Burkhardt-Cottingham sum rule and forward spin polarizabilities in Heavy Baryon Chiral Perturbation Theory

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Abstract

We study spin-dependent sum rules for forward virtual Compton scattering (VVCS) off the nucleon in heavy baryon chiral perturbation theory (HBChPT) at order $O(p^4)$. We show how these sum rules can be evaluated from low energy expansions (in the virtual photon energy) of the forward VVCS amplitude. We study in particular the Burkhardt-Cottingham sum rule in HBChPT and the higher terms in the low energy expansion, which can be related to generalized forward spin polarizabilities of the nucleon. The dependence of these observables on the photon virtuality $Q^2$ can be accessed, at small and intermediate $Q^2$ values, from existing and forthcoming data at Jefferson Lab.

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I. INTRODUCTION

Low energy (real) Compton scattering off the nucleon is a very useful tool to investigate global properties of nucleon structure. Low energy theorems \cite{1,2} express the leading terms in the energy of the forward Compton amplitude in terms of total charge, mass and anomalous magnetic moment of the system. The higher terms in such a low energy expansion can be identified with the response of the nucleon to an external electromagnetic field, parametrized by dipole and higher order nucleon polarizabilities (see Ref. \cite{3} for a recent review and references therein).

Sum rules for forward real Compton scattering directly connect these low energy quantities to the nucleon excitation spectrum. Some well known sum rules are the Baldin sum rule \cite{4} which connects the sum of electric and magnetic nucleon polarizabilities to the total photoabsorption cross section on the nucleon, as well as the Gerasimov-Drell-Hearn (GDH)
sum rule [5,6]. The latter sum rule establishes a connection between the nucleon anomalous magnetic moment and the helicity difference cross section for photon scattering with helicity parallel or anti-parallel to the nucleon helicity. The GDH sum rule has been the subject of several experiments in recent years [7,8].

It has also been emphasized that the above sum rules can be generalized to the virtual photon case, see e.g. Refs. [9,10,3]. For forward scattering of spacelike virtual photons (with virtuality $Q^2$) on the nucleon, the corresponding sum rules relate nucleon structure quantities to inclusive electroproduction cross sections. At large $Q^2$, they yield the sum rules studied in deep-inelastic scattering (DIS) experiments (see Ref. [11] for a recent review), such as the Bjorken sum rule [12] or the Burkhardt-Cottingham (BC) sum rule [13]. The Bjorken sum rule involves the difference of the first moments of the proton and neutron helicity structure functions $g_1^p - g_1^n$, and has been verified by several experiments [11]. The BC sum rule on the other hand, which involves the first moment of the second spin dependent structure function $g_2$ of the nucleon, has been addressed by a dedicated experiment only very recently [14]. So far, the experiment does not allow for a conclusive test of the BC sum rules due to an unmeasured small-$x$ region and awaits further experiments.

Having measured forward Compton sum rules at the real photon point and at large $Q^2$, through DIS, one may ask the question how these sum rules interpolate between both limits when one varies $Q^2$, which plays the role of the spatial resolution at which one probes the nucleon. The study of such sum rules for the virtual photon nucleon Compton amplitudes as function of $Q^2$ from the real photon point to large $Q^2$, opens up the perspective to map out in detail the transition from the resonance dominated regime at low $Q^2$ to the partonic regime at large $Q^2$, described by perturbative QCD. The theoretical tool to analyse the large $Q^2$ regime is given by the operator product expansion (OPE). At low $Q^2$, below about 0.5 GeV$^2$, where one is in the non-perturbative realm of QCD, a theoretical tool to investigate these sum rules is provided by chiral perturbation theory (ChPT). Using ChPT, the first moment of $g_1$ has been studied in Refs. [10,15,16]. It has been pointed out in Ref. [17] that in the first moment of the proton - neutron difference $g_1^p - g_1^n$ the $\Delta(1232)$ and other isospin $3/2$ resonances drop out, and therefore this might be a promising observable to look for a smooth transition between the ChPT result at low $Q^2$ and OPE description at large $Q^2$.

In this letter, we study in the framework of the heavy-baryon ChPT (HBChPT) to order $\mathcal{O}(p^4)$, the first moment of the structure function $g_2$ and the BC sum rule, as well as sum rules involving higher moments of the spin dependent structure functions, which have the physical interpretation of generalized forward spin polarizabilities.

In Section 2, we start with the general formalism for spin dependent forward virtual Compton scattering (VVCS). For the two spin-dependent VVCS amplitudes, we write down dispersion relations (DRs) and discuss the nucleon pole contribution. For the non-pole contributions, we show that one can proceed to write down a low energy expansion (LEX) in the virtual photon energy. The DRs allow us to write down sum rules for each term in such a LEX. The lowest terms in the LEX are given by the first moments of the nucleon structure functions $g_1$ and $g_2$, and we show how sum rules for the higher terms can be interpreted as generalized forward spin polarizabilities.

In Sections 3 and 4, we then proceed and calculate the terms in such a LEX in HBChPT. In particular in Section 3, we calculate the $Q^2$ dependence of the lowest moment of the nucleon structure function $g_2$ and discuss the validity of the BC sum rule in HBChPT at order
In Section 4, we present the results for the higher terms in the LEX in HBChPT at order $\mathcal{O}(p^4)$, and give the corresponding expressions for the generalized forward spin polarizabilities of the nucleon. We study the convergence of the HBChPT expansion by comparing the results at order $\mathcal{O}(p^3)$ and $\mathcal{O}(p^4)$. We also compare the $Q^2$ dependence of the HBChPT results with those of a phenomenological resonance estimate, using the MAID model [18].

Finally, in Section 5 we present our conclusions.

II. SPIN DEPENDENT FORWARD VIRTUAL COMPTON SCATTERING (VVCS)

We start with the spin-dependent doubly virtual Compton scattering amplitude in the forward direction (VVCS) as given in [3]:

$$ T(\nu, Q^2, \theta = 0)^{\text{spin}} = i \sigma \cdot (\varepsilon' \times \varepsilon) g_{TT}(\nu, Q^2) - i \sigma \cdot [\varepsilon' - \varepsilon] g_{LT}(\nu, Q^2), $$

where $\nu$ is the virtual photon energy, $Q^2$ its virtuality, and $\hat{q}$ the unit vector along the virtual photon momentum. Furthermore, $\varepsilon$ ($\varepsilon'$) are the polarization vectors of the initial (final) photons. In the VVCS amplitudes $g_{TT}$ and $g_{LT}$, $T$ (L) denote the transverse (longitudinal) virtual photon polarizations. In order to relate with the usual nucleon structure functions $g_1$ and $g_2$, it is also useful to cast Eq. (1) in the following covariant form:

$$ T(\nu, Q^2, \theta = 0)^{\text{spin}} = \varepsilon'^\mu \varepsilon_\nu \left\{ \frac{i}{M} \epsilon^{\mu \nu \alpha \beta} q_\alpha s_\beta S_1(\nu, Q^2) + \frac{1}{M^2} \epsilon^{\mu \nu \alpha \beta} q_\alpha (p \cdot q s_\beta s_\mu - s \cdot q p_\beta) S_2(\nu, Q^2) \right\}, $$

where $s^\alpha$ is the nucleon covariant spin vector satisfying $s^2 = -1$ and $s \cdot p = 0$, with $p$ ($q$) the nucleon (photon) four-momenta. The relation between the spin-dependent amplitudes $g_{TT}$, $g_{LT}$ and $S_1, S_2$ is given by [3]:

$$ S_1(\nu, Q^2) = \frac{\nu M}{\nu^2 + Q^2} \left( g_{TT}(\nu, Q^2) + \frac{Q}{\nu} g_{LT}(\nu, Q^2) \right), $$
$$ S_2(\nu, Q^2) = -\frac{M^2}{\nu^2 + Q^2} \left( g_{TT}(\nu, Q^2) - \frac{\nu}{Q} g_{LT}(\nu, Q^2) \right). $$

Next, we can write down dispersion relations (DR) for the VVCS amplitudes $g_{TT}$ and $g_{LT}$. Assuming that $g_{TT}(\nu, Q^2)$ and $g_{LT}(\nu, Q^2)$ drop sufficiently fast for $\nu \to \infty$ to ensure the convergence of the integrals (see Ref. [3] for a detailed discussion), one can write down unsubtracted DRs for $g_{TT}$ and $g_{LT}$, which take the form:

$$ \text{Re} g_{TT}(\nu, Q^2) = \text{Re} g_{TT}^{\text{pole}}(\nu, Q^2) + \frac{\nu}{2\pi^2} \mathcal{P} \int_{\nu_0}^{\infty} \frac{K(\nu', Q^2) \sigma_{TT}(\nu', Q^2)}{\nu'^2 - \nu^2} d\nu', $$
$$ \text{Re} g_{LT}(\nu, Q^2) = \text{Re} g_{LT}^{\text{pole}}(\nu, Q^2) + \frac{1}{2\pi^2} \mathcal{P} \int_{\nu_0}^{\infty} \frac{\nu' K(\nu', Q^2) \sigma_{LT}(\nu', Q^2)}{\nu'^2 - \nu^2} d\nu', $$

where
where the first terms in Eqs. (5,6) are the pole or elastic contributions due to the s- and u-channel singularities at \( \nu = \pm \nu_B \), with \( \nu_B \equiv Q^2/(2M_N) \), where \( M_N \) is the nucleon mass. These pole contributions are given by:

\[
\text{Re } g_{TT}^{\text{pole}}(\nu, Q^2) = -\frac{\alpha_{\text{em}}}{2M_N} \frac{\nu^2}{\nu^2 - \nu_B^2} G_M^2(Q^2),
\]

\[
\text{Re } g_{LT}^{\text{pole}}(\nu, Q^2) = -\frac{\alpha_{\text{em}} Q}{2M_N} \frac{Q^2}{\nu^2 - \nu_B^2} G_E(Q^2) G_M(Q^2),
\]

with \( \alpha_{\text{em}} \) the fine structure constant (\( \alpha_{\text{em}} = 1/137 \)), and \( G_E(G_M) \) the nucleon electric (magnetic) form factors. The dispersion integrals in Eqs. (5,6) correspond to the cut contributions along the real axis and start at the threshold for pion production \( \nu_0 = m_\pi + (m_\pi^2 + Q^2)/2M_N \), with \( m_\pi \) the pion mass. The integrals in Eqs. (5,6) involve the product of the partial cross sections \( \sigma_{TT} \) and \( \sigma_{LT} \), multiplied by a photon flux factor \( K \) (with dimension of energy)\(^1\).

These partial cross sections are related to the nucleon structure functions \( g_1 \) and \( g_2 \) as:

\[
K \cdot \sigma_{TT} = \frac{4\pi^2\alpha_{\text{em}}}{M_N} \left( g_1(x, Q^2) - \gamma^2 g_2(x, Q^2) \right),
\]

\[
K \cdot \sigma_{LT} = \frac{4\pi^2\alpha_{\text{em}}}{M_N} \gamma \left( g_1(x, Q^2) + g_2(x, Q^2) \right),
\]

with \( \gamma \equiv Q/\nu \) and \( x \equiv Q^2/(2M_N \nu) \).

For the non-pole contributions to \( g_{TT} \) and \( g_{LT} \), one can perform a low energy expansion (LEX) as follows [3] :

\[
\text{Re } g_{TT}(\nu, Q^2) - \text{Re } g_{TT}^{\text{pole}}(\nu, Q^2) = \left( \frac{2\alpha_{\text{em}}}{M_N^2} \right) I_{A}(Q^2) \nu + \gamma_0(Q^2) \nu^3 + \mathcal{O}(\nu^5),
\]

\[
\text{Re } g_{LT}(\nu, Q^2) - \text{Re } g_{LT}^{\text{pole}}(\nu, Q^2) = \left( \frac{2\alpha_{\text{em}}}{M_N^2} \right) Q I_3(Q^2) + Q \delta_{LT}(Q^2) \nu^2 + \mathcal{O}(\nu^4).
\]

For the \( \mathcal{O}(\nu) \) term in Eq. (11), one obtains from Eq. (5) a generalization of the GDH sum rule as:

\[
I_{A}(Q^2) = \frac{M_N^2}{4\pi^2\alpha_{\text{em}}} \int_{\nu_0}^{\infty} \frac{K(\nu, Q^2)}{\nu} \sigma_{TT}(\nu, Q^2) d\nu,
\]

\[
= \frac{2M_N^2}{Q^2} \int_0^{x_0} dx \left\{ g_1(x, Q^2) - \frac{4M_N^2}{Q^2} x^2 g_2(x, Q^2) \right\},
\]

and recovers the GDH sum rule at \( Q^2 = 0 \), as \( I_{A}(0) = -\kappa_N^2/4 \), with \( \kappa_N \) the nucleon anomalous magnetic moment (\( \kappa_p = 1.79 \), \( \kappa_n = -1.91 \)). For the \( \mathcal{O}(\nu^0) \) term in Eq. (12), one obtains from Eq. (6) the sum rule

\[^1\text{Note that the partial cross sections } \sigma_{TT} \text{ and } \sigma_{LT} \text{ depend on the virtual photon flux convention. However, in the dispersion integrals only the products } K \cdot \sigma_{TT} \text{ and } K \cdot \sigma_{LT} \text{ enter, which are independent of this convention.}\]
\[ I_3(Q^2) = \frac{M_N^2}{4\pi^2\alpha_{em}} \int_{\nu_0}^\infty \frac{K(\nu, Q^2)}{\nu} \frac{1}{Q} \sigma_{LT}(\nu, Q^2) d\nu , \]
\[ = \frac{2 M_N^2}{Q^2} \int_0^{x_0} dx \left\{ g_1(x, Q^2) + g_2(x, Q^2) \right\} . \]  

(14)

For the sum rule of Eq. (14) to exist, one sees that \( \sigma_{LT} \) should vanish faster than \( 1/\nu \) at large \( \nu \).

The higher order terms in Eqs. (11) and (12) can be expressed in terms of nucleon spin polarizabilities as [3] :

\[ \gamma_0(Q^2) = \frac{1}{2\pi^2} \int_{\nu_0}^\infty \frac{K(\nu, Q^2)}{\nu} \sigma_{TT}(\nu, Q^2) d\nu , \]
\[ = \frac{\alpha_{em} 16 M_N^2}{Q^6} \int_0^{x_0} dx x^2 \left\{ g_1(x, Q^2) - \frac{4 M_N^2}{Q^2} x g_2(x, Q^2) \right\} , \]  

(15)

and

\[ \delta_{LT}(Q^2) = \frac{1}{2\pi^2} \int_{\nu_0}^\infty \frac{K(\nu, Q^2)}{\nu} \sigma_{LT}(\nu Q^2) d\nu , \]
\[ = \frac{\alpha_{em} 16 M_N^2}{Q^6} \int_0^{x_0} dx x^2 \left\{ g_1(x, Q^2) + g_2(x, Q^2) \right\} . \]  

(16)

Analogously, we can also write down corresponding LEX and sum rules for the spin dependent Compton amplitudes \( S_1 \) and \( S_2 \) of Eq. (2). For these, one again first has to separate the pole contributions to \( S_1 \) and \( \nu S_2 \), which are given by

\[ \text{Re } S_1^{\text{pole}}(\nu, Q^2) = -\frac{\alpha_{em} Q^2}{2 M_N \nu^2 - \nu_B^2} F_D(Q^2) \left( F_D(Q^2) + F_P(Q^2) \right) , \]  

(17)

\[ \text{Re } (\nu S_2(\nu, Q^2))^{\text{pole}} = \frac{\alpha_{em} \nu_B^2}{2} \frac{\nu_B^2}{\nu^2 - \nu_B^2} F_P(Q^2) \left( F_D(Q^2) + F_P(Q^2) \right) , \]  

(18)

with \( F_D \) (\( F_P \)) the nucleon Dirac (Pauli) form factors. The resulting LEX for the non-pole contributions are then given by [3] :

\[ \text{Re } S_1(\nu, Q^2) - \text{Re } S_1^{\text{pole}}(\nu, Q^2) = \]

\[ \left( \frac{2 \alpha_{em}}{M_N} \right) I_1(Q^2) + \left[ \left( 2 \frac{\alpha_{em}}{M_N} \right) \frac{1}{Q^2} (I_A(Q^2) - I_1(Q^2)) + M_N \delta_{LT}(Q^2) \right] \nu^2 + \mathcal{O}(\nu^4) , \]  

(19)

and \(^2\):

\[ \text{Re } \nu S_2(\nu, Q^2) - \text{Re } (\nu S_2(\nu, Q^2))^{\text{pole}} = \]

\[ \left( 2 \frac{\alpha_{em}}{M_N} \right) I_2(Q^2) - \left( 2 \frac{\alpha_{em}}{M_N} \right) \frac{1}{Q^2} (I_A(Q^2) - I_1(Q^2)) \nu^2 \]
\[ + \frac{1}{Q^2} \left[ \left( 2 \frac{\alpha_{em}}{M_N} \right) \frac{1}{Q^2} (I_A(Q^2) - I_1(Q^2)) + M_N^2 \left( \delta_{LT}(Q^2) - \gamma_0(Q^2) \right) \right] \nu^4 + \mathcal{O}(\nu^6) . \]  

(20)

\(^2\)Note that the relation of Ref. [19], i.e., \( I'_A(0) - I'_1(0) = M_N^2/(2 \alpha_{em}) \cdot (\gamma_0(0) - \delta_{LT}(0)) \) ensures that the \( \nu^4 \) term in \( \nu S_2 \) has no singularity at \( Q^2 = 0 \).
The lowest order term in Eq. (19) is connected to the first moment of $g_1$ through the sum rule:

$$I_1(Q^2) = \frac{2M_N^2}{Q^2} \int_0^{x_0} g_1(x, Q^2) \, dx ,$$
$$= \frac{M_N^2}{4 \pi^2 \alpha_{em}} \int_{\nu_0}^\infty \frac{K(\nu, Q^2)}{\nu^2 + Q^2} \left\{ \sigma_{TT}(\nu, Q^2) + \frac{Q}{\nu} \sigma_{LT}(\nu, Q^2) \right\} \, d\nu , \quad (21)$$

where the integral over $x$ runs up to the value $x_0$ corresponding to $\nu_0$. At $Q^2 = 0$, Eq. (21) also reduces to the GDH sum rule, i.e., $I_1(0) = -\kappa_2^2/4$. Correspondingly, the lowest order term in Eq. (20) is connected to the first moment of $g_2$ through the sum rule:

$$I_2(Q^2) = \frac{2M_N^2}{Q^2} \int_0^{x_0} g_2(x, Q^2) \, dx ,$$
$$= \frac{M_N^2}{4 \pi^2 \alpha_{em}} \int_{\nu_0}^\infty \frac{K(\nu, Q^2)}{\nu^2 + Q^2} \left\{ -\sigma_{TT}(\nu, Q^2) + \frac{\nu}{Q} \sigma_{LT}(\nu, Q^2) \right\} \, d\nu . \quad (22)$$

Combining Eqs. (21) and (22), one obtains Eq. (14), i.e.,

$$I_3(Q^2) = I_1(Q^2) + I_2(Q^2) . \quad (23)$$

Furthermore, one sees that the higher order terms in Eqs. (19) and (20) are expressed in terms of the integrals $I_1, I_4$ and the spin polarizabilities $\gamma_0$ and $\delta_{LT}$ introduced before.

For the first moment of $I_2$ in Eq. (22) to converge, one has to assume, as in the case of $I_3$, the strong convergence condition that $\sigma_{LT}$ vanishes faster than $1/\nu$ at large $\nu$. If this assumption holds, then $I_2$ satisfies a sum rule, derived by Burkhardt and Cottingham [13] (BC sum rule), which allows one to express it at any $Q^2$ in terms of elastic nucleon form factors as:

$$I_2(Q^2) = \frac{1}{4} F_P(Q^2) \left( F_D(Q^2) + F_F(Q^2) \right) . \quad (24)$$

The BC sum rule has been shown to be satisfied in the case of quantum electrodynamics by a calculation in lowest order of $\alpha_{em}$ [20]. In perturbative QCD, the BC sum rule was calculated for a quark target to first order in the strong coupling and also shown to hold [21]. In the next section, we investigate next the BC sum rule at small momentum transfer within the framework of HBChPT to $O(p^4)$.

**III. BC SUM RULE IN HBCHPT TO $O(p^4)$**

The validity of the BC sum rule can be tested in HBChPT, by calculating both sides of Eq. (24). To this end, the complete one-loop calculation of the forward VVCS amplitudes at low energy $\nu$ and small momentum transfer $Q^2$ has been done, and some results have been presented in Refs. [10,15]. In particular, the first term in the expression of Eq. (19) for the inelastic part of $S_1$ has been found as [10,15]
\[ I_1(Q^2) = -\frac{1}{16} [(\kappa_s + \kappa_v \tau_3)^2] \]
\[ + \frac{g_A^2 m_{\pi} M_N}{(4\pi F_{\pi})^2} \cdot \frac{\pi}{32} \left\{ (-10 - 12 \kappa_v) + (-2 - 12 \kappa_s) \tau_3 \right\} \]
\[ + \left\{ (20 + 24 \kappa_v) + (4 + 24 \kappa_s) \tau_3 \right\} \cdot \frac{1}{w} \tan^{-1} \left[ \frac{w}{2} \right] \]
\[ + \left\{ (3 + 6 \kappa_v) + (3 + 10 \kappa_s) \tau_3 \right\} \cdot w \tan^{-1} \left[ \frac{w}{2} \right] \] (25)

with \( w = \sqrt{\frac{Q}{m_{\pi}}}. \) Eq. (25) is expressed in terms of the renormalized isoscalar (isovector) anomalous magnetic moments \( \kappa_s (\kappa_v), \) whose physical values are given by \( \kappa_s = -0.12 \) and \( \kappa_v = 3.70. \) Furthermore, throughout this paper we use the values : \( g_A = 1.267, \)
\( F_{\pi} = 0.0924 \text{ GeV}, \) and \( m_{\pi} = 0.14 \text{ GeV}. \)

For the inelastic part of the amplitude \( (\nu S_2), \) we found the first term of Eq. (20) to next-to-leading order in HBChPT to be given by :

\[ I_2(Q^2) = \frac{1}{16} [(\kappa_s + \kappa_v \tau_3)(1 + \tau_3) + (\kappa_s + \kappa_v \tau_3)^2] \]
\[ - \frac{g_A^2 m_{\pi} M_N}{(4\pi F_{\pi})^2} \cdot \frac{\pi}{16} \left\{ (-10 - 12 \kappa_v) + (-2 - 12 \kappa_s) \tau_3 \right\} \]
\[ + \left\{ (4 + 8 \kappa_v) + (4 + 8 \kappa_s) \tau_3 \right\} \cdot \frac{1}{w} \tan^{-1} \left[ \frac{w}{2} \right] \]
\[ + \left\{ (1 + 2 \kappa_v) + (1 + 2 \kappa_s) \tau_3 \right\} \cdot w \tan^{-1} \left[ \frac{w}{2} \right] \] (26)

This result is solely from the one-particle-reducible diagrams. To check the BC sum rule of Eq. (24), we need the corresponding expressions for the form factors in HBChPT which, to the order needed, are given by [22]

\[ F_D[Q^2] = \frac{1}{2} \left[ 1 + \tau_3 \right], \] (27)
\[ F_P[Q^2] = \frac{1}{2} \left[ \kappa_s + \kappa_v \tau_3 \right] - \tau_3 \frac{g_A^2 m_{\pi} M_N}{(4\pi F_{\pi})^2} \cdot 2\pi \left[ \frac{-1}{2} + \left( \frac{w}{4} + \frac{1}{w} \right) \tan^{-1} \left[ \frac{w}{2} \right] \right]. \] (28)

One therefore obtains for the rhs of Eq. (24) the expression :

\[ \frac{1}{4} F_P \cdot (F_D + F_P) = \frac{1}{16} \left[ \kappa_s (1 + \kappa_s) + \kappa_v (1 + \kappa_v) + [\kappa_s + \kappa_v + 2\kappa_s \kappa_v] \tau_3 \right] \]
\[ - \left[ (1 + 2 \kappa_v) + (1 + 2 \kappa_s) \tau_3 \right] \frac{g_A^2 m_{\pi} M_N}{(4\pi F_{\pi})^2} \cdot \frac{\pi}{4} \left[ \frac{-1}{2} + \left( \frac{w}{4} + \frac{1}{w} \right) \tan^{-1} \left[ \frac{w}{2} \right] \right]. \] (29)

By comparing Eq. (29) with Eq. (26), we verify the BC sum rule explicitly up to NLO in HBChPT, i.e., to \( O(p^4). \)

The HBChPT results for the integral \( I_2 \) for proton and neutron are shown in Fig. 1. They are compared with the results of a resonance estimate, and with the BC sum rule value (rhs of Eq. (24)), evaluated with the phenomenological proton and neutron form factors. The
resonance estimate is performed by calculating the absorption integrals on the rhs of Eq. (22), up to a maximum total c.m. energy \( W = 2 \) GeV, using the MAID model [18] for the one-pion production cross sections. The one-pion production channels are expected to dominate the integral in the low \( Q^2 \) region. From Fig. 1, one sees that when going to larger \( Q^2 \), the HBChPT result remains numerically close to the BC sum rule value up to \( Q^2 \approx 0.25 \) GeV\(^2\). Moreover, also the phenomenological MAID estimate for \( I_2 \) of the proton remains within 10 % of the BC sum rule value. For the neutron, the phenomenological estimate displays a larger deviation in particular at the real photon point, where it lies about 30 % above the BC sum rule value. This may originate from the poorly known contribution of \( \sigma_{LT}/Q \) in the integrand of Eq. (22), which survives in the limit \( Q^2 \to 0 \).

**IV. GENERALIZED FORWARD SPIN POLARIZABILITIES IN HBCHPT TO \( \mathcal{O}(P^4) \)**

The higher order terms in the LEX of Eqs. (11,12) or alternatively Eqs. (19,20) contain the information on the generalized forward spin polarizabilities of the nucleon. In this section, we present the results for these generalized forward spin polarizabilities in HBChPT to \( \mathcal{O}(p^4) \). Here we include all one-particle-reducible diagrams but remove their pole parts as in Refs. [15,24].

For the purely transverse generalized forward spin polarizability \( \gamma_0 \) defined in Eq. (11), we find the expression at \( \mathcal{O}(p^3) \) as:

\[
\gamma_0^{\mathcal{O}(p^3)}(Q^2) = \frac{\alpha_{em}^2 g_A^2}{(4\pi F_\pi)^2} \cdot \frac{4}{m_\pi^2} \left[ \frac{1}{3} + \frac{1}{w^2} - \frac{4w^2 + 12}{3w^2\sqrt{w^2 + 4}} \sinh^{-1} \left[ \frac{w}{2} \right] \right]. \tag{30}
\]

To test the convergence of the chiral expansion, we calculate the correction in HBChPT at \( \mathcal{O}(p^4) \), for which we obtain the result:

\[
\gamma_0^{\mathcal{O}(p^4)}(Q^2) = \frac{\alpha_{em} g_A^2}{(4\pi F_\pi)^2} \cdot \frac{\pi}{1152m_\pi} \left[ -576w^2 + \left[ (-1888 - 448\kappa_v) + (-416 - 128\kappa_s)\tau_3 \right] \right.

\[+\left[ (366 + 366\kappa_v) + (930 - 222\kappa_s)\tau_3 \right] \cdot \frac{1}{w^2} \]

\[+\left[ (123 + 123\kappa_v) + (-2787 + 93\kappa_s)\tau_3 \right] \cdot \frac{1}{w^2} \tan^{-1} \left[ \frac{w}{2} \right] \]

\[+\left[ (2724 - 732\kappa_v) + (-1860 + 444\kappa_s)\tau_3 \right] \cdot \frac{1}{w^2} \tan^{-1} \left[ \frac{w}{2} \right] \]

\[+\left[ (272 + 272\kappa_v) + (16 + 16\kappa_s)\tau_3 \right] \cdot \frac{1}{w^2 + 4} \]

\[ - 192\tau_3 \cdot \frac{w^4 - 4w^2 - 24}{w^2 + 4} + 576 \cdot \frac{w^4 - 12}{w^4 + 4w^2} \right]. \tag{31}
\]

Analogously, we find for the longitudinal-transverse generalized forward spin polarizability \( \delta_{LT} \), defined in Eq. (12), the expressions for the \( \mathcal{O}(p^3) \) term as:

\[
\delta_{LT}^{\mathcal{O}(p^3)}(Q^2) = \frac{\alpha_{em}^2 g_A^2}{(4\pi F_\pi)^2} \cdot \frac{4}{m_\pi^2} \left[ \frac{1}{3w\sqrt{w^2 + 4}} \sinh^{-1} \left[ \frac{w}{2} \right] \right], \tag{32}
\]
and for the $\mathcal{O}(p^4)$ correction as:

$$
\delta_{LT}^{\mathcal{O}(p^4)}(Q^2) = \frac{\alpha_{em} g_A^2}{(4\pi F_\pi)^2 M_N} \cdot \frac{\pi}{192 m_\pi} \left\{ (-48 + 24 \kappa_v) + (-24 + 48 \kappa_s) \tau_3 
+ [(-162 + 24 \kappa_v) + (-18 + 24 \kappa_s) \tau_3] \cdot \frac{1}{w^2} 
+ [(-27 - 36 \kappa_v) + (-27 - 12 \kappa_s) \tau_3] \cdot \frac{1}{w} \tan^{-1} \left[ \frac{w}{2} \right] 
+ [(-252 - 48 \kappa_v) + (36 - 48 \kappa_s) \tau_3] \cdot \frac{1}{w^3} \tan^{-1} \left[ \frac{w}{2} \right] 
+ [12 - (36 + 48 \kappa_s) \tau_3] \cdot \frac{1}{4 + w^2} + 384 \cdot \frac{3 + w^2}{4w^2 + w^4} \right\}. \tag{33}
$$

At the real photon point, these forward spin polarizabilities reduce to:

$$
\gamma_0(Q^2 = 0) = \frac{\alpha_{em} g_A^2}{(4\pi F_\pi)^2} \cdot \frac{2}{3m_\pi} \left[ 1 - \frac{\pi m_\pi}{8M_N} [15 + 3\kappa_v + (6 + \kappa_s) \tau_3] \right],
$$

$$
\delta_{LT}(Q^2 = 0) = \frac{\alpha_{em} g_A^2}{(4\pi F_\pi)^2} \cdot \frac{1}{3m_\pi} \left[ 1 + \frac{\pi m_\pi}{8M_N} [-3 + \kappa_v + (-6 + 4\kappa_s) \tau_3] \right]. \tag{34}
$$

For $\gamma_0$, our expressions at the real photon point reduce to the result which has been derived before in Refs. [24,23,25]. Furthermore, the slopes at $Q^2 = 0$ of the generalized forward spin polarizabilities are given by:

$$
\left[ \frac{d\gamma_0(Q^2)}{dQ^2} \right]_{Q^2 = 0} = \frac{\alpha_{em} g_A^2}{(4\pi F_\pi)^2} \cdot \frac{4}{45m_\pi} \left[ 1 - \frac{\pi m_\pi}{1024M_N} [5451 + 267\kappa_v + (-75 + 21\kappa_s) \tau_3] \right],
$$

$$
\left[ \frac{d\delta_{LT}(Q^2)}{dQ^2} \right]_{Q^2 = 0} = \frac{\alpha_{em} g_A^2}{(4\pi F_\pi)^2} \cdot \frac{-1}{18m_\pi} \left[ 1 + \frac{\pi m_\pi}{80M_N} [54 - 9\kappa_v - (27 + 24\kappa_s) \tau_3] \right]. \tag{35}
$$

Besides the $\pi N$ loop contribution, we also estimated the effects of the $\Delta$ contributions to the forward spin polarizabilities in the small scale expansion to order $\mathcal{O}(\varepsilon^3)$, where $\varepsilon$ stands for a soft momentum, the pion mass or the mass difference between $\Delta$ and nucleon, denoted by $\Delta = M_\Delta - M_N$. The $\Delta$ effects enter through both a $\Delta$-pole contribution as well as $\pi \Delta$ loop contributions (see Fig. 2). The leading order $\Delta$ contributions to the generalized forward spin polarizabilities are given as:

$$
\gamma_0^\Delta(Q^2) = \frac{-\alpha_{em}}{9} \left( \frac{G_1}{M_N} \right)^2 \cdot \frac{\Delta^2 + Q^2}{\Delta^4}
- \frac{32\alpha_{em}}{27} \frac{g_\pi^2 \Delta N}{(4\pi F_\pi)^2} \int_0^1 dx \frac{x^3}{m_0^2} (\mu_0^2 - 1)^2 \left[ (\mu_0^2 + 23 - 3\mu_0) \ln[\mu_0 + \sqrt{\mu_0^2 - 1}] \right],
$$

$$
- \frac{8\alpha_{em}}{27} \frac{g_\pi^2 \Delta N}{(4\pi F_\pi)^2} \int_0^1 dx \frac{x^4(1 - 2x)Q^2}{m_0^2} (\mu_0^2 - 1)^{-3}
\times \left[ 11\mu_0^2 - 4 - (6\mu_0^3 + 9\mu_0) \frac{\ln[\mu_0 + \sqrt{\mu_0^2 - 1}]}{\sqrt{\mu_0^2 - 1}} \right], \tag{36}
$$

$$
9}
\right.}
and

$$\delta_{LT}^\Delta(Q^2) = \frac{-32\alpha_{em}}{27} \frac{g_{\pi\Delta N}^2}{(4\pi F^\pi)^2} \int_0^1 dx \frac{x^3}{m_0^2}(\mu_0^2 - 1)^{-2} \left[ \mu_0^2 + 2 - 3\mu_0 \frac{\ln|\mu_0 + \sqrt{\mu_0^2 - 1}|}{\sqrt{\mu_0^2 - 1}} \right]
+ \frac{16\alpha_{em}}{9} \frac{g_{\pi\Delta N}^2}{(4\pi F^\pi)^2} \int_0^1 dx \frac{x^2(1-2x)}{m_0^2}(\mu_0^2 - 1)^{-1} \left[ 1 - \mu_0 \frac{\ln|\mu_0 + \sqrt{\mu_0^2 - 1}|}{\sqrt{\mu_0^2 - 1}} \right], \quad (37)$$

with $m_0 \equiv \sqrt{m^2 + x(1-x)Q^2}$ and $\mu_0 \equiv \frac{\Delta}{m_0}$. $G_1$ and $g_{\pi\Delta N}$ are the leading order $\gamma\Delta N$ and $\pi\Delta N$ coupling constants respectively. In the large $N_C$ limit of QCD, they are related with $\kappa_v$ and $g_A$ as:

$$G_1 = \frac{3}{2\sqrt{2}}\kappa_v, \quad g_{\pi\Delta N} = \frac{3}{2\sqrt{2}}g_A. \quad (38)$$

Note that Eqs. (36) and (37) apply when $Q^2 < 4(\Delta^2 - m_\pi^2) \approx 0.268$ GeV$^2$. Analogous formulas can be found for the range $Q^2 > 4(\Delta^2 - m_\pi^2)$, which we don’t specify here as they fall outside the range which we show in the following figures.

At the real photon point, the expressions for the $\Delta$ contributions reduce to:

$$\gamma_0^\Delta(Q^2 = 0) = -\frac{\alpha_{em}}{9} \left( \frac{M_N}{\Delta} \right)^2 \cdot \frac{1}{\Delta^2}
+ \frac{\alpha_{em}g_{\pi\Delta N}^2}{(4\pi F^\pi)^2} \cdot \frac{8}{27m_\pi^2}(\mu_0^2 - 1)^{-2} \left[ -\mu_0^2 - 2 + 3\mu_0 \frac{\ln|\mu_0 + \sqrt{\mu_0^2 - 1}|}{\sqrt{\mu_0^2 - 1}} \right], \quad (39)$$

$$\delta_{LT}^\Delta(Q^2 = 0) = \frac{\alpha_{em}g_{\pi\Delta N}^2}{(4\pi F^\pi)^2} \cdot \frac{8}{27m_\pi^2}(\mu_0^2 - 1)^{-2} \left[ -2\mu_0^2 - 1 + (\mu_0^2 + 2\mu_0) \frac{\ln|\mu_0 + \sqrt{\mu_0^2 - 1}|}{\sqrt{\mu_0^2 - 1}} \right],$$

with $\mu = \frac{\Delta}{m_\pi}$. Furthermore, the slopes at $Q^2 = 0$ of the generalized forward spin polarizabilities are given by:

$$\left[ \frac{d\gamma_0^\Delta(Q^2)}{dQ^2} \right]_{Q^2=0} = -\frac{\alpha_{em}}{9} \left( \frac{G_1}{M_N} \right)^2 \cdot \frac{1}{\Delta^4}
+ \frac{\alpha_{em}g_{\pi\Delta N}^2}{(4\pi F^\pi)^2} \cdot \frac{8}{405m_\pi^4} \left\{ (\mu_0^2 - 1)^{-2} \left[ 4\mu_0^2 + 4 - 9\mu_0 \frac{\ln|\mu_0 + \sqrt{\mu_0^2 - 1}|}{\sqrt{\mu_0^2 - 1}} \right]
+ (\mu_0^2 - 1)^{-3} \left[ -4\mu_0^4 + 11\mu_0^2 + 8
+ (3\mu_0^3 - 18\mu_0) \frac{\ln|\mu_0 + \sqrt{\mu_0^2 - 1}|}{\sqrt{\mu_0^2 - 1}} \right] \right\},$$

$$\left[ \frac{d\delta_{LT}^\Delta(Q^2)}{dQ^2} \right]_{Q^2=0} = \frac{\alpha_{em}g_{\pi\Delta N}^2}{(4\pi F^\pi)^2} \cdot \frac{2}{405m_\pi^4}
\times \left\{ (\mu_0^2 - 1)^{-1} \left[ 6 - 9\mu_0 \frac{\ln|\mu_0 + \sqrt{\mu_0^2 - 1}|}{\sqrt{\mu_0^2 - 1}} \right]
+ (\mu_0^2 - 1)^{-2} \left[ 7\mu_0^2 + 16 + (9\mu_0^3 - 36\mu_0) \frac{\ln|\mu_0 + \sqrt{\mu_0^2 - 1}|}{\sqrt{\mu_0^2 - 1}} \right] \right\},$$

10
In Fig. 3, we compare the HBChPT results for $\gamma_0(Q^2)$ with the phenomenological estimates using the MAID model. For $\gamma_0$, the experimental value at the real photon point has been obtained by the GDH experiment [7] and is indicated. The MAID estimate at $Q^2 = 0$ is in good agreement with the experimental value. By comparing the HBChPT results for $\gamma_0$ at $O(p^3)$ and at $O(p^4)$, one sees that for both proton and neutron, the $O(p^4)$ correction term is of opposite sign as the leading order term, and its magnitude is even larger than the leading order term, as has been noted before in Refs. [24,23]. By comparing the HBChPT result with the estimate as obtained with MAID, one sees that there is a large cancellation involved in $\gamma_0$ over the whole $Q^2$ region. Although the $O(p^3)$ and $O(p^4)$ terms are of opposite sign, and also display a large cancellation, their large magnitudes indicate that no convergence has been reached for $\gamma_0$ at $O(p^4)$. One may try to improve the HBChPT calculation by adding explicit $\Delta$ degrees of freedom. To this end, we also show in Fig. 3 the $Q^2$ dependence of the $\Delta$ contribution to $\gamma_0$ calculated in the small scale expansion to order $O(\varepsilon^3)$ according to Eq. (36). One sees from Fig. 3 that the $O(\varepsilon^3)$ correction term is sizeable and reduces the $O(p^3)$ result to come closer to the phenomenological result. However, quantitatively the result at order $O(\varepsilon^3)$ is still not satisfying when compared to the phenomenological result, and suggests that one needs to estimate the terms of the order $O(\varepsilon^4)$ in a future work.

The corresponding HBChPT results for the forward spin polarizability $\delta_{LT}(Q^2)$ are shown in Fig. 4. In contrast to $\gamma_0$, one sees that $\delta_{LT}$ receives a much smaller correction at $O(p^4)$. For the proton, the $O(p^4)$ correction term reduces the $O(p^3)$ result. By comparing the HBChPT result at $O(p^4)$ with the phenomenological estimate, one sees a reasonable agreement for the proton. For the neutron however, the correction at $O(p^4)$ increases the $O(p^3)$ result and lies about 20% above the value predicted by MAID. This discrepancy may however again be due to the uncertainty in the phenomenological estimate of $\sigma_{LT}$ for the neutron, which enters the rhs of Eq. (16), as was also seen for the neutron $I_2$ result in Fig. 1. Furthermore, we see that for $\delta_{LT}$ the correction due to explicit $\Delta$-resonance degrees of freedom is very small, making the forward spin polarizability $\delta_{LT}$ a very useful observable to study the transition from the low $Q^2$ regime to the perturbative regime at large $Q^2$.

Recently, doubly polarized asymmetries for inclusive electron nucleon scattering were measured at Jefferson Lab (JLab) using longitudinally polarized electrons incident on a longitudinally polarized nucleon [26,27]. These experiments are mainly sensitive to the nucleon structure function $g_1$, with some small admixture of $g_2$. For the proton, the CLAS Collaboration [26] measured the first moment of $g_1$ down to $Q^2 \approx 0.15$ GeV$^2$. For the neutron, the Hall A Collaboration [27] measured the first moment of $g_1^n$ down to $Q^2 \approx 0.10$ GeV$^2$, while a forthcoming Hall A experiment will extend this down to $Q^2 \approx 0.02$ GeV$^2$ [28]. These measurements map out the transition from the GDH sum rule value at $Q^2 = 0$ to the DIS value at large $Q^2$. In this way, they allow us to study quantitatively the transition from the resonance dominated regime at low $Q^2$ to the partonic regime at large $Q^2$. In terms of the virtual photon absorption cross sections $\sigma_{TT}$ and $\sigma_{LT}$ of Eqs. (9,10), the JLab experiments [26,27] with a longitudinally polarized target, are mainly sensitive to $\sigma_{TT}$. Therefore, it will be interesting to extract from these data the $Q^2$ dependence of $\gamma_0(Q^2)$ for the proton and neutron according to Eq. (15). Compared to the first moment of $g_1$, the forward spin

\[
+ (\mu^2 - 1)^{-3} \left\{ -16\mu^4 - 44\mu^2 + 60\mu^3 \ln \left[ \mu + \sqrt{\mu^2 - 1} \right] \right\}. \tag{40}
\]
polarizability involves the third moment of $g_1$ (see Eq. (15)). Therefore, this observable receives only a negligible small contribution from the unmeasured region at small-$x$, and can be directly extracted from the $x$-range as accessed by the present experiments at the lower $Q^2$. The extraction of $\delta_{LT}(Q^2)$ requires the measurement of $\sigma_{LT}$. This can be achieved by measuring the inclusive electroabsorption cross section for longitudinally polarized electrons on a transversely polarized nucleon target. Such measurements, giving access to the spin structure function $g_2$, are presently underway at JLab both in Hall A [29] and Hall C [30].

V. CONCLUSIONS

In conclusion, in this paper we studied spin-dependent sum rules for forward virtual Compton scattering (VVCS) off the nucleon in HBChPT at order $O(p^4)$. We showed in particular that the Burkhardt-Cottingham sum rule is satisfied in HBChPT to this order. We have also noticed that the $O(p^4)$ prediction remains close to the phenomenological sum rule evaluation, in the range up to $Q^2 \simeq 0.25$ GeV$^2$. This relatively wide range indicates that the first moment of $g_2$ is a promising observable to bridge the gap between the HBChPT description at the lower $Q^2$ and the perturbative QCD result at the larger $Q^2$.

Furthermore, we related the higher terms in the low energy expansion of the VVCS amplitude, to generalized forward spin polarizabilities of the nucleon. We then calculated these generalized forward spin polarizabilities of the nucleon at $O(p^4)$ in HBChPT. The result for $\gamma_0(Q^2)$ indicates a large $O(p^4)$ correction to the leading order result. The corresponding result for $\delta_{LT}(Q^2)$ however shows that the convergence of the heavy baryon and chiral expansion is better behaved, and receives only a negligible contribution from $\Delta$ degrees of freedom. It will be important to extract the information on the generalized forward spin polarizabilities from experiments at different $Q^2$, such as are presently performed at JLab. This will allow to quantify the transition from the resonance dominated regime at $Q^2 = 0$ to the partonic regime at large $Q^2$. For $\delta_{LT}(Q^2)$, our result also indicates that measuring the value of the integral related to $\delta_{LT}$ at $Q^2 = 0.02$ GeV$^2$, as accessible in the experiment of Ref. [28], will be enough to extrapolate to the value at $Q^2 = 0$.

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FIG. 1. $Q^2$ dependence of the integral $I_2$ for proton (upper panel) and neutron (lower panel). The solid curves represent the MAID estimate [18,19] for the $\pi$ channel. The dashed curves are the $\mathcal{O}(p^3) + \mathcal{O}(p^4)$ HBChPT results; and the dotted curves are the BC sum rule prediction for $I_2$, evaluated with phenomenological form factors.

FIG. 2. The $\mathcal{O}(\varepsilon^3)$ diagrams of photon-nucleon Compton scattering with the $\Delta$ as intermediate state. The crossed diagrams are not shown.
FIG. 3. $Q^2$ dependence of the forward spin polarizability $\gamma_0$ for proton (upper panel) and neutron (lower panel). The solid curves represent the MAID estimate [18,19] for the $\pi$ channel. The dotted curves are the $\mathcal{O}(p^3)$ HBChPT results; the dashed curves are the $\mathcal{O}(p^3) + \mathcal{O}(p^4)$ HBChPT results; and the dashed-dotted curves are the $\Delta$ contributions evaluated in $\mathcal{O}(\varepsilon^3)$. The solid circle for the proton at $Q^2 = 0$ corresponds with the experimental value extracted from the MAMI GDH experiment [7].
FIG. 4. $Q^2$ dependence of the forward spin polarizability $\delta_{LT}$ for proton (upper panel) and neutron (lower panel). The solid curves represent the MAID estimate [18,19] for the $\pi$ channel. The dotted curves are the $O(p^3)$ HBChPT results; the dashed curves are the $O(p^3) + O(p^4)$ HBChPT results; and the dashed-dotted curves (close to zero) are the $\Delta$ contributions evaluated in $O(\varepsilon^3)$. 