A Perturbative Window into Non-Perturbative Physics

Robbert Dijkgraaf

Institute for Theoretical Physics &
Korteweg-de Vries Institute for Mathematics
University of Amsterdam
1018 TV Amsterdam, The Netherlands

and

Cumrun Vafa

Jefferson Physical Laboratory
Harvard University
Cambridge, MA 02138, USA

Abstract

We argue that for a large class of $\mathcal{N} = 1$ supersymmetric gauge theories the effective superpotential as a function of the glueball chiral superfield is exactly given by a summation of planar diagrams of the same gauge theory. This perturbative computation reduces to a matrix model whose action is the tree-level superpotential. For all models that can be embedded in string theory we give a proof of this result, and we sketch an argument how to derive this more generally directly in field theory. These results are obtained without assuming any conjectured dualities and can be used as a systematic method to compute instanton effects: the perturbative corrections up to $n$-th loop can be used to compute up to $n$-instanton corrections. These techniques allow us to see many non-perturbative effects, such as the Seiberg-Witten solutions of $\mathcal{N} = 2$ theories, the consequences of Montonen-Olive $S$-duality in $\mathcal{N} = 1^*$ and Seiberg-like dualities for $\mathcal{N} = 1$ theories from
a completely perturbative planar point of view in the same gauge theory, without invoking a dual description.
1. Introduction

One of the important insights in recent years in theoretical physics has been the discovery of duality symmetries in gauge theory and string theory. In particular we have learned that the dynamics of supersymmetric gauge theories, in particular non-perturbative effects at strong coupling, are often captured by some weakly coupled dual theory. The Montonen-Olive duality of $\mathcal{N} = 4$ [1] and the strong coupling dynamics of $\mathcal{N} = 2$ gauge theory captured by a dual abelian gauge theory via the Seiberg-Witten geometry [2] are among the prime examples. These gauge theoretic dualities have been embedded in string theory dualities where the gauge theory is engineered by considering a suitable string background. In some cases, for example in the context of Matrix Theory [3] and the AdS/CFT correspondence [4], the string/gauge theory dualities are actually equivalent.

In this paper we wish to argue that in fact essentially all these gauge theoretic dualities can be seen perturbatively in the same gauge theory in the context of computations of exact superpotentials in $\mathcal{N} = 1$ gauge theories, that are obtained by adding deformations to the original gauge theory. We will give arguments that these superpotentials can be computed exactly by summing over all planar diagrams of the zero-momentum modes. Key to this all is the computation of the effective superpotential as a function of the glueball chiral superfield $S = \frac{1}{32\pi^2} \text{Tr} W_\alpha W^\alpha$, generalizing the Veneziano-Yankielowicz effective superpotential for the case of pure Yang-Mills [5].

That certain quantities can be computed exactly in perturbation theory has been encountered before in quantum field theories. In particular the axial anomaly can be computed exactly by a one loop computation. However, it is more rare to encounter computable, anomaly-like quantities at higher loops in perturbation theory. Such computable anomaly-like amplitudes were encountered in type II string perturbation theory [6,7], where it was shown that topological string amplitudes on Calabi-Yau threefolds compute F-type terms in the associated four-dimensional $\mathcal{N} = 2$ supersymmetric theory. Furthermore in [6] it was shown that in the context of type I string theory the topological string computes superpotential terms involving the glueball superfield at higher loops in open string perturbation theory, and it was speculated there that this must be important for aspects of gaugino condensation in $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions.

These open string computations of the effective superpotential can be exactly summed if a closed string large $N$ dual can be found. The study of large $N$ dualities in the context of topological strings were initiated in [8], and embedded in superstrings in [9].
idea was developed further in a series of papers beginning with [10] were highly non-trivial aspects of $\mathcal{N} = 1$ supersymmetric gauge theories were deduced using this duality [11,12,13,14,15,16,17]. More recently it was discovered in [18,19] that the relevant computations in [10] are perturbative even from the viewpoint of the corresponding gauge theory and reduce to matrix integrals. The aim of this paper is to explain the universality of the ideas in [18,19] and show its power in gaining insight into non-perturbative gauge theoretic phenomena from a perturbative perspective. In particular we explain how these ideas may be understood from first principles in purely gauge theoretic terms without appealing to string theory or any other dualities.

1.1. A chain of string dualities

Although we want to stress in the paper that we are making a purely field-theoretic statement about supersymmetric gauge theories, perhaps it can be helpful to summarize the chain of arguments in string theory that have led us to this result.

To actually compute F-terms explicitly in string theory one can make use of a series of powerful dualities. As a starting point one typically engineers the $\mathcal{N} = 1$ gauge theory in string theory by a D-brane configuration, for example by realizing it as a collection of D5-branes wrapped around two-cycles in a Calabi-Yau geometry in type IIB theory. This Calabi-Yau is typically a small resolution of a singular geometry. This open string theory can then have a large $N$ dual closed string realization, where the Calabi-Yau geometry goes through a so-called geometric transition in which the topology changes. The two-cycles around which the D-branes were wrapped are blown down and three-cycles appear. In the process of this transition the D-branes disappear and emerge as fluxes of a 3-form field strength $H = H_{RR} + \tau H_{NS}$. 

Fig. 1: The Calabi-Yau geometry dual to the gauge theory, with its canonical homology basis.
In the dual closed string theory on the deformed Calabi-Yau there is an elegant exact expression for the effective superpotential in terms of the geometry [20,21,22]

\[ W_{\text{eff}} = \int_X H \wedge \Omega \]

with \( \Omega \) the holomorphic \((3,0)\) form on the Calabi-Yau space \( X \). Introducing a canonical basis of homology three-cycles \( A_i, B_i \), where the \( A \)-cycles are typically compact and the \( B \)-cycles non-compact (see fig. 1) this can be written as

\[ W_{\text{eff}} = \sum_i \left[ \oint_{A_i} H \int_{B_i} \Omega - \int_{B_i} H \oint_{A_i} \Omega \right]. \]

The periods of the holomorphic three-form give the so-called special geometry relations

\[ \oint_{A_i} \Omega = 2\pi i S_i, \quad \int_{B_i} \Omega = \frac{\partial F_0}{\partial S_i}. \]

The variables \( S_i \) can be used as moduli that parametrize the complex structure of the CY. From the dual gauge theory perspective these moduli \( S_i \) are the glueball condensates that appear when the strongly coupled gauge theory confines [9]. To evaluate the superpotential one further needs the fluxes of the \( H \)-field through the cycles

\[ N_i = \oint_{A_i} H, \quad \tau_i = \int_{B_i} H. \]

The integers \( N_i \) correspond to the number of D-branes (the rank of the gauge group), and the cut-off dependent variables \( \tau_i \) correspond to the bare gauge couplings, \( \tau \sim 4\pi i/g^2 \).

With all this notation the final expression for superpotential can be simply written as

\[ W_{\text{eff}}(S) = \sum_i \left( N_i \frac{\partial F_0}{\partial S_i} - 2\pi i \tau_i S_i \right). \quad (1.1) \]

The only non-trivial ingredient is the prepotential \( F_0(S) \) that encodes the special geometry and that is completely determined by the CY geometry.

The prepotential \( F_0(S) \) is well-known to reduce to a computation in the closed topological string theory, that is purely defined in terms of the internal CY space. In fact, the full computation of the effective superpotential can be done within the topological string. The same geometric transition that related the closed type IIB string background to the D-brane system, also relates the closed topological string to an open topological string propagating in the background of a collection of branes. These so-called B-branes
are entirely located within the internal CY space—they are simply the internal part of the original D-branes. The open and closed topological strings are related by the same large $N$ duality as the physical theory. The planar diagrams of the open string reproduce the sphere diagram of the closed string.

But there is an important difference with the physical theory. In the topological setting the closed string moduli $S_i$ are expressed in terms of the number of branes $N_i$ and the string coupling $g_s$ through the 't Hooft couplings \[ S_i = g_s N_i, \]
defined in the limit $N_i \to \infty$, $g_s \to 0$. So in the topological context the variables $S_i$ and $N_i$ are not independent and the moduli $S_i$ are frozen to the above values. This should be contrasted with the physical string theory where the glueball fields $S_i$ are not frozen and independent of $N_i$—the expectation values of the $S_i$ are set by extremizing the effective superpotential (1.1) and thus depend on the gauge couplings.

Irrespective of the existence of a closed string theory dual defined by a geometric transition, one can study the open topological string on itself. Its amplitudes are directly related to F-term computations in the corresponding D-brane system. In particular its planar diagrams are directly related to the computation of the superpotential in the physical gauge theory. The world-volume theory on the B-brane is a dimensionally reduced version of the holomorphic Chern-Simons gauge theory introduced in [23]. As explained in [18] under suitable circumstances this model reduces to only zero-modes, and in this way the computation reduces to a zero-dimensional field theory—a generalized matrix model. However, once we have arrived at this point we have essentially gone through a $360^\circ$ rotation, since we can relate the diagrams of the matrix model directly to the original gauge theory, essentially bypassing all the previous dualities—and this will be the point of view in this paper. We summarize this chain of dualities in fig. 2

1.2. Outline

The plan of this paper is as follows: In section 2 we state the precise relation between $\mathcal{N} = 1$ four-dimensional supersymmetric gauge theories and matrix models. In particular this relation shows that one can use perturbative gauge theory techniques to gain exact information about highly non-trivial non-perturbative aspects of gauge theory. In section 3 we outline the arguments leading to this statement, both within string theory and in
field theory. In section 4 we discuss the implications of this result for deriving various exact consequences for gauge theories. This includes a perturbative derivation of $\mathcal{N} = 2$ Seiberg-Witten geometry as well as derivation of $\mathcal{N} = 1$ Seiberg-like dualities. Moreover we find that non-perturbative consequences of Olive-Montonen dualities, as seen in the $\mathcal{N} = 1^*$ deformation of $\mathcal{N} = 4$ Yang-Mills theory, can also be derived in this setup. In section 5 we conclude with some ideas which we are presently pursuing and summarize our main conclusions.

2. Matrix models and effective superpotentials

We will formulate a very general conjecture relating exact superpotentials and matrix
models. But before we do this let us first discuss an example that shows all the features.

2.1. The canonical example

The key example to keep in mind is $\mathcal{N} = 1 \ U(N)$ gauge theory with one adjoint chiral matter field $\Phi$. This theory is obtained by softly breaking $\mathcal{N} = 2$ super-Yang-Mills down to $\mathcal{N} = 1$ by means of the tree-level superpotential

$$\int d^2 \theta \ \text{Tr} \ W(\Phi).$$

Here we take $W(x)$ to be a polynomial of degree $n + 1$ in a single complex variable $x$. Since we do not want to limit ourselves exclusively to a superpotential of at most degree three, we typically think of this model as an effective theory obtained by integrating out other fields in an underlying renormalizable quantum field theory.

Since $W$ has (generically) $n$ isolated critical points at $x = a_1, \ldots, a_n$, the classical vacua of this theory are determined by distributing the eigenvalues of the matrix $\Phi$ over these critical points. If we choose a partition

$$N = N_1 + \ldots + N_n,$$

and put $N_i$ eigenvalues at the critical point $a_i$ then we have a symmetry breaking pattern

$$U(N) \to U(N_1) \times \cdots \times U(N_n).$$

The corresponding quantum vacua are described by the appearance of a gaugino condensate and confinement in the $SU(N_i)$ subgroups of these $U(N_i)$ factors. The gauge coupling becomes strong at a dynamically generated scale $\Lambda$ where the gauge group $SU(N_i)$ is completely broken down and a mass gap is generated. Let

$$S_i = \frac{1}{32\pi^2} \text{Tr}_{SU(N_i)} W_{\alpha}^2$$

denote the corresponding chiral superfield, whose lowest component is the gaugino bilinear $\text{Tr} \lambda_{\alpha}^2$ that gets a dynamical expectation value, and whose top component gives the (chiral half) of the super-Yang-Mills action. The condensate $\langle S_i \rangle$ breaks the $\mathbb{Z}_{2N_i}$ global symmetry down to $\mathbb{Z}_2$, and thus generates $N_i$ inequivalent vacua. The relevant physical quantities to compute in these confining vacua are the values of the gaugino condensate and the tensions of the domain walls interpolating between the different vacua, that can be
expressed as the differences of the values of the effective superpotential in these confining vacua. Of course, all these quantities will be highly non-trivial functions of the coupling constants of the bare superpotential $W_{\text{tree}}(\Phi)$, and it is these functions that we are after.

The effect of gaugino condensation is elegantly described by an effective superpotential which is a function of the chiral superfields $S_i$

$$\int d^2 \theta W_{\text{eff}}(S_i).$$

Minimizing this action with respect to the variables $S_i$ then describes the vacuum structure of the theory and determines the computable physical quantities. For a pure $SU(N)$ gauge theory this effective superpotential takes the Veneziano-Yankielowicz form [5] (up to non-universal terms that can be absorbed in the cut-off scale $\Lambda_0$)

$$W_{\text{eff}}(S) = NS \log(S/\Lambda_0^3) - 2\pi i \tau S,$$  \hspace{1cm} (2.2)

with $\tau$ the usual combination of the bare gauge coupling $g$ and the theta angle

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}.$$

Minimizing the action with respect to $S$ gives the value of the condensate in the $N$ distinct vacua labeled by $k = 0, 1, \ldots, N-1$

$$\langle S \rangle \sim \Lambda^3 e^{2\pi i k/N},$$  \hspace{1cm} (2.3)

with $\Lambda$ the dynamically generated scale. The gaugino condensate is interpreted as a non-perturbative effect due to a fractional instanton of charge $1/N$. Equivalently, the $N$-point function of $S$ is constant and saturated by a one-instanton field configuration, $\langle S^N \rangle \sim e^{2\pi i \tau}$ [24].

The $S \log S$ term in the action (2.2) strongly suggests a derivation of the superpotential by a perturbative—in fact one-loop—Coleman-Weinberg-like computation. As such it should perhaps be compared to the analogous one loop computation of the term $N \Sigma \log \Sigma$ in the two-dimensional linear sigma model for the supersymmetric $\mathbb{CP}^{N-1}$ sigma model [25]. The corresponding argument in the four-dimensional case uses the anomalous $R$-symmetry [5]: the anomalous axial $U(1)$ symmetry implies that rotating

$$S \rightarrow e^{2\pi i} S$$
shifts the theta angle by $2\pi N$ which implies that the superpotential shifts by
\[ W_{\text{eff}}(S) \rightarrow W_{\text{eff}}(S) + 2\pi i NS. \]
This multivaluedness then leads to the $NS \log S$ term in the superpotential (2.2). Note that this anomaly is closely related to the measure of the path-integral, in particular that of the chiral fermions. Field theoretic aspects of this have been discussed in [26].

In some sense minimizing the effective superpotential (2.2) thus turns a perturbative (one-loop) effect into the non-perturbative condensate (2.3). This is a more familiar story in two-dimensional supersymmetric linear sigma models such as the $\text{CP}^{N-1}$ model. There the 1-loop generated $N\Sigma \log \Sigma$ superpotential also captures, after minimizing, the instantons of the two-dimensional sigma model. In particular it gives rise to the quantum cohomology ring
\[ \Sigma^N = e^{-t}, \]
where $t$ is the Kahler parameter of $\text{CP}^{N-1}$ and $\Sigma$ represents the chiral field corresponding to the Kahler class. This two-dimensional example of the quantum cohomology ring clearly demonstrates that, even though we are in a massive vacuum, there is still interesting holomorphic information to be computed—a point that we want to stress also holds in the four-dimensional theory.

2.2. An exact superpotential

Now the question is how the form of the effective superpotential $W_{\text{eff}}(S)$ changes if we add the adjoint field $\Phi$ with the tree-level potential (2.1). The exact answer was given in [10] in terms of a dual Calabi-Yau geometry given by the algebraic variety
\[ u^2 + v^2 + y^2 + W'(x)^2 + f(x) = 0, \]  
where the polynomial $f(x)$ of degree $n-1$ is a deformation of the singular geometry. Its coefficients parametrize the moduli of the CY and thereby the gaugino condensates $S_i$ in the dual gauge theory. This answer was checked against many non-trivial field theoretic computations in [10].

Furthermore in [17] it was explicitly verified that this answer can also be derived from, and is in fact equivalent to, the Seiberg-Witten solution of the undeformed $\mathcal{N} = 2$ $U(N)$ super-Yang-Mills theory [2,27,28]. The idea in [17] is to consider a very special tree-level superpotential $W(\Phi)$ with degree $N+1$ for a $U(N)$ gauge group. Thus $W'(\Phi) = \ldots$
\( \epsilon \prod (\Phi - a_i) = 0 \) has \( N \) roots \( \Phi = a_1, \ldots, a_N \). If we place the \( N \) eigenvalues of \( \Phi \) at the \( N \) distinct values \( a_i \) (i.e. the multiplicities are \( N_i = 1 \) for all \( i \)), then as \( \epsilon \to 0 \) we go back to a particular point on the Coulomb branch of \( \mathcal{N} = 2 \) theory specified by the values of the roots \( a_i \). It was shown in [17] that the \( \mathcal{N} = 1 \) theory has certain non-trivial computable quantities that do not depend on \( \epsilon \) and therefore thus knows about the answer for \( \mathcal{N} = 2 \) which is obtained as \( \epsilon \to 0 \). In this way the full Seiberg-Witten geometry was derived from this \( \mathcal{N} = 1 \) deformation.

Then in [18] it was subsequently realized that this \( \mathcal{N} = 1 \) answer has an elegant formulation directly in terms of the perturbation theory of the original gauge theory. It was found that the full answer studied in [10] takes the form

\[
W_{\text{eff}}(S) = \sum_i \left[ N_i S_i \log(S_i/\Lambda^3_i) - 2\pi i \tau S_i + N_i \frac{\partial F_{\text{pert}}(S)}{\partial S_i} \right],
\]

(2.5)

where \( F_{\text{pert}}(S) \) is a perturbative expansion in the \( S_i \)

\[
F_{\text{pert}}(S) = \sum_{i_1, \ldots, i_n \geq 0} c_{i_1, \ldots, i_n} S_1^{i_1} \cdots S_n^{i_n}.
\]

This perturbation series is obtained from the tree-level superpotential \( W(\Phi) \) by an expansion in terms of ‘fat’ Feynman diagrams around the classical vacua, only taking into account the constant modes and the planar diagrams. Here every factor \( S_k \) corresponds to a single loop (hole in the corresponding Riemann surface) indexed by the critical point \( a_k \). The coefficients \( c_{i_1, \ldots, i_n} \) are the planar amplitudes of the matrix model with \( i_k \) holes ending on the \( a_k \) vacuum. Such a series of terms is not forbidden by non-renormalization effects because the explicit breaking of \( R \)-symmetry by the tree-level superpotential, whose coefficients have definite non-zero \( R \)-charge, gives a non-zero effect in the background of a \( S \) condensate. (We will argue for this more precisely in the next section.)

Since only the planar diagrams contribute to \( W_{\text{eff}} \) the \( N \)-dependence of the final answer is extremely simple. This unexpected result, that follows straightforwardly from string theory, is completely in accord with field theory arguments, as was shown in [10,17].

The above prescription can be conveniently formulated in terms of the following matrix model. Consider the saddle-point expansion of the holomorphic integral over complex \( N \times N \) matrices \( \Phi \)

\[
Z = \int d\Phi \cdot \exp\left(-\frac{1}{g_s} \text{Tr} W(\Phi)\right)
\]

(2.6)
Since we only integrate over $d\Phi$ and not $d\Phi d\Phi$, a non-perturbative definition of this integral will require a specific choice for the integration contour. But this choice will not influence the perturbative expansion. For a generic position of the contour the matrices $\Phi$ will be diagonalizable. The expansion parameter $g_s$ can be identified with the string coupling within a string theory realization (since the tree-level superpotential is given by a disc diagram) but within field theory it is simply a conveniently chosen overall scale in front of the tree-level superpotential.

The usual large $N$ techniques tell us that around the saddle-point where $N_i$ eigenvalues of $\Phi$ are at the $i$-th critical point, this matrix integral has a consistent saddle-point approximation of the form

$$Z = \exp \sum_{g \geq 0} g_s^{2g-2} \mathcal{F}_g(S)$$

in the ’t Hooft limit $N_i \gg 1$ and $g_s \ll 1$, while keeping finite the individual ’t Hooft couplings

$$S_i = g_s N_i.$$ 

We have one factor $S_i$ for each hole indexed by the index $i$. In this large $N$ expansion of the free energy the term $\mathcal{F}_g(S)$ is the sum of all diagrams of genus $g$. In particular the leading contribution $\mathcal{F}_0(S)$ is given by the planar diagrams.

Furthermore, the contribution of the measure in the matrix model, which is given by the volume of $U(N_1) \times \cdots \times U(N_n)$ (and some gaussian measure which goes into the definition of the scales $\Lambda_i$) gives exactly rise to the $S_i \log S_i/\Lambda_i^3$ term in the superpotential\textsuperscript{1}. If we factor out these measure contributions, we are left with

$$F_{\text{pert}}(S) = \mathcal{F}_{0,\text{pert}}(S)$$ (2.7)

where $\mathcal{F}_{0,\text{pert}}(S)$ is given by the sum of all planar diagrams contributing to the free energy of the matrix model.

To be completely specific, take for example the cubic superpotential

$$W(\Phi) = \frac{1}{2} m \Phi^2 + \frac{1}{3} g \Phi^3,$$

\textsuperscript{1} If one would have included matter fields transforming in the fundamental representation with mass matrix $m$, then simply integrating out these gaussian variables in the matrix model immediately gives the extra contribution $S \log \det(m/\Lambda)$ that upon minimizing reproduces the Affleck-Dine-Seiberg superpotential [29].
and consider the classical vacua where all eigenvalues sit at $\Phi = 0$. In that case the perturbative gauge group is still the full $U(N)$. We are now predicting that the perturbative contributions to the effective superpotential are obtained from the Feynman rules derived from the action $W(\Phi)$, with propagator $1/m$ and three-point vertex $g$. So the leading correction will take the form

$$F_{\text{pert}}(S) = \left( \frac{1}{6} + \frac{1}{2} \right) \frac{g^2}{m^3} S^3 + \mathcal{O}(S^4),$$

where the combinatorical weight in front is given by summing the two planar two-loop diagrams of fig. 3. This indeed agrees with the predictions in [10].

Quite remarkably, this perturbative sum of Feynman diagrams will turn into a non-perturbative sum over (fractional) instantons upon minimizing with respect to the glueball
superfields $S_i$. Even if one cannot find an exact large $N$ solution, one could still work order by order in the matrix loop expansion and translate this order by order computation into an order by order computation for the instanton sum. In particular if we keep up to $k$-th term in the $S_i$’s, after extremization we get corrections up to $k$-th fractional instanton. Moreover, summing all perturbative planar loops amounts to an exact instanton sum. Note that we have isolated a rare phenomenon, a gauge theory quantity for which the planar limit is exact!

As we showed in [18], in this specific example of a single adjoint chiral matter field one can solve for the exact large $N$ result directly using standard random matrices techniques. One diagonalizes the matrix $\Phi$ and writes an effective action for the eigenvalues $\lambda_1, \ldots, \lambda_N$

$$S_{\text{eff}}(\lambda) = \sum_I W(\lambda_I) - g_s \sum_{I<J} \log(\lambda_I - \lambda_J)^2.$$  

The derivative of this action for a given “probe” eigenvalue at a position $x$ in the complex plane is then given by

$$y(x) = W'(x) - g_s \sum_I \frac{2}{x - \lambda_I},$$

where the second term is a one-loop effect obtained by integrating out the angular variables in $\Phi$. The one-form $y dx$ is directly related to the eigenvalue density. In the large $N$ limit the variable $y$ can be shown to satisfy an algebraic equation

$$y^2 = W'(x)^2 + f_{n-1}(x) \quad \text{(2.8)}$$

that defines a hyperelliptic Riemann surface. Here $f_{n-1}(x)$ is a polynomial of degree $n - 1$ (the quantum deformation), whose $n$ coefficients can be used to parametrize the ’t Hooft couplings $S_i$. The solution of the matrix model then takes the form of a set of period integrals (special geometry) of the meromorphic one-form $y dx$

$$S_i = \frac{1}{2\pi i} \oint_{A_i} y dx, \quad \frac{\partial \mathcal{F}_0}{\partial S_i} = \int_{B_i} y dx, \quad \text{(2.9)}$$

where $A_i$ and $B_i$ are canonically conjugated cycles on the Riemann surface (2.8). This Riemann surface and the one-form $y dx$ are the reduction of the Calabi-Yau geometry (2.4) together with its holomorphic three-form $\Omega$ to the variables $(x, y)$. This then proves the equivalence between the perturbative computation of summing planar diagrams and the dual Calabi-Yau geometry.
2.3. The general conjectures

We are now in a position to formulate our general conjecture. Consider an $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $G$ a classical Lie group, i.e. $G$ is a product of the gauge groups of the type $U(N)$, $SO(N)$, or $Sp(N)$ (so no exceptional groups). Moreover we consider a matter content compatible with an 't Hooft Riemann surface expansion. This is essentially equivalent to the statement that the system has some kind of open string realization. In particular one can assume we have a quiver type gauge theory involving the product of series of $U(N_i)$ gauge groups with some bi-fundamental matter content and some (single-trace) tree-level superpotential terms. One might also possibly consider a $\mathbb{Z}_2$ orientifold of this system that gives rise in addition to $SO$ and $Sp$ gauge groups which can be formulated directly at the level of the quiver itself. So for example spinor representations are not allowed, and we can have up to second rank tensor products of fundamental representation. We also assume that we can add superpotential terms to give mass to all the matter fields.

This system will have some tree-level superpotential for the chiral matter multiplets $\Phi_a$

$$ \int d^2 \theta \ W_{\text{tree}}(\Phi_a). $$

The classical vacua correspond to the critical points of $W_{\text{tree}}(\Phi_a)$. We will further assume that we are working with a vacuum that is completely massive, possibly up to some pure $\mathcal{N} = 1$ Yang-Mills theory with group $G = \times_i G_i$ (each factor of which is associated to an integer $N_i$).

We want to determine the effective superpotential

$$ \int d^2 \theta \ W_{\text{eff}}(S_i) $$

in terms of the gaugino bilinear fields $S_i$ of the gauge groups $G_i$. Our claim is now the following: the effective superpotential is always of the form

$$ W_{\text{eff}}(S_i) = \sum_i (\tilde{N}_i S_i \log(S_i/\Lambda_i^3) - 2\pi i \tau_i S_i) + W_{\text{pert}}(S_i) $$

with $\tau_i$ the bare couplings. The leading terms is the usual one loop matching of scales. For the $U(N_i)$ gauge factors the integers $\tilde{N}_i$ are given by the rank $\tilde{N}_i = N_i$; for the case of $SO(N_i)$ and $Sp(N_i)$ they are given by $\tilde{N}_i = N_i \mp 2$. The perturbative contributions $W_{\text{pert}}(S_i)$ come from two sources: 't Hooft diagrams with the topology of $S^2$ or $\mathbb{RP}^2$. 

14
(the latter arises only when we have the orientifold operation and the resulting unoriented Riemann surfaces).

We will argue that these perturbative contributions are computed by the associated matrix model with action given by \( W_{\text{tree}}(\Phi_a) \). Consider the saddle-point expansion of this matrix theory corresponding to the vacuum with unbroken gauge group \( \times_i G_i \). Let \( \mathcal{F}_0(S_i) \) and \( \mathcal{G}_0(S_i) \) denote the contributions to the free energy of the matrix model of diagrams with the topology of \( S^2 \) and \( \mathbb{RP}^2 \) respectively. Here the \( S_i \) dependence of the diagrams captures the number of 't Hooft loops ending on the group \( G_i \) and thus can be identified with the coupling \( g_s \tilde{N}_i \). Then we claim that

\[
W_{\text{pert}}(S_i) = \tilde{N}_i \frac{\partial \mathcal{F}_0(S_i)}{\partial S_i} + \mathcal{G}_0(S_i).
\]

(The one loop \( S \log S \) piece can also absorbed into the above if we include the correct measure for the matrix theory.)

Moreover for the abelian \( U(1) \subset G_i \) factors, that remain after the non-abelian gauge factors develops a mass gap, one can find the matrix of coupling constants \( \tau_{ij} \). It is also given in terms of the planar diagrams as

\[
\tau_{ij} = \frac{\partial^2 F_0}{\partial S_i \partial S_j}.
\]

One can extend this conjecture to include certain gravitational corrections as will be discussed below.

Here we have formulated our conjecture in the context of \( \mathcal{N} = 1 \) theories with no massless moduli. However this correspondence also holds in the cases where we have massless moduli. To see this, note that we can always deform the superpotential and give mass to the massless moduli, reducing to the case considered in this conjecture, and then in the end removing the deformation. This idea will be important for us in this paper in the context of \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) supersymmetric gauge theories, which will have massless moduli. More generally, in the case of moduli there is still interesting holomorphic data to be computed, like the chiral ring of operators, and we claim that these data too should be captured by planar diagrams.
2.4. Calabi-Yau geometry as large $N$ master field

It is not necessary to solve for the exact large $N$ limit of the matrix model if one is content to work order by order in the instanton expansion. However, in case a large $N$ solution is possible we are conjecturing that the master field configuration that dominates in the large $N$ limit, will take the form of the special geometry associated to a (non-compact) Calabi-Yau three-fold. In the previous example, and also in the case of the quiver matrix models considered in [19], this three-fold is of the special form

$$u^2 + v^2 + F(x, y) = 0,$$

and is thus essentially given by the complex curve $F(x, y) = 0$. But in more complicated cases we expect this geometry to be a truly three-dimensional complex manifold. For example, as a first step in this direction, in [13] a quiver gauge theory is described with two adjoints and a tree-level superpotential

$$W_{\text{tree}} = \text{Tr}(\Phi_1 \Phi_2^2 + \Phi_1^n).$$

It is further shown that the exact effective superpotential is computed in terms of an associated dual CY geometry (“Laufer’s geometry”) that is of the more involved form

$$u^2 + F(v, x, y) = 0,$$

where $F(v, x, y) = 0$ defines a complex surface. It would be very interesting to derive this geometry directly from the matrix model, in particular to find the role of the three-form $\Omega$, that in this solution will reduce to a meromorphic two-form on the complex surface.

3. Derivation of our conjecture

We will now argue that for any $\mathcal{N} = 1$ gauge system that can be embedded in string theory, for example by engineering a suitable configuration of D-branes in a Calabi-Yau background in type II theory, we can give a proof of our main conjecture. As we will show below, that argument can be completely reduced to the $\alpha' \to 0$ field theory limit—no stringy effects play any role, otherwise than that the gauge should allow for a diagrammatic large $N$ expansion. We are confident therefore that this argument can ultimately be given completely in field theoretic terms.
One case where this strategy has been worked out successfully was the case of the conformally invariant orbifolds of $\mathcal{N} = 4$ gauge theory introduced in [30,31]. The vanishing of the beta function to leading order in large $N$ was later shown to be equivalent to that of the original $\mathcal{N} = 4$ theory, but the arguments for that remarkable result relied on string theory perturbation techniques [32]. However as noted in [32] one should have expected these results to also be derivable directly in field theory by taking the $\alpha' \to 0$ limit of string perturbation theory and in fact such a direct proof was later given entirely within field theory [33]. However the insight of string theory was crucial for coming up with such an argument. The same is true in the case we are studying.

### 3.1. F-terms in string theory

As was first discovered in [6,7], F-terms for a four-dimensional supersymmetric theory correspond within type II superstring theory to a particular class of amplitudes that are exactly computed by topological string theory. The topological string is entirely formulated in terms of the internal CY space, and so is in some sense localized to constant modes in space-time. For type I open strings there is also such a correspondence between F-terms and open topological strings that was found in [6,§8.2] where it was further shown that the superpotential for glueball fields receives contributions only from planar diagrams. The possibility of applying this powerful technique to the dynamics of four-dimensional supersymmetric gauge theories was already noted. The arguments in [6] easily generalizes from type I strings to arbitrary D-branes (whose relevance was not yet appreciated around that time). The derivation of these results is particularly simple in the Berkovits formalism, generalizing the same computation in the context of closed topological strings done in [34].

Let us briefly review why in the string context the superpotential is completely captured by the planar diagrams. There is a simple geometric argument given in [6]. One realizes the open string diagram with a very particular worldsheet metric, namely by a flat metric with cuts in it—very much as in light-cone gauge. This metric has curvature singularities at the two end points of each cut, where the deficit angle is $2\pi$ instead of $\pi$. These curvature insertions correspond exactly to insertions of the gaugino vertex operator at zero momentum, as sketched in fig. 4.

Since all insertions are at zero-momentum, the topological string amplitude is computing a non-derivative term in the effective action—the effective superpotential. The topological twisting produced by the insertions of these vertex operators induces a complete cancellation of contributions of the four-dimensional bosonic and fermionic fields.
The twist gives them the same worldsheet spins and therefore they cancel completely in the worldsheet path-integral—including the bosonic zero-modes, so there is no four-dimensional momentum flowing through the loops.

The combination of two of these insertions at each end point together with the sum over Chan-Patton indices gives exactly the required factor of a single $\text{Tr} \lambda_i^2 = S_i$ insertion for each hole in the string worldsheet. Zero-mode analysis gives that only for planar diagrams such an amplitude with only gaugino insertions can be non-vanishing. Moreover, each hole gets two gaugino fields except for one boundary hole. This is necessary to have two unabsorbed fermionic zero-modes, since we want a space-time term of the form $\int d^2 \theta W_{\text{eff}}(S)$.

So from this one concludes that the perturbative piece to the superpotential is given exactly by a series of the form (we work for convenience with one set of Chan-Paton indices of rank $N$)

$$W_{\text{pert}}(S) = \sum_{h>0} Nh \mathcal{F}_{0,h} S^{h-1} = N \frac{\partial \mathcal{F}_0(S)}{\partial S},$$

where $\mathcal{F}_{0,h}$ is the open topological string amplitude with $h$ holes. The factor $Nh$ arises from choosing a hole for which no gaugino $\lambda_\alpha$ is inserted and the factor $N$ represents the corresponding contribution from the Chan-Paton factor [9], see fig. 5. This combinatoric factor gives rise to the $N\partial/\partial S$ structure seen in the above formula.

Zero-modes analysis allows for another configuration. One can also distribute one of the gaugino fields from the inner holes to the outer hole. In other words, two holes will
have a single gluino vertex operator and the rest of the holes will each have two. These diagrams will compute the exact coupling constant of the low-energy $U(1)$ gauge fields. If we put a single gluino $\lambda_\alpha$ (instead of two) at two of the loops, the Chan-Paton factor does not kill it and this gives rise to a correction for the coupling constant of the corresponding $U(1)$, leading to the formula given before:

$$\tau_{ij} = \frac{\partial^2 F_0(S)}{\partial S_i \partial S_j}.$$

The typical diagram contributing to this gauge coupling is sketched in fig. 6.

Higher genus surfaces require bulk, i.e. gravitational, vertex operators (such as a graviphoton vertex) and thus do not contribute to the glueball superpotential or the coupling constant of the $U(1)$’s.

Within type IIB string theory there are no non-trivial worldsheet instantons or other $\alpha'$-effects that correct these result, and one can therefore take the $\alpha' \to 0$ (infinite tension) limit where all excited string modes decouple and the string diagrams reduce to ordinary “fat” Feynman diagrams. This gives the proof of the above conjecture for all the cases that can be engineered by type IIB string theory. However, it also looks feasible to reproduce the powerful stringy zero-mode arguments directly in four-dimensional field theory, thus generalizing the proof of our conjecture to cases beyond those realized in string theory. Let us sketch a possible approach to such a field theory argument.
3.2. Sketch of a field theoretic argument

Within field theory one basically considers what kind of path-integral configurations can contribute to superpotential terms. These come from configurations which preserve two of the four supercharges (giving rise to a $d^2 \theta$ integral). In this way the four-dimensional supersymmetric gauge theory path-integral automatically localizes to configurations where the $\Phi$ field is constant (by the usual argument that the anti-chiral supersymmetry variations of the fermionic fields gives total derivative terms $\partial_\mu \Phi$, which must thus be zero). Thus the computation reduces directly to an action of the form $\int d^2 \theta W_{\text{tree}}(\Phi)$. Let us for simplicity assume that

$$W_{\text{tree}}(\Phi) = \frac{1}{2} m \Phi^2 + \frac{1}{3} g \Phi^3.$$ 

To find the perturbative corrections to the Veneziano-Yancielowicz superpotential one could now argue as follows. Since by assumption we are in a massive vacuum, and since
we are computing a $d^2\theta$ term, we can work entirely in the zero-momentum sector. This allows one to completely discard the D-term and only work with the chiral F-term. From this point of view the propagator for the chiral superfield $\Phi(\theta)$ is coming directly from the quadratic part in the superpotential

$$\int d^2\theta \frac{1}{2} m^2 \Phi^2.$$ 

This is unconventional, since usually the kinetic terms come from integrating out the auxiliary fields using the D-term and couple $\Phi$ to $\bar{\Phi}$. Similarly the interacting vertices are coming from the higher order terms in $W_{\text{tree}}(\Phi)$. Working in a supercoordinate basis, we have the propagator

$$\langle \Phi(\theta)\Phi(\theta') \rangle = \frac{1}{m}(\theta - \theta')^2$$

with $\theta^2 \equiv \epsilon_{\alpha\beta} \theta^\alpha \theta^\beta$. Each vertex gives a factor $g \int d^2\theta$ which removes two $\theta$’s. A simple counting of factors of $\theta$ for a diagram with $V$ vertices and $P$ propagators gives a net factor of $n = 2(P - V)$ fermionic zero-modes. For a diagram with $h$ index loops and with topological genus $g$ we have $n = 2(h + 2g - 2)$.

We now claim that in a gaugino condensate background these extra $\theta$’s can be absorbed, but only for planar diagrams. If we naively couple to a background gaugino field $\lambda_{\alpha}(\theta)$ we have additional vertices of the form $\lambda_{\alpha}(\theta) \partial / \partial \theta_{\alpha}$ and these can absorb extra $\theta$’s. The insertion of the single-trace glueball field $S(\theta) = \text{Tr} \lambda_{\alpha}^2$ corresponds to two of these $\lambda_{\alpha}$ vertices, and by the index structure such an insertion is associated to a single index loop. So in this way we can absorb at most $2h$ zero-modes. Therefore only in the case of planar diagrams can insertions of $S(\theta)$’s completely remove the fermionic zero-modes. If one inserts in a planar diagram an $S(\theta)$ for all loops except one, one removes exactly all the remaining $\theta$’s from the propagators, leaving a final overall $\int d^2\theta$ integral, giving the required structure for a term of the form $\int d^2\theta W_{\text{eff}}(S)$.

It would be interesting to see whether this argument can be made into a more precise derivation; this is currently under investigation. As for the open string diagrams with the topology of $\mathbb{RP}^2$ that arise in the context of orientifolds, the arguments of [6] can be extended as discussed in [35,36] leading to the contribution $G_0(S)$ given above.
3.3. Gravitational couplings

One would wonder what the non-planar diagrams are good for? Again string theory provides the answer and thus extends our general conjecture to include certain gravitational correction. To obtain the topological amplitudes from ordinary superstrings in the case of non-planar diagrams we need to insert spin operators in the bulk of the worldsheet, which correspond to closed string or gravitational vertex operators. A completely similar analysis gives that for a worldsheet of genus \( g > 0 \), we obtain a term in the four-dimensional effective action of the form [9]:

\[
\int d^4x \int d^2\theta \sum_{h \geq 1} Nh \mathcal{F}_{g,h} S^{h-1}(W^2)^g + \int d^4x \int d^4\theta \sum_{h \geq 1} \mathcal{F}_{g,h} S^h W^{2g}.
\]

Here \( \mathcal{F}_{g,h} \) indicates a contribution of a worldsheet with \( g \) handles and \( h \) holes; \( W_{\alpha\beta} \) is the gravitational Weyl multiplet in \( \mathcal{N} = 2 \) supergravity. The lowest component of this chiral field is the self-dual part graviphoton field strength \( F_+ \), the highest component is the self-dual part of the Riemann tensor \( R_+ \).

Expanding the superspace integral in components for genus \( g = 1 \) we get a term that measures the response of coupling the \( \mathcal{N} = 1 \) gauge theory to a non-trivial four-dimensional gravitational background by the effective action

\[
\int d^4x \mathcal{F}_1(S) \text{Tr} R_+^2.
\]  

(3.1)

A typical worldsheet diagram contributing to this term is given in fig. 7.

More precisely, including the normalizations, we get a term

\[
\frac{1}{2} \mathcal{F}_1(S)(\chi - \frac{3}{2} \sigma),
\]

where \( \chi \) and \( \sigma \) denote the Euler characteristic and signature of the four manifold respectively. Here the coefficient \( \mathcal{F}_1(S) \) is given as a sum over all genus one diagrams with an arbitrary number of holes

\[
\mathcal{F}_1(S) = \sum_{h \geq 0} \mathcal{F}_{1,h} S^h.
\]

In particular we find from the planar limit the expectation value of the gaugino condensates \( S_i \) and plug it into this formula to find the gravitational corrections. It was suggested in [9] that the higher genus corrections for \( g > 1 \) in addition measure the response of the gauge
theory to a non-commutativity in spacetime represented by the (self-dual) graviphoton field strength $F_+$. This idea has found an interesting application in the recent computation of the instanton effects in the $\mathcal{N} = 2$ Yang-Mills theory [37].

If there is a dual Calabi-Yau geometry with moduli $S_i$ that captures the large $N$ limit, $\mathcal{F}(S)$ is given by the analytic torsion on that manifold [6]. If the geometry essentially reduces to a Riemann surface $\Sigma$ given by an algebraic curve

$$F(x, y) = 0,$$

as was the case for the perturbed $\mathcal{N} = 2$ theories in [18] and the quiver cases of [19], there are general arguments (basically from the fact that the full partition function is

---

**Fig. 7:** A genus one worldsheet diagram that contributes to the gravitational coupling $\mathcal{F}_1(S)W^2$. Note that there are now two bulk insertions of graviton vertex operators $W$, corresponding to the insertion of the extra handle compared to the planar diagrams.
given by a tau-function of an integrable hierarchy) that $\mathcal{F}_1$ is given by the logarithm of the determinant of a chiral boson on the Riemann surface $\Sigma$ (see e.g. [38])

$$\mathcal{F}_1(S) = -\frac{1}{2} \log \det \bar{\partial}_\Sigma. \quad (3.2)$$

We will make use of this result when we consider the $\mathcal{N} = 4$ theory later in this paper.

4. Perturbative derivation of dualities

In this paper we have proposed a general method to compute IR effects including exact instanton contributions to F-terms for a large class of $\mathcal{N} = 1$ theories. It is natural to ask if this perturbative perspective carries non-perturbative information such as $S$-dualities. Indeed we will point out how Seiberg-like dualities, are derivable from this perspective. It is also natural to ask if we can derive non-trivial exact results for theories with more supersymmetry, such as $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories. The answer to this also turns out to be positive.

For example, consider the $\mathcal{N} = 4$ theories deformed by mass terms for the adjoint fields. The deformation is known as the $\mathcal{N} = 1^*$ theory and was introduced in [39] to analyze topologically twisted $\mathcal{N} = 4$ theories on 4-manifolds. Here we will be able to make contact with the results in [39] from our direct perturbative analysis. Moreover the value of the superpotential has been studied for these theories in [40] and we will derive this result summing the planar diagrams below. Quite remarkably we will derive in this way, using only perturbative techniques, the modular properties of the superpotential.

We can also ask whether we can recover the exact results of Seiberg-Witten geometry for $\mathcal{N} = 2$ theories. This also turns out to be possible. We thus see that our perturbative methods are not only rather powerful, but we have also a derivation of these non-trivial conjectured dualities from first principles.

In this section we first briefly indicate how Seiberg-like dualities can be derived in this setup. We next discuss how the exact $\mathcal{N} = 2$ results are recovered. Then we turn to the main topic of this section, the $\mathcal{N} = 4$ theories and their deformations.
4.1. Seiberg-like dualities

Consider for definiteness the A-D-E $U(N_i)$ quiver theories with bi-fundamental matter and with adjoint matter with some superpotential for each adjoint field. Then, as shown in [14] Seiberg-like dualities correspond to Weyl reflections of the nodes of the Dynkin diagram. Moreover, it was shown in [14] that these dualities are manifest in the superpotential for the glueball fields. Basically, the glueball field superpotential is insensitive to Weyl reflections, up to appropriate field redefinition. As we have discussed here and in [19], the corresponding A-D-E matrix models at the planar limit compute the glueball superfield superpotential, and the large $N$ solution reproduces the geometry of [14]. It follows that Seiberg-like dualities are visible in the perturbative gauge theory analysis of the glueball superpotential which reduce to planar diagrams of the A-D-E matrix models.

4.2. The $\mathcal{N} = 2$ Seiberg-Witten solution

For simplicity let us consider the case of pure $\mathcal{N} = 2$ supersymmetric $U(N)$ Yang-Mills, though the statement can be easily generalized to quiver type theories. Consider deforming the theory by a superpotential $W(\Phi)$ which is a polynomial of degree $N+1$ in the adjoint field $\Phi$. In particular we have

$$W'(\Phi) = \epsilon P_N(\Phi) = \epsilon \prod_{i=1}^{N}(\Phi - a_i).$$

Choosing the branch where each eigenvalue of $\Phi$ is equal to one of the roots $a_i$, we can freeze to an arbitrary point on the Coulomb branch given by $a_i$. The $U(N)$ gauge group breaks to $U(1)^N$ for this branch. Then, as shown in [17], extremization of the glueball superpotential, which as we have argued is perturbatively computable by summing planar diagrams, leads to the special geometry on the curve

$$y^2 = P_N(x)^2 - \Lambda^{2N}$$

which is in exact agreement with the Seiberg-Witten curve for this case [2,27,28]. We stress once more that in our approach this geometry appears directly out of the planar limit of the perturbative diagrams—no non-perturbative duality is invoked. Moreover the coupling constant for the $U(1)^N$ theory is given by

$$\tau_{ij} = \frac{\partial^2 F_0}{\partial S_i \partial S_j}.$$
which just gives the period matrix for this geometry. Note in particular that this result is independent of $\epsilon$ and taking $\epsilon \to 0$ reduces the problem to that of unbroken $\mathcal{N} = 2$ gauge theory, in agreement with the predictions of the SW geometry.

Similarly one could ask about the computation of the BPS masses $a, a_D$. In this case it was found in [17] that there is a reduced 1-form $h = P_N'(x)dx/y$ (corresponding to the $H$-flux in the underlying Calabi-Yau geometry) which plays the role of the smeared density of the eigenvalues of the $\Phi$ field (which is the source of $H$-flux). In particular it has normalized periods $\oint_{A_i} h = 1$. Thus the average value of $\Phi$ which at the $i$-th vacuum is classically given by $a_i$ gets replaced by the quantum expression

$$a_i^{\text{quantum}} = \langle x \rangle = \oint_{A_i} x h$$

and as shown in [17] this exactly corresponds to the periods of the SW differential.

4.3. The $\mathcal{N} = 4$ theory and $S$-duality

A particular interesting case to apply our general philosophy is the $\mathcal{N} = 4$ super-Yang-Mills theory that is well-known to have a strong-weak coupling Montonen-Olive $S$-duality [1]. Some highly non-trivial aspects of this duality were checked in [39]. By now this $S$-duality is generally accepted and it has become the foundation on which many other dualities, both in field theory and string theory, rest. Can non-trivial consequences of this non-perturbative duality be seen using our perturbative large $N$ techniques? The answer is, surprisingly, yes.

The $\mathcal{N} = 4$ $U(N)$ gauge theory can be considered as a $\mathcal{N} = 1$ gauge theory with three adjoint chiral superfields $\Phi_1, \Phi_2, \Phi_3$ with a tree-level superpotential $\text{Tr}(\Phi_1[\Phi_2, \Phi_3])$. We will first consider breaking this theory softly down to $\mathcal{N} = 1$ by introducing masses for all the adjoints, i.e. working with the tree-level superpotential

$$W_{\text{tree}}(\Phi) = \text{Tr}(\Phi_1[\Phi_2, \Phi_3] + \sum_i m\Phi_i^2).$$

This is the so-called $\mathcal{N} = 1^*$ theory. This deformation was introduced in [39] to compute the topologically twisted $\mathcal{N} = 4$ theory on some manifolds including $K3$. In the context of AdS/CFT dualities this deformation was studied in [41].
According to our philosophy the effective superpotential of this model should be captured by the planar diagrams of the following three-matrix model, expanded around the relevant vacuum

$$\int d\Phi_1 d\Phi_2 d\Phi_3 \exp \text{Tr}(\Phi_1[\Phi_2, \Phi_3] + \sum_{i=1}^{3} m\Phi_i^2).$$ (4.1)

Remarkably this model has been studied—and solved—in [42] (building, as we understand, on earlier work in [43]) where it was considered in relation with the matrix models of M-theory. We will review the method of that paper, and present our own interpretation of the main result. In the process we will present the solution in a language that is more closely tailored to our needs.

For simplicity we will consider only the classical vacua given by the trivial configuration

$$\Phi_i = 0.$$

Here we have perturbatively still the full $U(N)$ gauge symmetry, but quantum effects will produce a mass gap and $N$ confining vacua. This choice of perturbative vacuum correspond to the usual small fluctuations saddle-point approximation of (4.1). The main fact that allows one to solve the matrix model is that, with this choice of vacuum, one can consider the action to be quadratic in the matrices $\Phi_2, \Phi_3$ and integrate them out. Indeed going to a basis $\Phi_\pm = \Phi_2 \pm i\Phi_3$, we can write the action as

$$W_{\text{tree}}(\Phi) = \text{Tr}(i\Phi_+ [\Phi_1, \Phi_-] + m\Phi_+ \Phi_- + m\Phi_1^2).$$

Therefore integrating out $\Phi_\pm$ as in a gaussian integral gives an extra determinant in the effective action for the remaining adjoint $\Phi = \Phi_1$. The matrix integral reduces in this way to the following holomorphic one-matrix model

$$\int d\Phi \frac{e^{m\text{Tr}\Phi^2}}{\det([\Phi, -] + im)}.$$

(Clearly the derivation this far would have worked for any potential $W(\Phi)$, not necessarily quadratic, for the first adjoint. We leave this generalization for further study.) The induced measure looks dangerously complicated but we can still go to an eigenvalue basis where it reduces to

$$\int \prod_I d\lambda_I \prod_{I < J} \frac{(\lambda_I - \lambda_J)^2}{(\lambda_I - \lambda_J + im)(\lambda_I - \lambda_J - im)} \exp \sum_I m\lambda_I^2.$$ (4.2)
This result has an obvious interpretation as a gas of $N$ eigenvalues $\lambda_I$ of charge +2 in a potential $W(x) = x^2$, interacting not only through the usual Coulomb potential with each other (as expressed by the numerator), but also with two charge $-1$ mirror images shifted by $\pm im$ in the complex plane (as indicated by the denominator). Note however that there are no mutual interaction among these two mirror images.

As a side remark we point out that, if we put the mass deformation for $\Phi_2$ and $\Phi_3$ to zero, which in field theory language corresponds to a flow to a non-trivial conformal fixed point [44], the total measures in (4.2) becomes completely trivial. This is a reflection of a more general fact that four-dimensional CFT’s correspond to net zero-charge Coulomb gas models.

4.4. The large $N$ solution

After rescaling the eigenvalues $\lambda_I \to m \lambda_I$ and writing $g_s = 1/m^3$ (a complex parameter) the equation of motion reads

$$2\lambda_I = g_s \sum_{J \neq I} \left\{ \frac{2}{\lambda_I - \lambda_J} - \frac{1}{\lambda_I - \lambda_J + i} - \frac{1}{\lambda_I - \lambda_J - i} \right\}.$$ 

The only parameter in the large $N$ limit will be the ’t Hooft coupling

$$S = g_s N.$$ 

The solution proceeds, as always, through study of the resolvent

$$\omega(x) = \frac{1}{N} \sum_I \frac{1}{x - \lambda_I}.$$ 

The general dynamics of large $N$ matrix models tells us that in this limit the eigenvalues will spread out from their classical locus $\lambda_I = 0$ at the minimum of the potential well, into (in this case) a single cut $(-a, a)$ along the real case. The size $a$ of the cut is determined by the ’t Hooft coupling, and is the only parameter in the solution. The continuous eigenvalue density $\rho(x) = \frac{1}{N} \sum_I \delta(x - \lambda_I)$ is given by the jump in the resolvent

$$\rho(x) = -\frac{1}{2\pi i} \left[ \omega(x + i\epsilon) - \omega(x - i\epsilon) \right].$$ 

The force $f(x)$ on a probe eigenvalue $\lambda_I = x$ in the complex plane takes the form

$$f(x) = 2x - S \left[ 2\omega(x) - \omega(x + i) - \omega(x - i) \right].$$
It is zero by the equation of motion along the cut. Following [42] we further introduce the function

\[ G(x) = x^2 + iS \left[ \omega(x + \frac{i}{2}) - \omega(x - \frac{i}{2}) \right], \]  

that is related to the force \( f(x) \) through the relation

\[ f(x) = -i \left[ G(x + \frac{i}{2}) - G(x - \frac{i}{2}) \right]. \]  

Note that on the cut \( x \in (-a, a) \) we have

\[ G(x + \frac{i}{2}) = G(x - \frac{i}{2}). \]

One furthermore has the relations \( G(-x) = G(x) \) and (for real \( S \)) \( G(\pi) = \overline{G(x)} \), so that \( G(x) \) is completely determined by its values in one quadrant of the \( x \)-plane.

4.5. The elliptic curve

The above relations actually imply somewhat surprisingly that \( G(x) \) is defined on an elliptic curve [42]. It is this elliptic curve, that emerges as the master field in the large \( N \) limit, that in the end will imply the modularity of the superpotential and the \( S \)-duality of the underlying \( \mathcal{N} = 4 \) theory.

This elliptic curve is determined by the behavior of \( G(x) \) on the \( x \)-plane. The function has discontinuities along the two cuts

\[ \mathcal{C}_+ = (-a + \frac{i}{2}, a + \frac{i}{2}), \quad \mathcal{C}_- = (-a + \frac{i}{2}, a + \frac{i}{2}). \]

These cuts are simply the two reflections of the original cut on which the eigenvalues condense. We can now think of the complex \( x \)-plane, compactified by adding the point at infinity, as a two-torus, by cutting the two cuts \( \mathcal{C}_+ \) and \( \mathcal{C}_- \) open and then gluing them together. In this gluing we identify the upper-half of \( \mathcal{C}_+ \) with the lower-half of \( \mathcal{C}_- \) and \textit{vice versa}. So, pictorially we make two incisions in the Riemann sphere and glue in a handle connecting the cuts to make a genus one Riemann surface, see fig. 8.

To work with this elliptic curve in a more traditional representation we follow [42] and observe that the function \( t = G(x) \) gives the \( x \)-plane as a branched double cover of the \( t \)-plane. More precisely \( G(x) \) maps the upper quadrant in the \( x \)-plane (folded around half of the cut \( \mathcal{C}_+ \)) to the upper-half-plane in \( t \). This can be seen by inverting \( G(x) = t \) and write

\[ x = G^{-1}(t) = A \int_{x_1}^{t} \frac{(t - x_3) \, dt}{\sqrt{(t - x_1)(t - x_2)(t - x_4)}} \]
Fig. 8: The emergence of the elliptic curve from the geometry of the matrix model.

for appropriate constants $A, x_1, x_2, x_3, x_4$. Here the branch points $t = x_1, x_2, x_4$ get mapped to $x = 0, \frac{i}{2} - i\epsilon, \frac{i}{2} + i\epsilon$, and the regular point $t = x_3$ gets mapped to $x = a + \frac{i}{2}$, the end point of the eigenvalue cut. All the constants are in principle completely fixed by the geometry, that is, the size $a$ of the cut in the $x$-plane (and thus indirectly by the 't Hooft coupling $S$). In particular one finds the relation $x_1 + x_2 + x_4 = 2x_3$. Also, from the large $x$ behaviour $t \sim x^2$ one derives $A = \frac{1}{2}$.

It will be convenient to shift $t \to t - 2c$ with $c = x_3/3$, so that we can write the relation between $x$ and $t$ as

$$x = \int_{t-2c}^{t-c} \frac{(t-c)dt}{y}. \quad (4.5)$$

Here $y(t)$ is given by the canonical Weierstrass form of the elliptic curve as branched over three point in the $t$-plane

$$y^2 = 4t^3 - g_2t - g_3.$$
Fig. 9: The $x$-plane is a double cover of the $t$-plane, branched at $t = x_1, x_2, x_4$. Note the identifications of the two cuts in the $x$-plane. The images of the canonical $A$ and $B$ cycles are indicated.

We recall that the conventional double periodic coordinate $z$ on the elliptic curve, that satisfies $z \sim z + \omega_1$ and $z \sim z + \omega_2$ with modulus $\omega_2/\omega_1 = \tau$, is related to the variables $t$ and $y$ through the Weierstrass $\wp$-function and its derivative

$$t = \wp(z), \quad y = \wp'(z), \quad \frac{dt}{y} = dz.$$

The canonical homology cycles $A$ and $B$ on the torus can now be described explicitly in the coordinates $x, t, z$ (see fig. 9). The $A$-cycle encircles clockwise the cut $C_+$ in the
x-plane (or, after a deformation, counterclockwise the cut \( C_- \)). In the \( t \)-plane it encircles the branch points \( x_2 \) and \( x_4 \). Finally in terms of \( z \) it runs from \( z = 0 \) to \( z = \omega_1 \).

The dual \( B \)-cycle is represented in the \( x \)-plane as the path from \( a - \frac{i}{2} \) to \( a + \frac{i}{2} \), i.e. as a path that connects the cut \( C_- \) to \( C_+ \). In the \( t \)-plane it encircles the branch points \( x_1 \) and \( x_2 \). It is also the path from \( z = 0 \) to \( z = \omega_2 = \tau \omega_1 \).

4.6. Periods and the superpotential

Let us now consider what quantities we have to compute to present the large \( N \) solution of the matrix model. First we claim that period of the one-form \( G(x)dx \) along the \( A \)-cycle measures the total number of eigenvalues, or more precisely

\[
\frac{1}{2\pi} \oint_A G(x)dx = g_s N = S.
\]

This follows directly from the fact that the jump in \( G(x) \) along the cut \( C_+ \) (going upwards) is \( 2\pi S \rho(x) \).

Secondly, the derivative of the planar free energy \( F_0(S) \) is obtained by changing the filling number by taking a fraction of eigenvalues from infinity to the cut. This change is computed as

\[
\frac{\partial F_0}{\partial S} = \int_\infty^a f(x)dx,
\]

with \( f(x) \) the force on the eigenvalue. By using relation (4.4) this can be written as

\[
\frac{\partial F_0}{\partial S} = i \int_B G(x)dx.
\]

Originally, the contour along which \( G(x)dx \) has to be integrated starts at the lower cut \( C_- \), goes to infinity, and then returns back to the upper cut \( C_+ \). However, since \( G(x) \) has no poles at finite \( x \), this contour can be smoothly deformed to the \( B \)-cycle that runs directly from \( C_- \) to \( C_+ \).

In order to perform these period integrals explicitly, we recall the following standard results. For the \( A \)-cycles we have

\[
\oint_A \frac{dt}{y} = \oint_A dz = \omega_1,
\]

\[
\oint_A \frac{t}{y} dt = \oint_A \wp(z)dz = \eta_1 = \frac{\pi^2}{3} E_2(\tau)\omega_1^{-1},
\]

\[
\oint_A \frac{t^2}{y} dt = \oint_A \wp^2(z)dz = \frac{g_2}{12} \omega_1 = \frac{\pi^4}{9} E_4(\tau)\omega_1^{-3}.
\]
and for the $B$-cycles we find

$$
\oint_B \frac{dt}{y} = \oint_B dz = \omega_2 = \tau \omega_1,
$$

$$
\oint_B \frac{tdt}{y} = \oint_B \varphi(z) dz = \eta_2 = \left[ \frac{\pi^2}{3} E_2(\tau) \tau - 2\pi i \right] \omega_1^{-1},
$$

(4.7)

$$
\oint_B \frac{t^2 dt}{y} = \oint_B \varphi^2(z) dz = g_2 \omega_2 = \frac{\pi^4}{9} E_4(\tau) \tau \omega_1^{-3}.
$$

Here $E_2(\tau), E_4(\tau)$ are the standard Eisenstein series, for example given by the $q$-series expansions ($q = e^{2\pi i \tau}$)

$$
E_2(\tau) = 1 - 24 \sum_{n} \frac{nq^n}{1 - q^n},
$$

$$
E_4(\tau) = 1 + 240 \sum_{n} \frac{n^3 q^n}{1 - q^n}.
$$

Returning to the large $N$ solution of our matrix model and in particular to the periods of the one-form $G(x)dx$, we recall the relations

$$
G(x) = t + 2c, \quad dx = \left( \frac{t - c}{y} \right) dt.
$$

We first compute the periods of the one-form $dx$. The period $\int_A dx = 0$ gives the relation

$$
c = \frac{\pi^2}{3} E_2 \omega_1^{-2}.
$$

The second period $\int_B dx = i$ fixes the gauge choice $\omega_1 = -1/2\pi$, so that $c = E_2/12$. Now we are finally in a position to compute the periods of meromorphic one-form $G(x)dx$. We find

$$
2\pi i S = \Pi_A, \quad \frac{\partial F_0}{\partial S} = \Pi_B,
$$

with the periods

$$
\Pi_A = i \int_A G(x) dx = i \int_A \frac{(t^2 + ct - 2c^2)}{y} dt = \frac{i\pi}{72} (E_2^2 - E_4),
$$

and

$$
\Pi_B = i \int_B G(x) dx = \frac{i\pi}{72} \tau (E_2^2 - E_4) - \frac{1}{12} E_2.
$$

33
Note the important relation between the two periods
\[ \Pi_B = \tau \Pi_A - \frac{1}{12} E_2(\tau). \]

The effective superpotential is obtained by inserting these periods in the expression
\[ \text{W}_{\text{eff}}(S) = N \frac{\partial F_0}{\partial S} - 2\pi i \tau_0 S = N \Pi_B - \tau_0 \Pi_A, \]
with \( \tau_0 \) the bare coupling. Now we have to extremize the superpotential
\[ \delta \text{W}_{\text{eff}} = N \delta \Pi_B - \tau_0 \delta \Pi_A = 0. \tag{4.8} \]

But the variations of the periods satisfy
\[ \delta \Pi_B = \tau \delta \Pi_A, \tag{4.9} \]
with \( \tau \) the modulus of the elliptic curve, i.e. we have a Seiberg-Witten-like geometry. This can be explicitly checked by using the relation
\[ \delta E_2(\tau) = \frac{i \pi}{6} \left( E_2^2 - E_4 \right) \delta \tau. \]

Plugging (4.9) back in (4.8) we find that
\[ \tau = (\tau_0 + k)/N, \quad k = 0, 1, \ldots, N - 1. \tag{4.10} \]

Here the \( \mathbb{Z}_N \) quantum number \( k \) distinguishes the different confining vacua. So, through the extremization procedure of the superpotential the modulus \( \tau \) of the elliptic curve that appeared as the master field for the matrix model is identified with \( 1/N \) times the bare coupling of the gauge theory! The natural geometric action of the modular group \( SL(2, \mathbb{Z}) \) on \( \tau \) in this way gives precisely the right action predicted by \( S \)-duality on the coupling \( \tau_0 \).

At these critical points the value of the superpotential is given by
\[ \text{W}_{\text{eff}} = N(\Pi_B - \tau \Pi_A) \]
with \( \tau \) given by (4.10). Inserting the values of the periods \( \Pi_A \) and \( \Pi_B \), and introducing again the scale \( m \) in the problem by the factor \( 1/g_s = m^3 \) (and matching the overall normalization to the \( \mathcal{N} = 1 \) result giving an extra factor \( \frac{1}{2} \)), we get the final result
\[ \text{W}_{\text{eff}} = -\frac{Nm^3}{24} E_2(\frac{\tau_0 + k}{N}). \]
in exact agreement with the results of [40,45,46] (up to an overall additive constant, but recall that only the differences $ΔW_{\text{eff}}$ between the vacua are physical). In particular differentiating with respect to $m$ one gets the one-point function

$$\langle \text{Tr}\Phi^2 \rangle = -\frac{Nm^2}{24}E_2(\tau_0 + k/N).$$

Similarly one can compute the value of the condensate

$$\langle S \rangle = \frac{m^3}{4\pi i}\Pi_A = \frac{m^3}{288}(E_2^2 - E_4) = -\frac{1}{2\pi i} \frac{∂}{∂τ_0} W_{\text{eff}}.$$

There are various generalizations of this result that we are currently investigating. For example, we can also consider the Leigh-Strassler deformation [47] of $N = 1^*$ given by

$$W(\Phi) = \text{Tr}(\Phi_1[\Phi_2, \Phi_3]_β + \sum_i m\Phi_i^2)$$

where

$$[\Phi_2, \Phi_3]_β = e^{iπβ}\Phi_2\Phi_3 - \Phi_3\Phi_2 e^{-iπβ}.$$ (The relation of this to non-commutativity in spacetime has been noted in [48].)

The corresponding matrix model with action $W(\Phi)$ given above corresponds to a matrix realization of the six-vertex model on random surfaces [49] and has been recently solved in the planar limit in [50]. Again an elliptic curve features in the solution, as expected from Montonen-Olive duality. We can in principal repeat the analysis we did above for the case at hand, which we shall not undertake in this paper.

## 4.7. $\mathcal{N} = 4$ Yang-Mills on $K3$

The partition function of $\mathcal{N} = 4$ topologically twisted Yang-Mills theory was studied on various four manifolds in [39] in order to check the Montonen-Olive conjecture, which predicts that the instanton sum should give rise to a modular form. Here we have considered the *untwisted* $\mathcal{N} = 4$ Yang-Mills theory. However, as noted in [39] on $K3$ the topological twisting is trivial and the topological theory is equivalent to the untwisted theory. For simplicity let us only consider this case, though it is likely that our result applies to all the Kahler manifolds with $b_2^+ > 1$ studied in [39].

As argued in [39] all one needs to compute is the correction to the $\mathcal{N} = 4$ Yang-Mills action proportional to $f(τ_0)\text{Tr} R_+^2$ (more precisely, including the normalizations, the term...
\[ \frac{1}{2} f(\tau_0)(\chi - \frac{3}{2} \sigma) \] needs to be computed. Moreover for the \( SU(2) \) case and for the vacuum with \( \Phi_i = 0 \) it was found that

\[ f(\tau_0) = - \log \eta(\tau_0/2) \]

As we have argued the same coupling can be determined by summing the genus one diagrams of the corresponding matrix model. We have found from (3.2) that

\[ f(\tau_0) = F_1(S) = - \log \eta(\tau), \]

as the determinant of the chiral boson on the torus is given by the \( \eta \)-function. Using the identification \( \tau = \tau_0/N \) and substituting \( N = 2 \) we reproduce the result of [39]. So in this way one can try to produce very non-trivial instanton sums by perturbative means. It would be very interesting to reproduce the results considered in [39] using this approach.

5. Generalizations and conclusions

Let us summarize the main conclusions of this paper. We have seen that chiral \( \int d^2 \theta \) contributions to the effective action of a \( \mathcal{N} = 1 \) supersymmetric gauge theory are computed exactly by summing diagrams of a given genus. In particular the superpotential is given by the sum of planar diagrams. Similarly the induced \( R^2 \)-coupling to gravity are given by genus one diagrams. Furthermore these diagrams are computed only using the constant modes and are thus captured by a generalized matrix model. The exact large \( N \) solution of the matrix model, in case such a solution can be found, will give an effective Calabi-Yau three-fold geometry. This Calabi-Yau geometry plays the role of the large \( N \) master field. In this way perturbative expansions can be directly converted to non-perturbative, fractional instanton expansions.

Many dualities, such as Seiberg-like dualities and Montonen-Olive duality can thus be seen from the planar limit of perturbation theory. Also Seiberg-Witten geometry can be derived from a purely perturbative perspective. It is interesting to note that the cases where there is an expected \( S \)-duality from field theory seem to correlate with the exact solvability of the matrix model, where we can sum up all the planar diagrams. In fact the exact sum of the planar diagrams is needed to see the structure of the \( S \)-duality group. For example in the \( \mathcal{N} = 1^* \) case modular transformations act on the \( A \) and \( B \) periods and therefore act essentially as a Legendre transformation on the superpotential, mixing
up the diagrams. However, even if the matrix model is not exactly solvable, matrix model perturbation theory still yields a systematic method to compute instanton corrections to F-terms in $\mathcal{N} = 1$ gauge theories.

Our results lead to many questions for further research. It would be interesting to study the dimensional reduction to three dimensions. Could it be that in this case we have to deal with matrix quantum mechanics; does compactification introduce extra dimensions in the matrix model, just as in matrix theory?

We have seen that perturbing theories to a massive vacuum, and then using localization techniques in perturbation theory, is a powerful way to probe the non-perturbative properties of the field theory. It would be interesting to apply this philosophy to more general problems, for example directly in superstring theory. In particular, could it be that for a certain class of problems string perturbation theory has enough information to yield non-perturbative results?

Finally this work might support a long-standing hope, that also in non-supersymmetric field theories, where we lack the power of holomorphy, resummations of certain classes of relevant perturbative diagrams might capture the essential dynamics of the gauge theory.

Acknowledgements

We would like to thank M. Aganagic, N. Arkani-Hamed, T. Banks, D. Berenstein, N. Berkovits, J. de Boer, F. Cachazo, N. Dorey, M. Douglas, D. Gross, M. Grisaru, S. Gukov, D. Kutasov, K. Intriligator, R. Leigh, J. Maldacena, G. Moore, N. Nekrasov, H. Ooguri, F. Quevedo, M. Rocek, N. Seiberg, A. Strominger, E. Verlinde, E. Witten and D. Zanon for useful discussions and the organizers of Strings 2002 and the Newton Institute in Cambridge, where part of this research was done, for providing a most stimulating research environment. R.D. also wishes to thank the participants of the 4th Amsterdam Summer Workshop for interesting comments. The research of R.D. is partly supported by FOM and the CMPA grant of the University of Amsterdam, C.V. is partly supported by NSF grants PHY-9802709 and DMS-0074329.

References


—, “Open/closed string dualities and Seiberg duality from geometric transitions in M-theory,” [arXiv:hep-th/0106040];


