish separately. Let us consider the term proportional to $F^{(15)}(v)$. Notice also that such a term is present only in $D_\gamma^{15}W|$, so there cannot be any cancellations involving terms with $D_n W|, n < 15$. Using (3.30) we get

$$\delta S_{15,F^{(15)}} = \frac{1}{2} \int d^{10}xe \zeta^{*\alpha}(2u)^{15} T_{\bar{\alpha}\delta}^\gamma T_{\gamma\delta}^\beta \lambda^{15,\beta} F^{(15)}(v)$$

Using the explicit form of the torsion coefficients we finally get

$$\delta S_{15,F^{(15)}} = -77 \int d^{10}xe (\zeta^{*\alpha})\lambda^{16}(2u)^{15} F^{(15)}(v)$$

which is non-zero. We conclude that the chiral measure does not exist and the action (5.1) is not supersymmetric. This computation still leaves the possibility that the measure exists at the linearized level. To check that, one needs to analyze the terms $D_n W|, n < 15$. Using (3.30) we get

In this section we investigated the existence of a scalar measure: all supercovariant derivatives in (5.2) are completely antisymmetrized. Terms similar to the ones in (5.33) will be generated if one relaxes the full antisymmetrization. In this case, however, the measure $d^{16}\Theta$ would have free indices, and therefore the superpotential should also carry indices. Such terms are also generated if the chirality constraint on the superpotential is relaxed. Perhaps there is a simple modification of the construction presented in this paper that will be supersymmetric and hopefully be related to the $R^4$ term that appears in string theory.

In the next section we consider the component form of the action in (5.1) with $\Delta$ determined by the cancellation of $\zeta$-terms, as described in this section. This action is not supersymmetric but as we shall see its properties are rather intriguing.

6 Components and $SL(2, Z)$ symmetry

6.1 The superpotential in components

We discuss in this section the computation of the superpotential in components. The computation consists of evaluating at $\theta = 0$ the terms in (5.5). Since we know all components of $V$ and we showed how to determine the components of $\Delta$ (and explicitly determined the first two), it is straightforward but tedious to obtain all components. We will discuss in detail the computation of the terms proportional to $\lambda^{16}, \psi^*\lambda^{15}$ and $r^4$.

First, notice that we can solve (3.15) to express $U^*$ in terms of $V$,

$$U^* = A + BV$$

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\( A = \frac{1}{U} \) and \( B = V^*/U \). Since \( A \) and \( B \) are annihilated by \( D \), they can be considered as constants for the purpose of evaluating the fermionic integral. We now define
\[
F(V) = W[V, U^* = A + BV] = (A + BV)^8 \tilde{W} \left( \frac{V}{A + BV} \right),
\]
where we used (5.3). With this definition we now schematically have
\[
S \sim \int d^{10}x \sum_{n=0}^{16-n} \sum_{k=1}^{16-n} F^{(k)}(v) D^n \Delta | \sum_{n_1=16-n} D^{n_1} V \cdots D^{n_k} V |
\]
where \( F^{(n)} = \partial^n F/\partial V^n \), and we suppress combinatorial factors (which however will be taken into account below).

We want to compute specific terms in the component expansion of the superpotential, namely the \( \lambda^{16}, \psi^* \lambda^{15} \) and \( r^4 \). To do this we will use the local \( U(1) \) symmetry and dimensional analysis. Let us first discuss the \( r^4 \) term. The Riemann tensor \( r \) is neutral under the local \( U(1) \). Therefore each of the projections contributing to \( r^4 \) should also be neutral under the local \( U(1) \). From the analysis in section (4) the form of the projections is
\[
D^n V| \sim ug ,
\]
where \( g \) is a function of the fields that does not depend on \( u \) or \( v^* \). Now, from table 1 we see that the covariant derivative has \( U(1) \) charge \(-1/2\), \( v \) charge 1 and \( u \) charge \(-1\). This implies that the function \( g \) has charge \( 2 - n/2 \). It follows that only the \( n = 4 \) projection is \( U(1) \) neutral. A similar argument applies to the projections of \( \Delta \). In this case only the leading component is \( U(1) \) neutral. Notice that terms of the form \( D^4 \Delta \) are excluded by a combination of dimensional and \( U(1) \) analysis. We therefore obtain that the \( r^4 \) term is given by
\[
S_{r^4} = \int d^{10}x e^{\frac{1}{(4!)^5}} F^{(4)}(v) (D^4 V)^4
\]
\[
= \int d^{10}x e^{\frac{1}{(4!)^5}} u^4 F^{(4)}(v) R^4
\]
where \( R \) is given in (4.20), and the exact index contractions are given in appendix E, see (E.2). In the same appendix we show that index contraction of the \( c^4 \) term in \( R^4 \) is the same as that of the \( c^4 \) term of string theory [1].

We thus see that the superpotential contain terms of the form \( c^k (Df_5)^l (f_5)^m \), for appropriate values of \( k, l, m \). One may extend the computation described in appendix E to obtain the exact index contractions, but we shall not do this here.

Let us now analyze the function \( F^{(4)}(v) \). Using the definition in (6.2) we obtain
\[
u^4 F^{(4)}(v) = \sum_{n=0}^{4} c^{(0)}_n (u^* v^*)^n \tilde{W}^{(4-n)}(a)
\]
where \( \tilde{W}^{(n)} = \partial^n \tilde{W}/\partial a^n \) and the combinatorial coefficients are given by
\[
c^{(0)}_0 = 1, \quad c^{(0)}_1 = 20, \quad c^{(0)}_2 = 180, \quad c^{(0)}_3 = 840, \quad c^{(0)}_4 = 1680.
\]
Let us now consider the $SL(2,R)$ Laplacian,

$$\nabla^2 = 4 \tau^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau^*} = (1 - aa^*)^2 \frac{\partial}{\partial a} \frac{\partial}{\partial a^*}$$  \hspace{1cm} (6.8)

where we also express it in terms of $SU(1,1)$ physical scalars. An easy computation yields,

$$\nabla^2((u^*v^*)^n \tilde{W}^{(m)}) = n(n+1)(u^*v^*)^n \tilde{W}^{(m)} + n(u^*v^*)^{n-1} \tilde{W}^{(m+1)}.$$  \hspace{1cm} (6.9)

Using this result we then obtain,

$$\nabla^2(u^4 F^{(4)}(v)) = 20(u^4 F^{(4)}(v)),$$  \hspace{1cm} (6.10)

where the exact values of the coefficients were crucial for $u^4 F^{(4)}(v)$ to be an eigenfunction of the Laplacian. Looking through the computation we see that the eigenvalue 20 is basically due to the fact that the $r^4$ term comes from the fourth power of the fourth projection of $V$.

The $c^4$ coupling receives a contribution from the complex conjugate of the superpotential term as well. The analysis is exactly the same (it is the cc analysis of what we presented). We thus finally define

$$t^{(0,0)}(\tau, \tau^*) = \frac{6^4}{(4!)^5 16^4} (u^4 F^{(4)}(v) + u^4 F^{(4)}(v^*)).$$  \hspace{1cm} (6.11)

The factors $1/16^4$ originate from a similar factor in (4.20) and the factor $6^4$ from the manipulations described in appendix E. We will shortly show that imposing $SL(2,Z)$ symmetry implies that $t^{(0,0)}(\tau, \tau^*)$ is equal to the non-holomorphic Eisenstein series $E_5$, but before we do this we will examine the $\lambda^{16}$ and $\lambda^{15} \psi^*$ terms and the $SL(2,R)$ properties of their coefficients.

Similar considerations as the ones above show that the only term that can contribute to the $\lambda^{16}$ term is $(DV|)^{16}$, and that the measure cannot contribute. We thus have

$$S_{\lambda^{16}} = \int d^{10} x \ e^{F^{(16)}} \frac{1}{16!} \epsilon_{\alpha_1 \ldots \alpha_{16}} D_{\alpha_1} V | \ldots D_{\alpha_{16}} V|$$

$$= \int d^{10} x \ e^{2^{16} u^4 F^{(16)}} \lambda^{16}$$  \hspace{1cm} (6.12)

where we define\(^\text{10}\)

$$(\lambda^n)_{\alpha_n+1 \ldots \alpha_{16}} = \frac{1}{n!} \epsilon_{\alpha_1 \ldots \alpha_{16}} \lambda^{\alpha_1} \ldots \lambda^{\alpha_n}.$$  \hspace{1cm} (6.13)

Let us call the coefficient of $\lambda^{16}$, $t^{(12,-12)}$ (the reason for the terminology will become apparent later). Then

$$t^{(12,-12)} = 2^{16} u^4 F^{(16)}.$$  \hspace{1cm} (6.14)

Let us now consider the terms $\psi^* \lambda^{15}$. There are two sources of such terms. One comes from $\Delta|D^2 V| (\Delta V)^{14}$ and another receives a contribution from the measure $(D\Delta)(D V)^{15}$. The former contributes because $D^2 V$ is proportional to the supercovariant field strength

\(^{10}\text{Here we follow the conventions in [45]. This definition differs by a sign when } n \text{ is odd from the similar definition in (5.6).} \)
\( \hat{f}_3 \) (4.14), and the latter contains a \( \psi^* \lambda \) term, see (3.42). This contribution is also discussed by [45], but we get an additional term from the measure. Combing the two we obtain

\[
S_{\psi^* \lambda^{15}} = \int d^{10} x \left( -i e 2^{18} u^{15} F^{(15)} \right) \left( \psi_c^* \gamma^c \lambda^{15} \right). \tag{6.15}
\]

Let us define

\[
t^{(11,-11)} = -i 2^{18} u^{15} F^{(15)} \tag{6.16}
\]

to be the coefficient of the \( \psi^* \lambda^{15} \) term. Comparing (6.14) and (6.16) we see that they satisfy

\[
u \frac{\partial}{\partial v} t^{(11,-11)} = -4 i t^{(12,-12)}. \tag{6.17}
\]

The coefficients \( t^{(12,-12)} \) and \( t^{(11,-11)} \) are analogous to the coefficients \( f^{(12,-12)} \) and \( f^{(11,-11)} \) introduced by Green and Sethi in [45] (but we view \( f^{(11,-11)} \) as the coefficient of \( \psi^* \lambda^{15} \) rather than \( \hat{f}_3 \lambda^{14} \), i.e. we view (3.3) of [45] rather than (3.1) as the starting point of their analysis. As discussed above, the coefficient of \( \psi^* \lambda^{15} \) receives a contribution from the measure as well). A supersymmetry analysis that uses only the lowest order supersymmetry rules leads them to the constraint (3.5) which after rescaling their \( f^{(11,-11)} \) by \((-3 \cdot 144)\) and adapting their result to our conventions reads\(^{11}\)

\[
D_{11} f^{(11,-11)} = 2 i f^{(12,-12)} \tag{6.18}
\]

where \( D_{11} \) is a modular covariant derivative. For later use, we introduce the modular covariant derivatives

\[
D_w = i (\tau_2 \frac{\partial}{\partial \tau} - \frac{i w}{2}), \quad D_w^* = -i (\tau_2 \frac{\partial}{\partial \tau^*} + \frac{i \hat{w}}{2}), \quad (6.19)
\]

\( D_w \) and \( D_w^* \) acting on a modular form of weight \((w, \hat{w})\) (see footnote 6 for the definition) gives a form of weight \((w+1, \hat{w}-1)\) and \((w-1, \hat{w}+1)\), respectively. Our superpotential term is not supersymmetric. We shall show, however, that our coefficients automatically satisfy (6.18). In particular, we will show that (6.17) is exactly (6.18). Our computations so far were all done in the gauge invariant formulation, but the ones in [45] in a specific gauge, so to compare our formulas with theirs we first need to express our results in the gauge used in [45].

### 6.2 Gauge fixing the \( U(1) \)

We discuss in this subsection how to express our results in the gauge used in [45]. This gauge is described in a real basis in section 2.1 of [72]. Expressing (3.10) in \( SL(2, R) \) variables using (3.12) and comparing with the results of section 2.1 of [72] we find that the gauge fixing condition is

\[
\cos \phi = \frac{1 + a_1}{\sqrt{(1 + a_1)^2 + a_2^2}}. \tag{6.20}
\]

\(^{11}\)To check this, one needs to work out the supersymmetry variation given in appendix C in the gauge (6.20) used in [45] and compare with the corresponding transformations in appendix A of [45]. In particular, the supersymmetry transformation rule of the dilaton differs by a factor of \( 2i \), and of the vielbein by a sign.
It will be convenient in what follows to consider \( u \) and \( v \) as the independent variables. Then in the gauge (6.20) we have

\[
u^* = \frac{1}{u-v} - v, \quad v^* = \frac{1}{u-v} - u\]

We next work out the modular covariant derivatives in terms of these independent variables,

\[
D_w = -\frac{1}{4} \left( (2u - v) \frac{\partial}{\partial v} + u \frac{\partial}{\partial u} \right) + \frac{w}{2}
\]

\[
D^*_w = -\frac{1}{4} \left( (2v - u) \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} \right) + \frac{w}{2}
\]

(6.22)

Recalling that \( F(v) \) is a function of two variables \( v \) and \( u^* \) but with \( u^* = Av + B \) (see (6.2)), we can further manipulate these formulas as

\[
D_w f(v, u^* = A + Bv) = \left[ \frac{1}{2} u \partial_v f(v, u^*) + Pf(v, u^*) + \frac{w}{2} f(v, u^*) \right]_{u^* = A + Bv}
\]

\[
D^*_w f(v, u^* = A + Bv) = \left[ Qf(v, u^*) + \frac{w}{2} f(v, u^*) \right]_{u^* = A + Bv}
\]

(6.23)

where

\[
P = \frac{1}{4} (v \partial_v - u \partial_u - v^* \partial_{v^*})
\]

\[
Q = -\frac{1}{4} ((2v - u) \partial_u - v \partial_v + (2u^* - v^*) \partial_{v^*})
\]

(6.24)

where after the differentiation one is instructed to substitute \( u^* = A + Bv \) and impose the gauge fixed relations (6.21). The operators \( P \) and \( Q \) satisfy the following relations,

\[
Pu^* = \frac{1}{4} u^*, \quad Qu^* = u^*,
\]

\[
P(u^* v^*) = 0, \quad Q(u^* v^*) = 2(u^*)^2,
\]

\[
Pa = 0, \quad QA = 0.
\]

(6.25)

### 6.3 \( SL(2, Z) \) invariance

After this detour we go back to the computation of \( t^{(11,-11)} \) and \( t^{(12,-12)} \). Using the definition in (6.2) we obtain

\[
t^{(11,-11)} = -4i2^{16} \frac{1}{u^{*22}} \sum_{n=0}^{7} c_n^{(11)} (u^* v^*)^n \tilde{W}^{(15-n)}(a)
\]

\[
t^{(12,-12)} = 2^{16} \frac{1}{u^{*24}} \sum_{n=0}^{6} c_n^{(12)} (u^* v^*)^n \tilde{W}^{(16-n)}(a)
\]

(6.26)

where the combinatorial coefficients are given by

\[
c_0^{(11)} = 1, \quad c_1^{(11)} = -90, \quad c_2^{(11)} = 3150, \quad c_3^{(11)} = -54600, \quad c_4^{(11)} = 491400,
\]

\[
c_0^{(12)} = 1, \quad c_1^{(12)} = -60, \quad c_2^{(12)} = 180, \quad c_3^{(12)} = -2880, \quad c_4^{(12)} = 23040.
\]
\[c_5^{(11)} = -2162160, \quad c_6^{(11)} = 3603600,\]
\[c_0^{(12)} = 1, \quad c_1^{(12)} = -112, \quad c_2^{(12)} = 5040, \quad c_3^{(12)} = -117600, \quad c_4^{(12)} = 1528800,\]
\[c_5^{(12)} = -11007360, \quad c_6^{(12)} = 40360320, \quad c_7^{(12)} = -57657600.\]  

Notice that the overall factors of \(u^*\) in (6.26) carry the local \(U(1)\) charge of \(t^{(w,-w)}\). Recall that \(\lambda\) and \(\psi^*\) carry local \(U(1)\) charge 3/2 and \(-1/2\), respectively. Thus the \(\lambda^{16}\) and \(\psi^*\lambda^{15}\) have \(U(1)\) charge 24 and 22, respectively. It follows that \(t^{(12,-12)}\) and \(t^{(11,-11)}\) should have local \(U(1)\) charge \(-24\) and \(-22\), and this is indeed the case.

It is now easy to check that (6.18) follows from (6.17). Indeed,
\[
D_{11}t^{(11,-11)} = -\frac{1}{2} u \partial_u t^{(11,-11)} + \left( P + \frac{11}{2} \right) t^{(11,-11)} = 2it^{(12,-12)}
\]
where we used (6.17) and the fact that \(t^{(w,-w)}\) is an eigenfunction of \(P\) with eigenvalue \(-w/2\). This follows from (6.25), and it is independent of the combinatorial factors \(c_n^{(w)}\). In this respect, gauge invariance is crucial in getting (6.18). Notice also that it was crucial to incorporate the contribution of the measure. The numerical coefficient in (6.17) and thus (6.29) depends on this contribution.

Let us now examine whether the coefficients are eigenfunctions of the appropriate Laplacian. Notice that for \((w,-w)\) forms one can define two Laplacians,
\[
\nabla^2_{(-)w} = 4D_{w-1}D_{w}^* = 4\tau_2^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau^*} - 2iw\tau_2 (\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \tau^*}) - w(w-1) \\
\nabla^2_{(+)w} = 4D_{-w}D_{-w}^* = 4\tau_2^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau^*} - 2iw\tau_2 (\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \tau^*}) - w(w+1)
\]
and the eigenfunctions satisfy
\[
\nabla^2_{(-)w} t^{(w,-w)} = \sigma_{w} t^{(w,-w)} \\
\nabla^2_{(+)w} t^{(w-1,-w+1)} = \sigma_{w} t^{(w-1,-w+1)}.
\]

The easiest way to check that (6.26) are eigenfunctions is to compute \(D_{-12}^* t^{(12,-12)}\). A short computation using (6.25) shows that, \(D_{-12}^* t^{(12,-12)} = Ct^{(11,-11)}\), if and only if the combinatorial factors satisfy
\[
c_n^{(11)} = -\frac{i}{8C} (n+1)c_{n+1}^{(12)}
\]
Inspection of the coefficients in (6.27) shows that this is indeed the case with \(C = 14i\), and we get
\[
D_{-12}^* t^{(12,-12)} = 14it^{(11,-11)}.
\]

It follows from these results that
\[
\nabla^2_{(-)12} t^{(12,-12)} = -112 t^{(12,-12)} \\
\nabla^2_{(-)11} t^{(11,-11)} = -90 t^{(11,-11)}.
\]
These results were also checked by a direct computation using the expression of the Laplacian in (6.30). Notice that the eigenvalue of the Laplacian in these cases, as well as
for $t^{(0,0)}$, is given by $c_1^{(w)}$. It can be seen from (6.9) and the fact that the superpotential $W$ is a holomorphic function of $a$ that $c_1^{(w)}$ has to be equal to the eigenvalue. Of course, the remaining combinatorial coefficients should be consistent with this fact too.

As is discussed in section 2.2 of [45], the eigenvalues of two modular forms that are related to each other by modular covariant derivatives are related in a specific manner. In particular, if $t^{(w-m,-w+m)}$ is related to $t^{(w,-w)}$ by the application of $m$ modular covariant derivatives and $\sigma_w$ is the eigenvalue of $t^{(w,-w)}$ then

$$\Delta_{(w-m,-w-m)}t^{(w-m,-w+m)} = (\sigma_w + 2mw - m^2 - m)t^{(w-m,-w+m)}$$ (6.35)

The case relevant for us is $w = 0$ and $m = -11$ and $m = -12$. We have computed earlier that $\sigma_0 = 20$ (see (6.10)). Applying (6.35) we precisely get (6.34)!

To summarize, we have computed the coefficient $t^{(0,0)}$, $t^{(11,-11)}$ and $t^{(12,-12)}$ of $r^4$, $\psi^*\lambda^5$ and $\lambda^6$ as a function of the superpotential and its derivatives, and we have shown by independent computations that all three are eigenfunctions of appropriate $SL(2, R)$ Laplacians, related to each other by modular covariant derivatives, and the corresponding eigenvalues are consistent with this fact. We have also checked that the supersymmetry constraint (6.18) derived earlier in [45] is automatically satisfied in our case.

The discussion so far was at the supergravity level. In string theory, the $SL(2, R)$ symmetry is believed to be replaced by a local $SL(2, Z)$ symmetry. The effective action should now be $SL(2, Z)$ symmetric, and as we next discuss this implies that $t^{(0,0)}$, $t^{(11,-11)}$ and $t^{(12,-12)}$ are uniquely fixed. In particular, the unique (up to multiplicative constants) non-holomorphic modular form which is an eigenfunction of the $SL(2, Z)$ Laplacian with eigenvalue 20 is the Eisenstein series [73]

$$E_5(\tau) = \sum_{\gamma \in \Gamma \setminus \Gamma_\infty} (\text{Im}(\gamma \tau))^5 \frac{1}{2} (\tau_2)^5 \sum_{(m,n)=1} \frac{1}{|m\tau + n|^10}$$ (6.36)

where $(m, n)$ is the greatest common divisor of $m$ and $n$ and

$$\Gamma_\infty = \{(\pm 1 \ n \ \pm 1) \in SL(2, Z) = \Gamma\}$$ (6.37)

($n$ is any integer). In other words, $E_5$ is obtained by starting from $\tau_2^7$, which manifestly is an eigenfunction of the Laplacian, and taking the $SL(2, Z)$ orbit. Since $\tau_2$ is invariant under the $T$ transformations $\tau \rightarrow \tau + 1$ generating $\Gamma_\infty$, the orbit excludes these elements. We thus conclude that (up to an overall constant)

$$t^{(0,0)}(\tau, \tau^*) = E_5(\tau, \tau^*)$$ (6.38)

The asymptotic form of $E_5$ is given by [73]

$$E_5 = c_1\tau_2^5 + c_2\tau_2^{-4} + \sum_{n \neq 0} a_n\tau_2^{1/2}K_{9/2}(2\pi|n|\tau_2) \exp(2\pi n\tau_1)$$ (6.39)

where $c_1$, $c_2$ and $a_n$ are (known) constants (see [73] p. 208), and $K_{9/2}$ is a modified Bessel function of the third kind. Notice also that there are no solutions with $c_1 = c_2 = 0$ (i.e. there are no cusp forms). The analysis so far was in the Einstein frame. Going over to the string frame leads to the result (1.2) given in the introduction. We discuss the significance of this result in the next section.
7 Discussion

We investigated in this paper the construction of a superinvariant in type IIB supergravity that contains the well-known $R^4$ term among its components. We looked for a superinvariant that can be constructed as an integral over half of IIB on-shell superspace. The construction involved the following steps. We first showed that the type IIB superspace admits a chiral superfield whose leading component is the dilaton. The linearized version of this superfield is well-known [25], but the non-linear version has not appeared before (although its existence was undoubtedly known by the experts). We presented an iterative construction of all its components, and explicitly discussed the first four. The iterative formulas originate from the solution of the Bianchi identities. The second step involves the construction of the appropriate chiral measure. We showed that one can systematically investigate the existence of a chiral measure by requiring supersymmetry. Supersymmetry leads to 32 conditions, and we showed that 16 of those can be used to uniquely determine the components of the chiral measure, but the other 16 should be satisfied automatically. It turns out that there is an obstruction at the next-to-leading order at the non-linear level. The obstruction is the same as the obstruction to the existence of chiral superfields.

IIB supergravity has an $SL(2, R)$ symmetry, and this symmetry can be realized linearly by the introduction of an extra auxiliary scalar and an extra $U(1)$ gauge invariance. It is useful to keep the extra gauge invariance because it constrains the couplings in the effective action. We constructed an 8-derivative term by integrating an arbitrary holomorphic function of the chiral superfields over half of superspace. Gauge invariance demands that this function is a product of a $U(1)$ compensator times an arbitrary holomorphic function of $a$ (or equivalently of $\tau$). The superpotential term is given in (5.1).

We further discussed the computation of the superpotential in components. In particular, we computed the complete dependence on the Riemann curvature (which actually enters only through the Weyl tensor $C$). We find that there are possible terms of the form $C^k(DF^5)^lF^m_5$, for appropriate $k, l, m$. In particular, the index contractions in the $R^4$ terms are exactly the ones appearing in the four-point graviton computation [1] (i.e. they involve the $t_8$ tensor).

We also computed the moduli dependent coefficients of the $R^4$, $\lambda^{16}$ and $\psi^*\lambda^{15}$. We showed that these coefficients are eigenfunctions of (appropriate) $SL(2, R)$ Laplacians with correlated eigenvalues. Imposing $SL(2, Z)$ invariance uniquely fixes these functions. In particular, the coefficient of the $R^4$ term turns out to be the non-holomorphic Eisenstein series $E_5$. Using the known asymptotics of $E_5$ we obtained that at weak coupling the coefficient of the $R^4$ term, in the string frame, consists of two “perturbative” terms $g_s^{-11/2}$ and $g_s^{7/2}$, see (1.2), plus non-perturbative corrections. As we discussed in the introduction, this asymptotic behavior cannot be generated by closed string perturbation theory, or any known non-perturbative effects. Notice that for sufficiently small string coupling constant the $R^4$ term dominates over the “leading” $R$ term. It thus seems that, if such a (non-supersymmetric) $R^4$ term is present, then in order to stay within the regime of the low energy effective actions, the dilaton would have to be stabilized at some non-zero value that depends on the curvature of the background.

As is well known, the $R^4$ term is generated in string perturbation theory at tree and one-loop level [1]. One may ask what is the superinvariant that is associated with these terms. Such a superinvariant has been constructed at the linearized level, i.e. in terms
of linearized on-shell superfields, for the $T^2$ reduction of the theory in [48]. The super-invariant was given as a sum of two “superpotential” terms. The first one involves an integral of an arbitrary function of a chiral superfield over 16 $\theta$'s, and the second is an integral of an arbitrary function of a linear superfield over a different set of 16 $\theta$'s. In total the integration involves 24 different $\theta$'s. In eight dimensions it is possible to choose two different sets of $\theta$'s and construct two different Lorentz invariant measures. In ten dimensions, however, there is only one Lorentz invariant measure that involves 16 $\theta$'s. This suggests that the oxidation of the eight dimensional construction to ten dimensions would involve non-scalar “superpotentials”. In other words, the integrand should transform under Lorentz transformations such that the action is Lorentz invariant. Such construction has been presented in [33], but in that paper the measure and the integrand transformed under an internal symmetry. In our case we would like them to transform under the Lorentz group.

Notice that our construction of the projections of $V$ also leads to an iterative construction of all projections of all other superfields that appear in the solution of the Bianchi identities, since the latter are related to $V$ by the application of fermionic covariant derivatives. The term we investigated here involves an arbitrary scalar function of the chiral superfield. It would be interesting to study more general constructions that may involve other superfields.

Our discussion so far was in IIB supergravity. The method, however, should be applicable to all other string theories. In particular, on-shell $N = 1$ supergravity [58] is completely determined by a scalar superfield $\Phi$. The coupling of the latter to Yang-Mills requires an additional vector superfield. The superspace of IIA supergravity has been discussed in [57]. In string theories, one can in principle compute the higher derivative corrections by means of string amplitude computations and $\sigma$-model computations. In M-theory, however, the absence of a microscopic formulation makes even more important the construction of superinvariants. Eleven dimensional supergravity is described on-shell by a single superfield $W_{rstu}$, totally antisymmetric in the flat Lorentz indices, that satisfies the constraint, $(\Gamma_{rst}D)W_{rstu} = 0$ [64, 65]. The leading component of $W_{rstu}$ is the field strength of the three form. It will be interesting to investigate whether one can use $W_{rstu}$ in order to construct a superinvariant associated with the higher derivative corrections to eleven dimensional supergravity.

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A Notation and conventions

We use the notation and conventions of [25], but we denote complex conjugation by $\ast$ (rather than a bar), and the $U(1)$ charges are normalized as in [69], i.e. they are half of the ones given in [25].

The superspace coordinates are

$$z^M = (x^m, \theta^\mu, \theta^{\bar \mu})$$

(A.1)

where $m = 1, \ldots, 10$ are the 10 bosonic coordinates, and $\theta^\mu \mu = 1, \ldots, 16$ are 16 complex fermionic coordinates. They form a 16 component Weyl spinor of $SO(9,1)$. We use the notation $(\theta^\mu)^* = \theta^{\bar \mu}$. Curved space vector and spinor indices are denoted by $m,n,\ldots$ and $\mu,\nu,\ldots$, respectively, while for tangent space indices we use $a,b,\ldots$ and $\alpha,\beta,\ldots$.

The vielbein one-form superfield is

$$E^A = dz^M E_M^A$$

(A.2)

with non-zero lowest $\theta = 0$ components

$$E_m^a| = e_m^a$$

$$E_m^\alpha| = \psi_m^\alpha$$

$$E_\mu^\alpha| = \delta_\mu^\alpha$$

$$E_\mu^\bar \alpha| = -\delta_\mu^\bar \alpha.$$  

(A.3)

Here $e_m^a$ is the bosonic vielbein, and $\psi_m^\alpha$ is the gravitino.

A $p$-form may be written as

$$\phi = E_{A_p}^{A_1 \ldots A_p} \phi_{A_1 \ldots A_p} = E_{M_p}^{M_1 \ldots M_p} \phi_{M_1 \ldots M_p}$$

$$E^{A_p \ldots A_1} = E^{A_p} \wedge \ldots \wedge E^{A_1}, \quad E^{M_p \ldots M_1} = dz^{M_p} \wedge \ldots \wedge dz^{M_1}.$$  

(A.4)

The exterior derivative is given by

$$d\phi = E_{M_{p+1} \ldots M_1} \partial M_1 \phi_{M_2 \ldots M_{p+1}} = E_{A_{p+1} \ldots A_1} (D_{A_1} \phi_{A_2 \ldots A_{p+1}} + \frac{1}{2} T_{A_1 A_2} B \phi_{B A_3 \ldots A_{p+1}}).$$  

(A.5)

The superspace scalar product of two vectors is

$$U^A V_A = U^a V_a + U^\alpha V_\alpha - U^{\bar \alpha} V_{\bar \alpha}$$

(A.6)

where the minus sign in the last term is for reality of the scalar product.

Following [25], we use a mostly minus metric,

$$\eta_{ab} = \text{diag}(+1, -1, \ldots, -1)$$

(A.7)

and we use the $\Gamma$ matrices

$$\Gamma^0 = \sigma_1 \otimes 1_{16}$$

$$\Gamma^i = i \sigma_2 \otimes \gamma^i \quad i = 1, \ldots, 9$$

$$\Gamma_{11} = \sigma_3 \otimes 1_{16}$$

(A.8)
where $\sigma_k$ are the Pauli matrices, and $\gamma^i$ are the 16 dimensional hermitian gamma matrices. The 16-dimensional gamma matrices $\gamma^i$, $i = 1, \ldots, 8$, decompose under $SO(8)$ as follows:

$$\gamma^i = \begin{pmatrix} 0 & \gamma^i_{\dot{a}a} \\ \gamma^i_{\dot{a}a}^\dagger & 0 \end{pmatrix},$$ (A.9)

where $a$ and $\dot{a}$ are 8, and 8s indices, respectively. The matrices $\gamma^i_{\dot{a}a}$ are given in appendix 5.B of [74]. This decomposition will be useful in appendix E. The ninth $16 \times 16$ gamma matrix is given by

$$\gamma^9 = \gamma_1 \gamma_2 \cdots \gamma_8 = \begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix}.$$ (A.10)

The representation (A.8) can be rewritten as

$$\Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ \dot{\gamma}^i & 0 \end{pmatrix}, \quad i = 0, \ldots, 9$$

$$\Gamma_{11} = \sigma_3 \otimes 1_{16}$$ (A.11)

where $\dot{\gamma}$ is defined by (A.8). The $\gamma$'s used in the text are the $\gamma^i$ given here. The antisymmetrized product of $\gamma$ matrices has the property that $\gamma_{a}, \gamma_{abcd}$ and $\gamma_{abcde}$ are symmetric, and $\gamma_{ab}$ and $\gamma_{abc}$ are antisymmetric. In all explicit formulas it is only $\gamma$ that appears.

For arbitrary $32 \times 32$ matrices $M, N$ and 32-dimensional complex spinors $\chi, \lambda$, the Fierz rearrangement formula reads [75]:

$$M\chi\bar{\lambda}N = -\frac{1}{32} \sum_{n=0}^{5} c_n (\bar{\lambda}NG^{(n)}M\chi)\Gamma^{(n)},$$ (A.12)

where $c_0 = 2, c_1 = 2, c_2 = -1, c_3 = -\frac{1}{3}, c_4 = \frac{1}{12}$ and $c_5 = \frac{1}{120}$ and $\Gamma^{(n)} = \Gamma^{i_1 \cdots i_n}$.

One may derive from (A.12) useful lemmas. Consider a complex chiral spinor $\chi$. The only non-vanishing bilinear is the axial current $\bar{\chi}^\sigma \Gamma^{\mu\nu\sigma} \chi$. Combining results obtained by Fierzing $\bar{\chi}^\sigma \Gamma^{\mu\nu\sigma} \chi$ and $\bar{\chi}^\lambda \Gamma^{\lambda\mu\nu} \chi \bar{\chi}^\sigma \Gamma^{\sigma}$, we obtain

$$\bar{\chi}^\sigma \Gamma^{\mu\nu\sigma} \chi = \frac{1}{2} \bar{\chi}^\lambda \Gamma^{\lambda\mu\nu} \chi \bar{\chi}^\sigma \Gamma^{\sigma}.$$ (A.13)

This is the analog of (A.3) in [75] but for complex spinors. Now multiplying from the right by $\Gamma^{(3)} \chi$ we get

$$\ddot{\chi}^\nu \Gamma^{abc} \chi \bar{\chi}^\lambda \Gamma^{d} \chi = 0 .$$ (A.14)

Using this formula we will show in appendix E that the fourth projection of the $V$ contains only the Weyl tensor.

For the graded commutator of two covariant derivatives we have

$$[D_A, D_B] = -T_{AB}^C D_C + \frac{1}{2} R_{ABC}^D L_D^C + 2i M_{AB}\kappa,$$ (A.15)

where $T_{AB}^C$ is the torsion, $R_{ABC}^D$ is the curvature tensor, $M_{AB}$ is the $U(1)$ curvature, $\kappa$ is the $U(1)$ generator, and $L_{AB}$ are the generators of $SO(9,1)

$$L_{ab} = -L_{ba}, \quad L_{\alpha}^\beta = \frac{1}{4} (\gamma^ab)_{\alpha}^\beta L_{ab}, \quad L_{\dot{\alpha}}^{\dot{\beta}} = -L_{\dot{\alpha}}^{\dot{\beta}}, \quad L_{ab} = L_{a\beta} = L_{\alpha}^{\beta} = L_{\dot{\alpha}}^{\dot{\beta}} = 0.$$ (A.16)
The covariant derivative acting on a field $\phi_B$ of $U(1)$ charge $q$ is given by

$$D_A \phi_B = E^M_A D_M \phi_B = E^M_A (\partial_M \phi_B + \Omega_{MB}^C \phi_C + 2iqQ_M \phi_B)$$

$$= \partial_A \phi_B + \Omega_{AB}^C \phi_C + 2iqQ_A \phi_B . \quad (A.17)$$

The spin-connection $\Omega_B^C = dz^M \Omega_{MA}^B$ has the same symmetry properties as $L_A^B$.

Complex conjugation (more properly Hermitian conjugation) reverses the order of elements,

$$(AB)^* = B^* A^* . \quad (A.18)$$

It acts on derivatives as

$$(\partial_A)^* = -(-)^A \partial_A . \quad (A.19)$$

## B IIB supergeometry

In [25] the torsion and curvature tensors are computed from Bianchi identities. We list here without derivation all non-zero components. We closely follow their notations.

$$T^c_{\alpha \beta} = -i \gamma^c_{\alpha \beta}$$

$$T^\gamma_{\alpha \beta} = (\gamma^c)_{\alpha \beta} (\gamma_a)^\gamma \Lambda^*_a - 2 \delta^\gamma_{(\alpha} \Lambda_{\beta)}^*$$

$$T^\gamma_{a \beta} = -\frac{3}{16} (\gamma^{bc})_{\beta}^\gamma F^*_{abc} - \frac{1}{48} (\gamma_{abcd})_{\beta}^\gamma F^{*bcd}$$

$$T^\gamma_{a \beta} = i \left( \frac{21}{2} X_{a \beta}^\gamma + \frac{3}{2} (\gamma_{ab})_{\beta}^\gamma X^b + \frac{5}{4} (\gamma^{bc})_{\beta}^\gamma X_{abc} + \frac{1}{4} (\gamma_{abcd})_{\beta}^\gamma X^{bcd} + (\gamma^{bcde})_{\beta}^\gamma Z_{abcde} \right)$$

$$T^\alpha_{ab} = \Psi^\alpha_{ab}$$

$$T^\gamma_{a \beta} = -(T^\alpha_{a \beta})^\gamma$$

$$T^\gamma_{a \beta} = -(T^\alpha_{a \beta})^\gamma$$, \quad (B.1)

with

$$Z_{abcde} = Z^+_{abcde} + \frac{1}{48} X_{abcde} = \frac{1}{192} F_{abcde} + \frac{1}{16} X_{abcde}$$

$$X_{a_1..a_i} = \frac{1}{16} \Lambda^*_{a_1..a_i} A \ . \quad (B.2)$$

The nonzero curvature components are

$$R_{\alpha \beta ab} = i \left( \frac{3}{4} (\gamma^c)_{\alpha \beta} F^*_{abc} + \frac{1}{24} (\gamma_{abcde})_{\alpha \beta} F^{*cde} \right)$$

$$R_{\alpha \beta ab} = - \left( 3(\gamma_{abc})_{\alpha \beta} X^c + 5(\gamma^c)_{\alpha \beta} X_{abc} + \frac{1}{2} (\gamma_{abcde})_{\alpha \beta} X^{cde} \right.$$

$$+ \left. \frac{1}{2} (\gamma^{cde})_{\alpha \beta} \left( \frac{1}{12} F_{abcde} + X_{abcde} \right) \right)$$

$$R_{\alpha \beta \gamma \delta} = \frac{1}{4} (\gamma^{ab})_{\gamma \delta} R_{\alpha \beta ab}$$

$$R_{\alpha \beta \gamma \delta} = \frac{1}{4} (\gamma^{ab})_{\gamma \delta} R_{\alpha \beta ab}$$

$$R_{abcd} = -\frac{1}{2} i \left( (\gamma_b)_{\alpha \beta} \Psi_{cd} \bar{\beta} + (\gamma_c)_{\alpha \beta} \Psi_{bd} \bar{\beta} - (\gamma_d)_{\alpha \beta} \Psi_{bc} \bar{\beta} \right), \quad (B.3)$$

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and the non-zero U(1) curvature components are
\[
M_{a\beta} = 2i\Lambda_a\Lambda^*_\beta \\
M_{ab} = -i\hat{P}_a\Lambda_b \\
M_{ab} = -iP_{[a}P_{b]}.
\] (B.4)

The non-zero components of the tensors \(P_A, F_{ABC}\) and \(F_{ABCDE}\) are
\[
P_a = -2\Lambda_a, \quad P_a \\
F_{a\beta\gamma} = F^{*}_{a\beta\gamma} = -i(\gamma_a)_{\beta\gamma}, \quad F_{abc} \\
F_{a\delta\gamma} = -i(\gamma_{ab})_\delta\gamma \quad F^{*}_{a\delta\gamma} = -(\gamma_{ab})_\delta\Lambda^*_\gamma \\
F_{a\delta\epsilon\gamma} = (\gamma_{abc})_{\alpha\delta} \\
F_{abcde} = F^{+}_{abcde} - 8X_{abcde},
\] (B.5)

where \(F^+\) denotes the self-dual part.

C Supersymmetry transformation rules

In this appendix we list the supersymmetry transformation rules. In these formulae, we corrected several typos in the original paper of Howe and West [25]\(^{12}\), but we note that despite considerable effort we were unable to exactly match the transformation rules below to the ones reported in [70, 69], indicating that there maybe typos in [70, 69] and/or in the solution of Bianchi’s given in the previous appendix.

\[
\delta e_m{}^a = -i ((\zeta^*\gamma^a\psi_m) + (\zeta\gamma^a\psi^*_m))
\] (C.1)

\[
\delta\psi_m = \nabla_m\zeta - \frac{3}{16}\hat{f}_{mba}\gamma^{ab}\zeta^* + \frac{1}{48}\hat{f}_{abcd}\gamma_{mbcd}\zeta^* - \frac{1}{192}i\hat{g}_{mbcd}\gamma^{abcd}\zeta \\
+ \frac{1}{16}i \left[ -\frac{21}{2}(\lambda^*\gamma_m\lambda) + \frac{3}{2}(\lambda^*\gamma^a\lambda)\gamma_{ma} + \frac{5}{4}(\lambda^*\gamma_{mab}\lambda)\gamma^{ab} - \frac{1}{4}(\lambda^*\gamma^{abc}\lambda)\gamma_{mabc} \\
- \frac{1}{16}(\lambda^*\gamma_{mbcd}\lambda)\gamma^{abcd} \right] \zeta - (\zeta^*\gamma^a\psi^*)\gamma_{a\lambda} + (\psi^*_m\lambda)\zeta^* - (\zeta\gamma^a\psi^*_m) 
\] (C.2)

\[
\delta u = 2(\zeta^*\lambda^*)u \quad \delta v = -2(\zeta\lambda)u
\] (C.3)

\[
\delta\lambda = \frac{1}{24}i\hat{f}_{abc}\gamma^{abc}\zeta + \frac{1}{2}i\hat{p}_a\gamma^a\zeta^*
\] (C.4)

\[
\delta(a_{mn}^*, a_{mn}) = -\left( (\zeta\gamma_{mn}\lambda) + 2i(\zeta^*\gamma_{[m}\psi_{n]}), (\zeta^*\gamma_{mn}\lambda) - 2i(\zeta\gamma_{[m}\psi_{n]}) \right) \mathbb{V}^{-1}
\] (C.5)

\[
\delta b_{mnrs} = -4(\lambda^*\gamma_{[mn}\psi_{s]}^*) + 4(\lambda^*\gamma_{[mn}\psi_{s]}^*) + 12i \left( a_{[mn}\delta a_{rs]}^* - a_{[ma}\delta a_{rs]}' \right)
\] (C.6)

where \(\mathbb{V}\) is given in (3.2).

Here \(\hat{p}_a, \hat{f}_{abc}\) and \(\hat{f}_{abcde}\) are the leading components of the corresponding superfields
\[
P_a| = \hat{p}_a = p_a + 2(\psi_a\lambda) \\
F_{abc}| = \hat{f}_{abc} = f_{abc} - 3(\psi^*_a\gamma_{bc}\lambda) - 3i(\psi_{[a}\gamma_{b}\psi_{c]}) \\
F_{abcde}| = \hat{f}_{abcde} = f_{abcde} + 20(\psi^*_a\gamma_{bcd}\psi_{e}]).
\] (C.7)

\(^{12}\)We thank Paul Howe and Peter West for confirming the corrections.
D $F_5^2$ terms in the dilaton superfield

We provide in this appendix some details leading to (4.18). Starting from (4.16) and collecting only the $F_5^2$ terms we get

\[
D_{[\beta} D_{\gamma} D_{\delta]} V |_{f_3 f_5} = - \frac{1}{4} u \left\{ \frac{1}{16912} (\gamma_{abc}) [\beta (\gamma^{ijk}) \gamma \delta] \epsilon_{aijkmnpn'p'} f^{bmnpl} f^{cm'n'p'} I - \frac{1}{192} (\gamma_{abc}) [\beta (\gamma_a^{de}) \gamma \delta] f^{bemnp} f^{cd mnp} \right\}. \tag{D.1}
\]

To derive this we expressed products of gamma matrices into completely antisymmetric combinations and we used the fact that only $\gamma^{ab}$ and $\gamma^{abc}$ (and $\gamma^{(8)}$ and $\gamma^{(7)}$) are antisymmetric in the spinor indices. Furthermore, we used the gamma matrix identity $\gamma_{a_1 a_7} = -\frac{1}{32} \epsilon_{a_1 a_7 b_1 b_2 b_3} \gamma^{b_1 b_2 b_3}$ in the first term on the right hand.

Let us define

\[
\begin{align*}
A &= (\gamma^{abc}) [\beta (\gamma^{def}) \gamma \delta] f_{ba fmn} f_{ced}^{mn} \\
B &= (\gamma^{abc}) [\beta (\gamma_a^{de}) \gamma \delta] f_{bemnp} f_{cd mnp} \\
C &= (\gamma^{abc}) [\beta (\gamma_{ade}) \gamma \delta] f_{bemnp} f_{demnp} \\
D &= (\gamma^{abc}) [\beta (\gamma^{de}) \gamma \delta] f_{abcmn} f_{def mn} \\
E &= (\gamma^{a bc}) [\beta (\gamma^{ijk}) \gamma \delta] \epsilon_{aijkmnpn'p'} f^{bmnpl} f^{cm'n'p'} I.
\end{align*}
\]

$B$ and $E$ appear in (D.1).

$B$ and $C$ can be shown to be equal to zero due to the self-duality of $f_{(5)}$. We show this for $C$,

\[
\begin{align*}
C &= \gamma^{abc} \gamma_{ade} f_{bemnp} f_{demnp} \\
&= \gamma^{abc} \gamma_{ade} \frac{1}{(5!)^2} \epsilon_{bemnpa_1 a_5} \epsilon^{demnpb_1 b_5} f^{a_1 a_5} f_{b_1 b_5} \\
&= -\gamma^{abc} \gamma_{ade} \frac{3!}{(5!)^2} \delta_{bca_1 a_5} f^{a_1 a_5} f_{b_1 b_5} \\
&= -\gamma^{abc} \gamma_{ade} \frac{2! 3!}{5!} \left( \delta_{bc} \delta_{a_1 a_5} - 10 \delta_{bc} \delta_{a_2 a_5} + 10 \delta_{bc} \delta_{a_3 a_5} \right) f^{a_1 a_5} f_{b_1 b_5} \\
&= -\gamma^{abc} \gamma_{ade} f_{bemnp} f_{demnp} \\
&= -C, \tag{D.3}
\end{align*}
\]

where we supress the spinor indices. Notice that we used (A.14), i.e. $(\gamma^{abc}) [\beta (\gamma_{abd}) \gamma \delta] = 0$. A similar computation establishes that $B = 0$. Using the self-duality of $f_{(5)}$ in $A$ and $D$ one finds again $B = C = 0$ (and no constraint on $A$ and $D$). Furthermore, similar manipulations yield

\[
E = 18(3A + 2B + C - D). \tag{D.4}
\]

We thus obtain

\[
E = 18 \gamma^{abc} \gamma^{def} (3 f_{ba fmn} f_{ced}^{mn} - f_{abcmn} f_{def mn}). \tag{D.5}
\]

Combining these formulas we finally get

\[
D_{[\beta} D_{\gamma} D_{\delta]} V |_{f_3 f_5} = - \frac{u}{1536} \gamma^{abc} \gamma^{def} (3 f_{ba fmn} f_{ced}^{mn} - f_{abcmn} f_{def mn}) \tag{D.6}
\]

which is (4.18).
In this appendix we show that the \( R^4 \) term appearing among the components of the superpotential term has the same structure as the usual \( R^4 \) term that appears in string scattering amplitudes. The computation presented here follows closely the analysis in [41] and it is included for completeness.

In section 6 we showed that this term comes from the fourth projection of the scalar superfield. The relevant terms are given by

\[
S_{r^4} = \int d^{10}x e \frac{1}{(4!)^5} \epsilon^{a_1...a_{16}}(D_{a_1}...D_{a_4}V)(D_{a_5}...D_{a_8}V)(D_{a_9}...D_{a_{12}}V)(D_{a_{13}}...D_{a_{16}}V)
\]

(E.1)

Using (4.20) and keeping only the term proportional to \( c^4 \) we get

\[
S_{r^4} = \int d^{10}x e \frac{1}{(4!)^5 16^4} u^4 F^{(4)}(v) \epsilon^{a_1...a_{16}}(\gamma^{ai_1i_2}_{a_1a_2} \gamma^{i_3i_4}_{a_3a_4} c_{i_1i_2i_3i_4}) \times
\]

\[
\times \cdots (\gamma^{d_1l_2}_{a_13a_{14}} \gamma^{l_3l_4}_{a_15a_{16}} c_{l_1l_2l_3l_4})
\]

(E.2)

One may similarly manipulate the \( F_5 \)-dependent terms, but we shall not do this here.

The aim of this appendix is to eliminate the gamma matrices from (E.2). The resulting expression will have only vector indices contracted among themselves. This can be done in several ways, and we will choose to go via a route motivated by the light-cone computations. A covariant computation is done in appendix B of [24]. We first go to light-cone coordinates by decomposing space-time indices into longitudinal and transverse indices, \((+, -, i)\), where \( i = 1, \ldots, 8 \). We will also decompose SO(9,1) spinors into SO(8) spinors in the \( 8 \) and \( 8^\ast \) representations, which we will denote by undotted/dotted indices, respectively. So, in the representation (A.11) a spinor of negative chirality, \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \theta \), decomposes under SO(8) as \( \theta = (\theta_a, \dot{\theta}^a) \). The epsilon tensor in (E.2) also factorizes into a product of two epsilon tensors. Since chiral spinors have \( 8 + 8 \) independent components, we can use them to represent the epsilon tensors:

\[
\int d^8 \theta \theta^{a_1} \cdots \theta^{a_8} = \epsilon^{a_1...a_8}, \quad \int d^8 \dot{\theta} \dot{\theta}^{\dot{a}_1} \cdots \dot{\theta}^{\dot{a}_8} = \epsilon^{\dot{a}_1...\dot{a}_8}.
\]

(E.3)

This will be a useful bookkeeping device that simplifies our intermediate expressions. We also decompose the gamma matrices as in (A.9). In the remaining of this appendix we deal with the \( 8 \times 8 \)-dimensional gamma matrices only. The expression (E.2) then reduces to

\[
S_{r^4} = \int d^{10}x d^8 \theta d^8 \dot{\theta} e \frac{1}{(4!)^5 16^4} u^4 F^{(4)}(v) \Phi^4[c]
\]

(E.4)

where

\[
\Phi[c] = 2(-\theta \gamma^{ij} \hat{\theta} \gamma^{kl} \dot{\hat{\theta}} + \theta \gamma^{ijk} \hat{\theta} \gamma^{mkl} \dot{\hat{\theta}}) c_{ijkl}.
\]

(E.5)

This is the transverse contribution, and we have omitted longitudinal terms which contribute to covariantize the final expression. This is the first line in formula (56) of [41], but already rewritten in terms of the Weyl tensor. Indeed, either from the SO(9,1) Fierzing or
directly in light-cone gauge one can check that all the terms that include the Ricci tensor cancel out.

To remove the space-time indices \( m \) that are summed over in the above expression, we perform SO(8) Fierzings:

\[
\theta_a \theta_b = \frac{1}{16} (\theta \gamma_{ij} \theta) \gamma^{ij}_{ab}, \quad \hat{\theta}_a \hat{\theta}_b = \frac{1}{16} (\hat{\theta} \gamma_{ij} \hat{\theta}) \gamma^{ij}_{ab}
\]  

(E.6)

and after computing the traces over products of SO(8) \( \gamma \)'s we get:

\[
(\theta \gamma^{ijk} \hat{\theta})(\hat{\theta} \gamma^{lm} \theta) c_{ijklm} = -2(\theta \gamma^{ij} \hat{\theta})(\hat{\theta} \gamma^{kl} \theta) c_{ijkl},
\]  

(E.7)

and again we suppress terms proportional to traces of the Weyl tensor. Since these are traces over transverse indices, they are not zero but rather recombine with remaining terms from the longitudinal part to give a vanishing contribution to the final result. Therefore the contribution in (E.7) is the one that covariantizes to the final 10d expression. Filling (E.7) into (E.5), we get

\[
\Phi[c] = -6(\theta \gamma^{ij} \hat{\theta})(\hat{\theta} \gamma^{kl} \theta) c_{ijkl},
\]  

(E.8)

which is the second line of (56) in [41] (but with the overall coefficient corrected).

Filling (E.8) in the expression for the action (E.4), and using the representation (E.3), we get

\[
S_{R^4} = \int d^{10}x \, e^{t^{(0,0)}} \epsilon^{a_1 \ldots a_8} \epsilon^{\hat{a}_1 \ldots \hat{a}_8} \left( \gamma^{i_1 j_1 \hat{i}_1 \hat{j}_1} c_{i_1 j_1 k_1 l_1} \right) \cdots \left( \gamma^{i_4 j_4 \hat{i}_4 \hat{j}_4} c_{i_4 j_4 k_4 l_4} \right)
\]  

(E.9)

where \( t^{(0,0)} \) is given in (6.11).

As shown in [41], this is equal to

\[
S_{r^4} = \int d^{10}x \, e^{t^{(0,0)}} \epsilon^{i_1 j_1 \ldots i_4 j_4} \left( \epsilon^{k_1 \ldots k_4 l_1 \ldots l_4} - \frac{1}{2} \epsilon^{k_1 \ldots k_4} c_{i_1 j_1 k_1 l_1} \cdots c_{i_4 j_4 k_4 l_4} \right)
\]  

(E.10)

The tensor \( t_8 \) is the well-known kinematical factor appearing in the tree-level and one-loop string scattering amplitudes. The explicit expression can be found in, e.g., appendix 9.A of [74]. It is now straightforward to write the 10d covariant expression. As advertised, we get the usual combination of Weyl tensors:

\[
S_{r^4} = \int d^{10}x \, e^{t^{(0,0)}} \left( c^{hmn} c_{pqmn} c_{h^{rsp}} c_{rsk} + \frac{1}{2} c^{hkn} c_{pqmn} c_{h^{rsp}} c_{rsk} \right).
\]  

(E.11)

References


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