Hopf algebras of canonical commutation relations

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Abstract
Given a Heisenberg algebra $A$ of canonical commutation relations modelled over an infinite-dimensional nuclear space, a Hopf algebra of its quantum deformations is also an algebra of canonical commutation relations whose Fock representation recovers some non-Fock representation of $A$.

1 Introduction

By virtue of the well-known Stone–von Neumann uniqueness theorem, all irreducible representations of the canonical commutation relations (henceforth the CCR) of finite degree of freedom are equivalent. On the contrary, the infinite-dimensional CCR possess many non-equivalent irreducible representations (see [1] for a survey). Here, we restrict our consideration to the CCR modelled over a nuclear space. They include the CCR of finite degrees of freedom, but we focus on the infinite-dimensional CCR. In particular, this is the case of field theory [5].

Let $A$ be the Heisenberg algebra of the CCR modelled over a nuclear space. Since $A$ is a Lie algebra, one can associate to $A$ a Hopf algebra, regarded as an algebra of $q$-deformed CCR (see [3] for the case of finite-dimensional CCR). We show that this Hopf algebra is the enveloping algebra of another CCR algebra $A_{q,c}$. Moreover, $A$ and $A_{q,c}$ possess the same set of representations. Herewith, operators of the Fock representation of $A_{q,c}$ carry out some non-Fock representation of $A$.

2 The nuclear CCR

Let us recall the notion of a nuclear space (see, e.g., [4]). Let a complex vector space $V$ be provided with a countable set of non-degenerate Hermitian forms $\langle \cdot | \cdot \rangle_k$, $k = 1, \ldots$, such that

$$\langle v|v\rangle_1 \leq \cdots \leq \langle v|v\rangle_k \leq \cdots$$

for all $v \in V$. Let $V$ be complete in the topology defined by the set of norms $\| \cdot \|_k^{1/2} = \langle \cdot | \cdot \rangle_k$. Then $V$ is called a countably Hilbert space. Let $V_k$ denote the completion of $V$ with respect
to the norm $\|\cdot\|_k$. There is the chain of injections $V_1 \supset V_2 \supset \cdots V_k \supset \cdots$, and $V = \cap_k V_k$. Let $T^m_n, m \leq n$, be a prolongation of the map $V_n \supset V \ni v \mapsto v \in V \subset V_m$ to the continuous map of $V_n$ onto a dense subset of $V_m$. A countably Hilbert space $V$ is called a nuclear space if, for any $m$, there exists $n$ such that $T^m_n$ is a nuclear map, i.e.,

$$T^m_n(v) = \sum_i \lambda_i \langle v_i | v_i \rangle v_i v_m^i,$$

where: (i) $\{v_n^i\}$ and $\{v_m^i\}$ are bases for the Hilbert spaces $V_n$ and $V_m$, respectively, (ii) $\lambda_i \geq 0$, and (iii) the series $\sum \lambda_i$ converges. Note that a Hilbert space is not nuclear, unless it is finite-dimensional.

Let $V$ be a real nuclear space provided with still another non-degenerate Hermitian form $\langle \cdot | \cdot \rangle$, which is separately continuous. This form makes $V$ to a separable pre-Hilbert space. Let us consider the group $G$ of the triples $(v_1, v_2, \lambda)$ of elements $v_1, v_2$ of $V$ and complex numbers $\lambda$ of unit modulus which are subject to multiplications

$$(v_1, v_2, \lambda)(v'_1, v'_2, \lambda') = (v_1 + v'_1, v_2 + v'_2, \exp[i(v_2, v'_1)]\lambda\lambda'). \quad (1)$$

It is a Lie group whose group space is a nuclear manifold modelled over

$$W = V \oplus V \oplus \mathbb{R}. \quad (2)$$

Let us denote $T(v) = (v, 0, 0)$ and $P(v) = (0, v, 0)$. Then the multiplication law (1) takes the form

$$T(v)T(v') = T(v + v'), \quad P(v)P(v') = P(v + v'),$$

$$P(v)T(v') = \exp[i(v | v')]T(v')P(v). \quad (3)$$

Written in this form, $G$ is called the Weyl CCR group.

The Lie algebra of the nuclear Lie group $G$ is the above mentioned Heisenberg algebra $A$. It is generated by the Hermitian elements $I, \phi(v), \pi(v), v \in V$, which obey the commutation relations

$$[\phi(v), I] = [\pi(v), I] = [\phi(v), \phi(v')] = [\pi(v), \pi(v')] = 0, \quad (4)$$

$$[\pi(v), \phi(v')] = -i\langle v | v' \rangle I. \quad (5)$$

Given a countable orthonormal basis $\{v_i\}$ for the pre-Hilbert space $V$, the CCR (4) – (5) take the form

$$[\phi(v_j), \phi(v_k)] = [\pi(v_k), \pi(v_j)] = 0, \quad [\pi(v_j), \phi(v_k)] = -i\delta_{jk}I.$$

One also introduces the creation and annihilation operators

$$a^\pm(v) = \frac{1}{\sqrt{2}}[\phi(v) \mp i\pi(v)]. \quad (6)$$

They obey the conjugation rule $(a^\pm(v))^* = a^\mp(v)$ and the commutation relations

$$[a^-(v), a^+(v')] = \langle v | v' \rangle I, \quad [a^+(v), a^+(v')] = [a^-(v), a^-(v')] = 0.$$
3 Hopf algebras of the CCR

Let us consider the tensor algebra $\otimes W$ of the vector space $W$ (2) generated by elements $\phi(v), \pi(v)$ and $I$. It is provided with a unique Hopf algebra structure, characterized by the comultiplication

$$\Delta(w) = w \otimes 1 + 1 \otimes w, \quad w \in W,$$

the counit $\epsilon(w) = 0$, the antipode $S(w) = -w$, and the universal matrix $R = 1 \otimes 1$. It is a cocommutative quasi-triangular Hopf algebra, called the classical Hopf algebra.

Let $A$ be the enveloping algebra of the Heisenberg CCR algebra $A$. It is the quotient of the tensor algebra $\otimes W$ by the commutation relations (4) – (5), written with respect to the tensor product $\otimes$, and by the relation

$$I \otimes I = I. \quad (7)$$

The $A$ inherits the structure of the classical Hopf algebra on $\otimes W$. We denote it $B_{cl}(A)$.

Now let us consider the quotient $A_{q,c}$ of the tensor algebra $\otimes W$ by the relations (4), (7) and the commutation relations

$$[\pi(v), \phi(v')] = -i \langle v|v' \rangle \frac{q^c I - q^{-c} I}{c(q - q^{-1})}, \quad (8)$$

where $q$ and $c$ are strictly positive real numbers. Due to the relation (7), the right-hand side of the relations (8) is well defined on $\otimes W$, and we have

$$[\pi(v), \phi(v')] = -i \langle v|v' \rangle \frac{q^c - q^{-c}}{c(q - q^{-1})} I. \quad (9)$$

Hence, $A_{q,c}$ is the enveloping algebra of the Heisenberg CCR algebra $A_{q,c}$ given by the commutation relations (4) and (9). This CCR algebra is modelled over the same nuclear space $V$, but provided with the Hermitian form

$$\langle v|v' \rangle_{q,c} = C_{q,c} \langle v|v' \rangle, \quad C_{q,c} = \frac{q^c - q^{-c}}{c(q - q^{-1})}. \quad (10)$$

The enveloping algebra $A_{q,c}$ admits both the structure of the classical Hopf algebra $B_{cl}(A_{q,c})$ and the Hopf algebra $B(A_{q,c})$, which differs from the classical one in the comultiplication law

$$\Delta(\phi(v)) = \phi(v) \otimes q^{c I/2} + q^{-c I/2} \otimes \phi(v), \quad \Delta(\pi(v)) = \pi(v) \otimes q^{c I/2} + q^{-c I/2} \otimes \pi(v),$$

$$\Delta(I) = I \otimes 1 + 1 \otimes I.$$
One can think of $B(A_{q,c})$ as being a Hopf algebra of the $q$-deformed CCR. It is readily observe that, if $c = 1$, the CCR algebras $A$ and $A_{q,1}$ coincide for any $q$, but the Hopf algebra $B(A_{q,1})$ differs from the classical one $B_{cl}(A_{q,1}) = B_{cl}(A)$. If $q = 1$, then $A_{1,c} = A$ and $B(A_{1,c}) = B_{cl}(A)$ for any $c$.

Since the Hopf algebra $B(A_{q,c})$ is the enveloping algebra of the CCR algebra $A_{q,c}$, its representations are determined in full by representations of $A_{q,c}$. Let us compare the representations of the CCR algebras $A$ and $A_{q,c}$.

### 4 Representations of the nuclear CCR

The CCR group $G$ contains two Abelian subgroups $T$ and $P$. Following the representation algorithm in [2], we first construct representations of the nuclear Abelian group $T$ [5].

Its cyclic strongly continuous unitary representation $\rho$ in a Hilbert space $(E, \langle .| . \rangle_E)$ with a (normed) cyclic vector $\theta \in E$ defines the complex function

$$Z(v) = \langle \rho(T(v))\theta|\theta\rangle_E$$

on $V$. This function is continuous and positive-definite, i.e., $Z(0) = 1$ and

$$\sum_{i,j} Z(v_i - v_j)c_ic_j \geq 0$$

for any finite set $v_1, \ldots, v_m$ of elements of $V$ and arbitrary complex numbers $c_1, \ldots, c_m$. By virtue of the well-known Bochner theorem, such a function on a nuclear space $V$ is the Fourier transform

$$Z(v) = \int \exp[i\langle v, u\rangle]\mu$$

(11)

of a positive measure $\mu$ of total mass 1 on the topological dual $V'$ of $V$. Then the above mentioned representation $\rho$ of $T$ can be given by the operators

$$T_Z(v)f(u) = \exp[i\langle v, u\rangle]f(u)$$

(12)

in the Hilbert space $L^2(V', \mu)$ of classes of $\mu$-equivalent square integrable complex functions $f(u)$ on $V'$. The cyclic vector $\theta$ of this representation is the $\mu$-equivalence class $\theta \approx_\mu 1$ of the constant function $f(u) = 1$. Conversely, every positive measure $\mu$ of total mass 1 on the dual $V'$ of $V$ (and, consequently, every continuous positive-definite function $Z(v)$ on $V$) defines a cyclic strongly continuous unitary representation (12) of the nuclear group $T$.

We agree to call $Z$ a generating function of this representation. One can show that distinct generating functions $Z$ and $Z'$ determine equivalent representations $T_Z$ and $T_{Z'}$ (12) of $T$ in the Hilbert spaces $L^2(V', \mu)$ and $L^2(V', \mu')$ iff they are the Fourier transform of equivalent measures on $V'$.
The representation $T_Z$ (12) of the group $T$ can be extended to the CCR group $G$ if the measure $\mu$ possesses the following property. Let $u, v \in V$, denote an element of $V'$ given by the condition

$$\langle v', u v \rangle = \langle v', v \rangle, \quad \forall v' \in V. \tag{13}$$

These elements form the image of the monomorphism $V \to V'$ determined by the Hermitian form $\langle \cdot, \cdot \rangle$ on $V$. Let the measure $\mu$ in (11) remain equivalent under translations $u \mapsto u + u_v$ of $V'$ by any element $u_v$ of $V \subset V'$, i.e.,

$$\mu(u + u_v) = a^2(v, u)\mu(u), \quad \forall u_v \in V \subset V', \tag{14}$$

where a function $a(v, u)$ is square $\mu$-integrable and strictly positive almost everywhere on $V'$. This function fulfils the relations

$$a(0, u) = 1, \quad a(v + v', u) = a(v, u)a(v', u + u_v). \tag{15}$$

A measure on $V'$ obeying the condition (14) is called translationally quasi-invariant. Let the generating function $Z$ of a cyclic strongly continuous unitary representation of the nuclear group $T$ be the Fourier transform (11) of such a measure $\mu$ on $V'$. Then the representation (12) of $T$ is extended to the representation of the nuclear CCR group $G$ in the Hilbert space $L^2(V', \mu)$ by operators

$$P_Z(v)f(u) = a(v, u)f(u + u_v). \tag{16}$$

Moreover, one can show that if $\mu'$ is a $\mu$-equivalent positive measure of total mass 1 on $V'$, it is also translationally quasi-invariant and provides an equivalent representation of $G$.

A strongly continuous unitary representation $T_Z$ (12), $P_Z$ (16) of the nuclear CCR group $G$ implies a representation of its Lie algebra $A$ by (unbounded) operators

$$I = 1, \quad \phi(v)f(u) = \langle v, u \rangle f(u), \quad \pi(v)f(u) = -i(\delta_v + \eta(v, u))f(u), \tag{17}$$

$$\delta_{v'}f(u) = \lim_{\alpha \to 0} \alpha^{-1}[f(u + \alpha u_v) - f(u)], \quad \alpha \in \mathbb{R},$$

$$\eta(v, u) = \lim_{\alpha \to 0} \alpha^{-1}[a(\alpha v, u) - 1], \tag{18}$$

in the same Hilbert space $L^2(V', \mu)$. With the aid of the formulas

$$\delta_v \delta_{v'} = \delta_{v'} \delta_v, \quad \delta_v(\eta(v', u)) = \delta_{v'}(\eta(v, u)),$$

$$\delta_v = -\delta_{-v}, \quad \delta_v(\langle v', u \rangle) = \langle v'|v \rangle,$$

$$\eta(0, u) = 0, \quad \forall u \in V', \quad \delta_v \theta = 0, \quad \forall v \in V,$$

derived from the relations (15), it is easily justified that the operators (17) fulfil the Heisenberg CCR (4).
Gaussian measures exemplify a physically relevant class of translationally quasi-invariant measures on the dual \( V' \) of a nuclear space \( V \). The Fourier transform of a Gaussian measure reads

\[
Z(v) = \exp \left[ -\frac{1}{2} M(v) \right],
\]

(19)

where \( M(v) \) is a seminorm on \( V' \) called the covariance form. Let \( \mu_K \) denote a Gaussian measure on \( V' \) whose Fourier transform is the generating function

\[
Z_K = \exp \left[ -\frac{1}{2} M_K(v) \right]
\]

(20)

with the covariance form \( M_K(v) = (K^{-1}v|K^{-1}v) \), where \( K \) is a bounded invertible operator in the Hilbert completion \( \tilde{V} \) of \( V \) with respect to the Hermitian form \( \langle .| . \rangle \). The Gaussian measure \( \mu_K \) is translationally quasi-invariant:

\[
\mu_K(u + u_v) = a_K^2(v,u) \mu_K(u),
\]

\[
a_K(v,u) = \exp \left[ -\frac{1}{4} M_K(Cv) - \frac{1}{2} (Cq, u) \right],
\]

(21)

where \( C = KK^* \) is a bounded Hermitian operator in \( \tilde{V} \).

Let us construct the representation of the CCR algebra \( A \) determined by the generating function \( Z_K \) (20). Substituting the function (21) into the formula (18), we find

\[
\eta(v, u) = -\frac{1}{2} (Cv, u).
\]

Hence, the operators \( \phi(v) \) and \( \pi(v) \) (17) take the form

\[
\phi(v) = \langle v, u \rangle, \quad \pi(v) = -i \delta_v - \frac{1}{2} (Cv, u)).
\]

(22)

Accordingly, the creation and annihilation operators (6) read

\[
a^\pm(v) = \frac{1}{\sqrt{2}} \pm \delta_v \pm \frac{1}{2} (Cv, u) + \langle v, u \rangle.
\]

(23)

In particular, let us put \( K = \sqrt{2} \cdot 1 \). Then the generating function (20) takes the form

\[
Z_F(v) = \exp \left[ -\frac{1}{4} \langle v|v \rangle \right],
\]

(24)

and determines the Fock representation of the CCR algebra \( A \) by the operators

\[
\phi(v) = \langle v, u \rangle, \quad \pi(v) = -i(\delta_v - \langle v, u \rangle),
\]

\[
a^+(v) = \frac{1}{\sqrt{2}} [-\delta_v + 2 \langle v,u \rangle], \quad a^-(v) = \frac{1}{\sqrt{2}} \delta_v.
\]
Note that the Fock representation up to an equivalence is characterized by the existence of a cyclic vector \( \theta \) such that
\[
a^-(v)\theta = 0, \quad \forall v \in V.
\] (25)

An equivalent condition is that there exists the particle number operator \( N \) possessing a lower bounded spectrum. This operator is defined by the conditions
\[
[N, a^\pm(v)] = \pm a^\pm(v)
\]
up to a summand \( \lambda 1 \). With respect to a countable orthonormal basis \( \{v_k\} \), it is given by the sum
\[
N = \sum_k a^+(v_k) a^-(v_k).
\]

A glance at the expression (23) shows that the condition (25) does not hold, unless \( Z_K \) is \( Z_F \) (24). For instance, the particle number operator in the representation (23) reads
\[
N = \sum_j a^+(v_j) a^-(v_j) = \sum_j [-\delta_{v_j}\delta v_j + C^j_k(v_k, u)\partial v_j +
\]
\[
(\delta_{km} - \frac{1}{4} C^j_k C^j_m)(v_k, u)\langle v_m, u \rangle - (\delta_{jj} - \frac{1}{2} C^j_j)].
\]

One can show that this operator is defined and is lower bounded only if the operator \( C \) is a sum of the scalar operator \( 2 \cdot 1 \) and a nuclear operator in \( \tilde{V} \). For instance, the generating function
\[
Z_c(v) = \exp[-\frac{c^2}{2}\langle v|v \rangle], \quad c^2 \neq \frac{1}{2},
\]
determines a non-Fock representation of the nuclear CCR.

At the same time, the non-Fock representation (22) of the CCR algebra (4) is the Fock representation
\[
\phi_K(v) = \phi(v) = \langle v, u \rangle,
\]
\[
\pi_K(v) = \pi(S^{-1}v) = -i(\delta^K_v - \frac{1}{2}\langle v, u \rangle), \quad \delta^K_v = \delta_{S^{-1}v},
\]
of the CCR algebra \{\phi_K(v), \pi_K(v), I\}, where
\[
[\phi_K(v), \pi_K(v)] = i(K^{-1}v|K^{-1}v')I.
\]

Bearing in mind this fact, turn now to the CCR algebra \( A_{q,c} \) in Section 3. Comparing the commutation relations (5) and (9), one can show that, given a representation \( \rho \) of the CCR algebra \( A \), the CCR algebra \( A_{q,c} \) admits a representation \( \rho_{q,c} \) by the operators
\[
\rho_{q,c}(\phi(v)) = \rho(\phi(v)), \quad \rho_{q,c}(\pi(v)) = \rho(\pi(C_{q,c}v)), \quad \rho_{q,c}(I) = \rho(I) = 1,
\]
where \( C_{q,c} \) is the real number given by the expression (10). For instance, if \( \rho \) is the Fock representation of the CCR algebra \( A \), the representation \( \rho_{q,c} \) is not equivalent to the Fock representation of the CCR algebra \( A_{q,c} \), unless \( V \) is finite-dimensional.
References


[5] G.Sardanashvily, Non-equivalent representations of nuclear algebras of canonical com-