Renormalizable Quantum Gauge Theory of Gravity

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Abstract

The quantum gravity is formulated based on gauge principle. The model discussed in this paper has local gravitational gauge symmetry and gravitational field is represented by gauge potential. A preliminary study on gravitational gauge group is presented. Path integral quantization of the theory is discussed in the paper. A strict proof on the renormalizability of the theory is also given. In leading order approximation, the gravitational gauge field theory gives out classical Newton’s theory of gravity. It can also give out an Einstein-like field equation with cosmological term. The prediction for cosmological constant given by this model is well consistent with experimental results. For classical tests, it gives out the same theoretical predictions as those of general relativity. Combining cosmological principle with the field equation of gravitational gauge field, we can also set up a cosmological model which is consistent with recent observations.

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1 Introduction

In 1686, Isaac Newton published his book *MATHEMATICAL PRINCIPLES OF NATURAL PHILOSOPHY*. In this book, through studying the motion of planet in solar system, he found that gravity obeys the inverse square law and the magnitude of gravity is proportional to the mass of the object[1]. The Newton’s classical theory of gravity is kept unchanged until 1916. At that year, Einstein proposed General Relativity[2, 3]. In this great work, he founded a relativistic theory on gravity, which is based on principle of general relativity and equivalence principle. Newton’s classical theory for gravity appears as a classical limit of general relativity.

One of the biggest revolution in human kind in the last century is the foundation of quantum theory. The quantum hypothesis was first introduced into physics by Max Plank in 1900. In 1916, Albert Einstein points out that quantum effects must lead to modifications in the theory of general relativity[4]. Soon after the foundation of quantum mechanics, physicists try to found a theory that could describe the quantum behavior of the full gravitational field. In the 70 years attempts, physicists have found two theories based on quantum mechanics that attempt to unify general relativity and quantum mechanics, one is canonical quantum gravity and another is superstring theory. But for quantum field theory, there are different kinds of mathematical infinities that naturally occur in quantum descriptions of fields. These infinities should be removed by the technique of perturbative renormalization. However, the perturbative renormalization does not work for the quantization of Einstein’s theory of gravity, especially in canonical quantum gravity. In superstring theory, in order to make perturbative renormalization to work, physicists have to introduce six extra dimensions. But up to now, none of the extra dimensions have been observed. To found a consistent theory that can unify general relativity and quantum mechanics is a long dream for physicists.

The ”relativity revolution” and the ”quantum revolution” are among the greatest successes of twentieth century physics, yet two theories appears to be fundamentally incompatible. General relativity remains a purely classical theory which describes the geometry of space and time as smooth and continuous, on the contrary, quantum mechanics divides everything into discrete quanta. The underlying theoretical incompatibility between two theories arises from the way that they treat the geometry of space and time. This situation makes some physicists still wonder whether quantum theory is a truly fundamental theory of Nature, or just a convenient description of some aspects of the microscopic world. Some physicists even consider the twentieth century as the century of the incomplete revolution. To set up a consistent quantum theory of gravity is considered to be the last challenge of quantum theory[5, 6]. In other words, combining general relativity with quantum mechanics
is considered to be the last hurdle to be overcome in the "quantum revolution".

In 1921, H. Weyl introduced the concept of gauge transformation into physics[7, 8], which is one of the most important concepts in modern physics, though his original theory is not successful. Later, V. Fock, H. Weyl and W. Pauli found that quantum electrodynamics is a gauge invariant theory[9, 10, 11]. In 1954, Yang and Mills proposed non-Abel gauge field theory[12]. This theory was soon applied to elementary particle physics. Unified electroweak theory[13, 14, 15] and quantum chromodynamics are all based on gauge field theory. The predictions of unified electroweak theory have been confirmed in a large number of experiments, and the intermediate gauge bosons $W^\pm$ and $Z^0$ which are predicted by unified electroweak model are also found in experiments. The $U(1)$ part of the unified electroweak model, quantum electrodynamics, now become one of the most accurate and best-tested theories of modern physics. All these achievements of gauge field theories suggest that gauge field theory is a fundamental theory that describes fundamental interactions. Now, it is generally believed that four kinds of fundamental interactions in Nature are all gauge interactions and they can be described by gauge field theory. From theoretical point of view, the principle of local gauge invariance plays a fundamental role in particle’s interaction theory.

Gauge treatment of gravity was suggested immediately after the gauge theory birth itself[16, 17, 18]. In the traditional gauge treatment of gravity, Lorentz group is localized, and the gravitational field is not represented by gauge potential[19, 20, 21]. It is represented by metric field. The theory has beautiful mathematical forms, but up to now, its renormalizability is not proved. In other words, it is conventionally considered to be non-renormalizable. There is also some other attempts to use Yang-Mills theory to reformulate quantum gravity[22, 23, 24, 25, 26]. In these new approaches, the importance of gauge fields is emphasized. Some physicists also try to use gauge potential to represent gravitational field, some suggest that we should pay more attention on translation group.

I will not talk too much on the history of quantum gravity and the incompatibilities between quantum mechanics and general relativity here. Materials on these subject can be widely found in literatures. Now we want to ask that, except for traditional canonical quantum gravity and superstring theory, whether exists another approach to set up a fundamental theory, in which general relativity and quantum mechanics are compatible.

In literatures [27, 28, 29], a new formulation of quantum gauge theory of gravity is proposed By N.Wu. In this new attempt, the quantum gravity is based on gauge principle, which is a renormalizable quantum theory. In literature [28], another
model, which uses different selections of $\eta_2$ and $J(C)$, is discussed. In this paper, we will give a systematic formulation of the model which is discussed in literature [28]. The path integral quantization of the theory and the proof the renormalizability of the theory are discussed. Because the selections of the Lagragian are different, the proof on the renormalizability of the theory is different from that of the model in literature [27].

As we have mentioned above, gauge field theory provides a fundamental tool to study fundamental interactions. In this paper, we will use this tool to study quantum gravity. We will use a completely new language to express the quantum theory of gravity. In order to do this, we first need to introduce some transcendental foundations of this new theory, which is the most important thing to formulate the whole theory. Then we will discuss a new kind of non-Abel gauge group, which will be the fundamental symmetry of quantum gravity. For the sake of simplicity, we call this group gravitational gauge group. After that, we will construct a Lagrangian which has local gravitational gauge symmetry. In this Lagrangian, gravitational field appears as the gauge field of the gravitational gauge symmetry. Then we will discuss the gravitational interactions between scalar field (or Dirac field or vector field ) and gravitational field. Just as what Albert Einstein had ever said in 1916 that quantum effects must lead to modifications in the theory of general relativity, there are indeed quantum modifications in this new quantum gauge theory of gravity. In other words, the local gravitational gauge symmetry requires some additional interaction terms other than those given by general relativity. This new quantum theory of gravity can even give out an exact relationship between gravitational fields and space-time metric in generally relativity. The classical limit of this new quantum theory will give out classical Newton’s theory of gravity and general relativity. In other words, the leading order approximation of the new theory gives out classical Newton’s theory of gravity. It can also give out the Einstein’s field equation with cosmological constant. A rough estimation shows that the theoretical expectation of cosmological constant is of the order of $10^{-52}m^{-2}$, which is well consistent with experimental results. For classical tests, it gives out the same theoretical predictions as those of general relativity. Then we will discuss quantization of gravitational gauge field. Something most important is that this new quantum theory of gravity is a renormalizable theory. A formal strict proof on the renormalizability of this new quantum theory of gravity is given in this paper. I hope that the effort made in this paper will be beneficial to our understanding on the quantum aspects of gravitational field. The relationship between this new quantum theory of gravity and traditional canonical quantum gravity or superstring theory is not study now, and I hope that this work will be done in the near future. Because the new quantum theory of gravity is logically independent of traditional quantum gravity, we need not discuss traditional quantum gravity first. Anyone who is familiar with tradi-
2 The Transcendental Foundations

It is known that action principle is one of the most important fundamental principle in quantum field theory. Action principle says that any quantum system is described by an action. The action of the system contains all interaction information, contains all information of the fundamental dynamics. The least of the action gives out the classical equation of motion of a field. Action principle is widely used in quantum field theory. We will accept it as one of the most fundamental principles in this new quantum theory of gravity. The rationality of action principle will not be discussed here, but it is well known that the rationality of the action principle has already been tested by a huge amount of experiments. However, this principle is not a special principle for quantum gravity, it is a ubiquitous principle in quantum field theory. Quantum gravity discussed in this paper is a kind of quantum field theory, it’s naturally to accept action principle as one of its fundamental principles.

We need a special fundamental principle to introduce quantum gravitational field, which should be the foundation of all kinds of fundamental interactions in Nature. This special principle is gauge principle. In order to introduce this important principle, let’s first study some fundamental laws in some fundamental interactions other than gravitational interactions. We know that, except for gravitational interactions, there are strong interactions, electromagnetic interactions and weak interactions, which are described by quantum chromodynamics, quantum electrodynamics and unified electroweak theory respectively. Let’s study these three fundamental interactions one by one.

Quantum electrodynamics (QED) is one of the most successful theory in physics which has been tested by most accurate experiments. Let’s study some logic in QED. It is known that QED theory has $U(1)$ gauge symmetry. According to Noether’s theorem, there is a conserved charge corresponding to the global $U(1)$ gauge transformations. This conserved charge is just the ordinary electric charge. On the other hand, in order to keep local $U(1)$ gauge symmetry of the system, we had to introduce a $U(1)$ gauge field, which transmits electromagnetic interactions. This $U(1)$ gauge field is just the well-know electromagnetic field. The electromagnetic interactions between charged particles and the dynamics of electromagnetic field are completely determined by the requirement of local $U(1)$ gauge symmetry. The source of this
electromagnetic field is just the conserved charge which is given by Noether’s theorem. After quantization of the field, this conserved charge becomes the generator of the quantum $U(1)$ gauge transformations. The quantum $U(1)$ gauge transformation has only one generator, it has no generator other than the quantum electric charge.

Quantum chromodynamics (QCD) is a prospective fundamental theory for strong interactions. QCD theory has $SU(3)$ gauge symmetry. The global $SU(3)$ gauge symmetry of the system gives out conserved charges of the theory, which are usually called color charges. The local $SU(3)$ gauge symmetry of the system requires introduction of a set of $SU(3)$ non-Abel gauge fields, and the dynamics of non-Abel gauge fields and the strong interactions between color charged particles and gauge fields are completely determined by the requirement of local $SU(3)$ gauge symmetry of the system. These $SU(3)$ non-Abel gauge fields are usually call gluon fields. The sources of gluon fields are color charges. After quantization, these color charges become generators of quantum $SU(3)$ gauge transformation. Something which is different from $U(1)$ Abel gauge symmetry is that gauge fields themselves carry color charges.

Unified electroweak model is the fundamental theory for electroweak interactions. Unified electroweak model is usually called the standard model. It has $SU(2)_L \times U(1)_Y$ symmetry. The global $SU(2)_L \times U(1)_Y$ gauge symmetry of the system also gives out conserved charges of the system. The requirement of local $SU(2)_L \times U(1)_Y$ gauge symmetry needs introducing a set of $SU(2)$ non-Abel gauge fields and one $U(1)$ Abel gauge field. These gauge fields transmit weak interactions and electromagnetic interactions, which correspond to intermediate gauge bosons $W^\pm$, $Z^0$ and photon. The sources of these gauge fields are just the conserved Noether charges. After quantization, these conserved charges become generators of quantum $SU(2)_L \times U(1)_Y$ gauge transformation.

QED, QCD and the standard model are three fundamental theories of three kinds of fundamental interactions. Now we want to summarize some fundamental laws of Nature on interactions. Let’s first ruminate over above discussions. Then we will find that our formulations on three different fundamental interaction theories are almost completely the same, that is the global gauge symmetry of the system gives out conserved Noether charges, in order to keep local gauge symmetry of the system, we have to introduce gauge field or a set of gauge fields, these gauge fields transmit interactions, and the source of these gauge fields are the conserved charges and these conserved Noether charges become generators of quantum gauge transformations after quantization. These will be the main content of gauge principle.

Before we formulate gauge principle formally, we need to study something more
on symmetry. It is known that not all symmetries can be localized, and not all symmetries can be regarded as gauge symmetries and have corresponding gauge fields. For example, time reversal symmetry, space reflection symmetry, \cdots are those kinds of symmetries. We can not find any gauge fields or interactions which correspond to these symmetries. It suggests that symmetries can be divided into two different classes in nature. Gauge symmetry is a special kind of symmetry which has the following properties: 1) it can be localized; 2) it has some conserved charges related to it; 3) it has a kind of interactions related to it; 4) it is usually a continuous symmetry. This symmetry can completely determine the dynamical behavior of a kind of fundamental interactions. For the sake of simplicity, we call this kind of symmetry dynamical symmetry or gauge symmetry. Any kind of fundamental interactions has a gauge symmetry corresponding to it. In QED, the $U(1)$ symmetry is a gauge symmetry, in QCD, the color $SU(3)$ symmetry is a gauge symmetry and in the standard model, the $SU(2)_L \times U(1)_Y$ symmetry is also a gauge symmetry. The gravitational gauge symmetry which we will discuss below is also a kind of gauge symmetry. The time reversal symmetry and space reflection symmetry are not gauge symmetries. Those global symmetries which can not be localized are not gauge symmetries either. Gauge symmetry is a fundamental concept for gauge principle.

Gauge principle can be formulated as follows: Any kind of fundamental interactions has a gauge symmetry corresponding to it; the gauge symmetry completely determines the forms of interactions. In principle, the gauge principle has the following four different contents:

1. **Conservation Law**: the global gauge symmetry gives out conserved current and conserved charge;

2. **Interactions**: the requirement of the local gauge symmetry requires introduction of gauge field or a set of gauge fields; the interactions between gauge fields and matter fields are completely determined by the requirement of local gauge symmetry; these gauge fields transmit the corresponding interactions;

3. **Source**: qualitative speaking, the conserved charge given by global gauge symmetry is the source of gauge field; for non-Abel gauge field, gauge field is also the source of itself;

4. **Quantum Transformation**: the conserved charges given by global gauge symmetry become generators of quantum gauge transformation after quantization, and for this kind of interactions, the quantum transformation can not have generators other than quantum conserved charges given by global gauge symmetry.

It is known that conservation law is the objective origin of gauge symmetry, so gauge symmetry is the exterior exhibition of the interior conservation law. The
conservation law is the law that exists in fundamental interactions, so fundamental interactions are the logic precondition and foundation of the conservation law. Gauge principle tells us how to study conservation law and fundamental interactions through symmetry. Gauge principle is one of the most important transcendental fundamental principles for all kinds of fundamental interactions in Nature; it reveals the common nature of all kinds of fundamental interactions in Nature. It is also the transcendental foundation of the quantum gravity which is formulated in this paper. It will help us to select the gauge symmetry for quantum gravitational theory and help us to determine the Lagrangian of the system. In a meaning, we can say that without gauge principle, we can not set up this new renormalizable quantum gauge theory of gravity.

Another transcendental principle that widely used in quantum field theory is the microscopic causality principle. The central idea of the causality principle is that any changes in the objective world have their causation. Quantum field theory is a relativistic theory. It is know that, in the special theory of relativity, the limit spread speed is the speed of light. It means that, in a definite reference system, the limit spread speed of the causation of some changes is the speed of light. Therefore, the special theory of relativity exclude the possibility of the existence of any kinds of non-local interactions in a fundamental theory. Quantum field theory inherits this basic idea and calls it the microscopic causality principle. There are several expressions of the microscopic causality principle in quantum field theory. One expression say that two events which happen at the same time but in different space position are two independent events. The mathematical formulation for microscopic causality principle is that

$$\{O_1(\vec{x}, t), O_2(\vec{y}, t)\} = 0, \quad (2.1)$$

when $\vec{x} \neq \vec{y}$. In the above relation, $O_1(\vec{x}, t)$ and $O_2(\vec{y}, t)$ are two different arbitrary local bosonic operators. Another important expression of the microscopic causality principle is that, in the Lagrangian of a fundamental theory, all operators appear in the same point of space-time. Gravitational interactions are a kind of physical interactions, the fundamental theory of gravity should also obey microscopic causality principle. This requirement is realized in the construction of the Lagrangian for gravity. We will require that all field operators in the Lagrangian should be at the same point of space-time.

Because quantum field theory is a kind of relativistic theory, it should obey some fundamental principles of the special theory of relativity, such as principle of special relativity and principle of invariance of light speed. These two principles conventionally exhibit themselves through Lorentz invariance. So, in constructing the Lagrangian of the quantum theory of gravity, we require that it should have
Lorentz invariance. This is also a transcendental requirement for the quantum theory of gravity. But what we treat here that is different from that of general relativity is that we do not localize Lorentz transformation. Because gauge principle forbids us to localize Lorentz transformation, asks us only to localize gravitational gauge transformation. We will discuss this topic in details later. However, it is important to remember that global Lorentz invariance of the Lagrangian is a fundamental requirement. The requirement of global Lorentz invariance can also be treated as a transcendental principle of the quantum theory of gravity.

It is well-known that two transcendental principles of general relativity are principle of general relativity and principle of equivalence. It should be stated that, in the new gauge theory of gravity, the principle of general relativity appears in another way, that is, it realized itself through local gravitational gauge symmetry. From mathematical point of view, the local gravitational gauge invariance is just the general covariance in general relativity. In the new quantum theory of gravity, principle of equivalence plays no role. In other words, we will not accept principle of equivalence as a transcendental principle of the new quantum theory of gravity, for gauge principle is enough for us to construct quantum theory of gravity. We will discuss something more about principle of equivalence later.

3 Gravitational Gauge Group

Before we start our mathematical formulation of gravitational gauge theory, we have to determine which group is the gravitational gauge group, which is the starting point of the whole theory. It is know that, in the traditional quantum gauge theory of gravity, Lorentz group is localized. We will not follow this way, for it contradicts with gauge principle. Now, we use gauge principle to determine which group is the exact group for gravitational gauge theory.

Some of the most important properties of gravity can be seen from Newton’s classical gravity. In this classical theory of gravity, gravitational force between two point objects is given by:

$$f = G \frac{m_1 m_2}{r^2}$$

(3.1)

with $m_1$ and $m_2$ masses of two objects, $r$ the distance between two objects. So, gravity is proportional to the masses of both objects, in other words, mass is the source of gravitational field. In general relativity, Einstein’s gravitational equation is the equation which gives out the relation between energy-momentum tensor and space-time curvature, which is essentially the relation between energy-momentum tensor and gravitational field. In the Einstein’s gravitational equation, energy-momentum
is treated as the source of gravity. This opinion is inherited in the new quantum theory of gravity. In other words, the starting point of the new quantum gauge theory of gravity is that the energy-momentum is the source of gravitational field. According to rule 3 and rule 1 of gauge principle, we know that, energy-momentum is the conserved charges of the corresponding global symmetry, which is just the symmetry for gravity. According to quantum field theory, energy-momentum is the conserved charge of global space-time translation, the corresponding conserved current is energy-momentum tensor. Therefore, the global space-time translation is the global gravitational gauge transformation. According to rule 4, we know that, after quantization, the energy-momentum operator becomes the generator of gravitational gauge transformation. It also states that, except for energy-momentum operator, there is no other generator for gravitational gauge transformation, such as, angular momentum operator $M_{\mu\nu}$ can not be the generator of gravitational gauge transformation. This is the reason why we do not localize Lorentz transformation in this new quantum gauge theory of gravity, for the generator of Lorentz transformation is not energy-momentum operator. According to rule 2 of gauge principle, the gravitational interactions will be completely determined by the requirement of the local gravitational gauge symmetry. These are the basic ideas of the new quantum gauge theory of gravity, and they are completely deductions of gauge principle.

We know that the generator of Lorentz group is angular momentum operator $M_{\mu\nu}$. If we localize Lorentz group, according to gauge principle, angular momentum will become source of a new filed, which transmits direct spin interactions. This kind of interactions does not belong to traditional Newton-Einstein gravity. It is a new kind of interactions. Up to now, we do not know that whether this kind of interactions exists in Nature or not. Besides, spin-spin interaction is a kind of non-renormalizable interaction. In other words, a quantum theory which contains spin-spin interaction is a non-renormalizable quantum theory. For these reasons, we will not localize Lorentz group in this paper. We only localize translation group in this paper. We will find that go along this way, we can set up a consistent quantum gauge theory of gravity which is renormalizable. In other words, only localizing space-time translation group is enough for us to set up a consistent quantum gravity.

From above discussions, we know that, from mathematical point of view, gravitational gauge transformation is the inverse transformation of space-time translation, and gravitational gauge group is space-time translation group. Suppose that there is an arbitrary function $\phi(x)$ of space-time coordinates $x^\mu$. The global space-time translation is:

$$x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu.$$  \hspace{1cm} (3.2)
The corresponding transformation for function $\phi(x)$ is

$$\phi(x) \rightarrow \phi'(x') = \phi(x) = \phi(x' - \epsilon). \quad (3.3)$$

According to Taylor series expansion, we have:

$$\phi(x - \epsilon) = \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \epsilon^{\mu_1} \cdots e^{\mu_n} \partial_{\mu_1} \cdots \partial_{\mu_n}\right) \phi(x), \quad (3.4)$$

where

$$\partial_{\mu_i} = \frac{\partial}{\partial x^{\mu_i}}. \quad (3.5)$$

Let’s define a special exponential operation here. Define

$$E^{a_{\mu} \cdot b_{\mu}} \triangleq 1 + \sum_{n=1}^{\infty} \frac{1}{n!} a^{\mu_1} \cdots a^{\mu_n} \cdot b_{\mu_1} \cdots b_{\mu_n}. \quad (3.6)$$

This definition is quite different from that of ordinary exponential function. In general cases, operators $a^\mu$ and $b_\mu$ do not commute each other, so

$$E^{a_{\mu} \cdot b_{\mu}} \neq E^{b_{\mu} \cdot a_{\mu}}, \quad (3.7)$$

$$E^{a_{\mu} \cdot b_{\mu}} \neq e^{a_{\mu} \cdot b_{\mu}}, \quad (3.8)$$

where $e^{a_{\mu} \cdot b_{\mu}}$ is the ordinary exponential function whose definition is

$$e^{a_{\mu} \cdot b_{\mu}} \equiv 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (a^{\mu_1} \cdot b_{\mu_1}) \cdots (a^{\mu_n} \cdot b_{\mu_n}). \quad (3.9)$$

If operators $a^\mu$ and $b_\mu$ commute each other, we will have

$$E^{a_{\mu} \cdot b_{\mu}} = E^{b_{\mu} \cdot a_{\mu}}, \quad (3.10)$$

$$E^{a_{\mu} \cdot b_{\mu}} = e^{a_{\mu} \cdot b_{\mu}}. \quad (3.11)$$

The translation operator $\hat{U}_\epsilon$ can be defined by

$$\hat{U}_\epsilon \triangleq 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \epsilon^{\mu_1} \cdots e^{\mu_n} \partial_{\mu_1} \cdots \partial_{\mu_n}. \quad (3.12)$$

Then we have

$$\phi(x - \epsilon) = (\hat{U}_\epsilon \phi(x)). \quad (3.13)$$
In order to have a good form which is similar to ordinary gauge transformation operators, the form of $\hat{U}_\epsilon$ can also be written as

$$\hat{U}_\epsilon = E^{-i\epsilon^\mu \dot{P}_\mu},$$

(3.14)

where

$$\dot{P}_\mu = -i \frac{\partial}{\partial x^\mu}.$$  

(3.15)

$\dot{P}_\mu$ is just the energy-momentum operator in space-time coordinate space. In the definition of $\hat{U}_\epsilon$ of eq.(3.14), $\epsilon^\mu$ can be independent of of space-time coordinate or a function of space-time coordinate, in a ward, it can be any functions of space time coordinate $x$.

Some operation properties of translation operator $\hat{U}_\epsilon$ are summarized below.

1. Operator $\hat{U}_\epsilon$ translate the space-time point of a field from $x$ to $x - \epsilon$,

$$\phi(x - \epsilon) = (\hat{U}_\epsilon \phi(x)),$$

(3.16)

where $\epsilon^\mu$ can be any function of space-time coordinate. This relation can also be regarded as the definition of the translation operator $\hat{U}_\epsilon$.

2. If $\epsilon$ is a function of space-time coordinate, that is $\partial_\mu \epsilon^\nu \neq 0$, then

$$\hat{U}_\epsilon = E^{-i\epsilon^\mu \dot{P}_\mu} \neq E^{-i\dot{P}_\mu \epsilon^\mu},$$

(3.17)

and

$$\hat{U}_\epsilon = E^{-i\epsilon^\mu \dot{P}_\mu} \neq e^{-i\epsilon^\mu \dot{P}_\mu}.$$  

(3.18)

If $\epsilon$ is a constant, that is $\partial_\mu \epsilon^\nu = 0$, then

$$\hat{U}_\epsilon = E^{-i\epsilon^\mu \dot{P}_\mu} = E^{-i\dot{P}_\mu \epsilon^\mu},$$

(3.19)

and

$$\hat{U}_\epsilon = E^{-i\epsilon^\mu \dot{P}_\mu} = e^{-i\epsilon^\mu \dot{P}_\mu}.$$  

(3.20)

3. Suppose that $\phi_1(x)$ and $\phi_2(x)$ are two arbitrary functions of space-time coordinate, then we have

$$\left(\hat{U}_\epsilon (\phi_1(x) \cdot \phi_2(x))\right) = (\hat{U}_\epsilon \phi_1(x)) \cdot (\hat{U}_\epsilon \phi_2(x))$$

(3.21)

4. Suppose that $A^\mu$ and $B_\mu$ are two arbitrary operators in Hilbert space, $\lambda$ is an arbitrary ordinary c-number which is commutate with operators $A^\mu$ and $B_\mu$, then we have

$$\frac{d}{d\lambda} E^{\lambda A^\mu \cdot B_\mu} = A^\mu \cdot E^{\lambda A^\mu \cdot B_\mu} \cdot B_\mu.$$  

(3.22)
5. Suppose that $\epsilon$ is an arbitrary function of space-time coordinate, then
\[
(\partial_\mu \hat{U}_\epsilon) = -i(\partial_\mu \epsilon^\nu)\hat{U}_\epsilon \hat{P}_\nu.
\] (3.23)

6. Suppose that $A^\mu$ and $B_\mu$ are two arbitrary operators in Hilbert space, then
\[
tr(E^{A^\mu \cdot B_\mu} E^{-B_\nu \cdot A^\nu}) = tr\mathbf{I},
\] (3.24)
where $tr$ is the trace operation and $\mathbf{I}$ is the unit operator in the Hilbert space.

7. Suppose that $A^\mu$, $B_\mu$ and $C^\mu$ are three operators in Hilbert space, but operators $A^\mu$ and $C^\nu$ commutate each other:
\[
[A^\mu , C^\nu] = 0,
\] (3.25)
then
\[
tr(E^{A^\mu \cdot B_\mu} E^{B_\nu \cdot C^\nu}) = tr(E^{(A^\mu + C^\nu) \cdot B_\mu}).
\] (3.26)

8. Suppose that $A^\mu$, $B_\mu$ and $C^\mu$ are three operators in Hilbert space, they satisfy
\[
\begin{align*}
[A^\mu , C^\nu] &= 0, \\
[B_\mu , C^\nu] &= 0,
\end{align*}
\] (3.27)
then
\[
E^{A^\mu \cdot B_\mu} E^{C^\nu \cdot B_\nu} = E^{(A^\mu + C^\nu) \cdot B_\mu}.
\] (3.28)

9. Suppose that $A^\mu$, $B_\mu$ and $C^\mu$ are three operators in Hilbert space, they satisfy
\[
\begin{align*}
[A^\mu , C^\nu] &= 0, \\
[[B_\mu , C^\nu] , A^\rho] &= 0, \\
[[B_\mu , C^\nu] , C^\rho] &= 0,
\end{align*}
\] (3.29)
then,
\[
E^{A^\mu \cdot B_\mu} E^{C^\nu \cdot B_\nu} = E^{(A^\mu + C^\nu) \cdot B_\mu} + [E^{A^\mu \cdot B_\mu} , C^\sigma] E^{C^\nu \cdot B_\nu} B_\sigma.
\] (3.30)

10. Suppose that $\hat{U}_{\epsilon_1}$ and $\hat{U}_{\epsilon_2}$ are two arbitrary translation operators, define
\[
\hat{U}_{\epsilon_3} = \hat{U}_{\epsilon_2} \cdot \hat{U}_{\epsilon_1},
\] (3.31)
then,
\[
\epsilon_{3}^\mu (x) = \epsilon_{2}^\mu (x) + \epsilon_{1}^\mu (x - \epsilon_{2}(x)).
\] (3.32)
This property means that the product to two translation operator satisfy closure property, which is one of the conditions that any group must satisfy.
11. Suppose that $\hat{U}_\epsilon$ is a non-singular translation operator, then

$$\hat{U}_\epsilon^{-1} = E^{\mu\nu}(f(x)) \hat{P}_\mu,$$

(3.33)

where $f(x)$ is defined by the following relations:

$$f(x - \epsilon(x)) = x.$$ 

(3.34)

$\hat{U}_\epsilon^{-1}$ is the inverse operator of $\hat{U}_\epsilon$, so

$$\hat{U}_\epsilon^{-1} \hat{U}_\epsilon = \hat{U}_\epsilon \hat{U}_\epsilon^{-1} = 1,$$

(3.35)

where 1 is the unit element of the gravitational gauge group.

12. The product operation of translation also satisfies associative law. Suppose that $\hat{U}_{\epsilon_1}$, $\hat{U}_{\epsilon_2}$ and $\hat{U}_{\epsilon_3}$ are three arbitrary translation operators, then

$$\hat{U}_{\epsilon_3} \cdot (\hat{U}_{\epsilon_2} \cdot \hat{U}_{\epsilon_1}) = (\hat{U}_{\epsilon_3} \cdot \hat{U}_{\epsilon_2}) \cdot \hat{U}_{\epsilon_1}.$$ 

(3.36)

13. Suppose that $\hat{U}_\epsilon$ is an arbitrary translation operator and $\phi(x)$ is an arbitrary function of space-time coordinate, then

$$\hat{U}_\epsilon \phi(x) \hat{U}_\epsilon^{-1} = f(x - \epsilon(x)).$$

(3.37)

This relation is quite useful in following discussions.

14. Suppose that $\hat{U}_\epsilon$ is an arbitrary translation operator. Define

$$\Lambda^\alpha_\beta = \frac{\partial x^\alpha}{\partial (x - \epsilon(x))^\beta},$$

(3.38)

$$\Lambda_\alpha^\beta = \frac{\partial (x - \epsilon(x))^\beta}{\partial x^\alpha}.$$ 

(3.39)

They satisfy

$$\Lambda^\alpha_\mu \Lambda^\mu_\nu = \delta^\alpha_\nu,$$

(3.40)

$$\Lambda^\mu_\nu \Lambda^\nu_\alpha = \delta^\mu_\alpha.$$ 

(3.41)

Then we have following relations:

$$\hat{U}_\epsilon \hat{P}_\alpha \hat{U}_\epsilon^{-1} = \Lambda^\beta_\alpha \hat{P}_\beta,$$

(3.42)

$$\hat{U}_\epsilon \hat{d}x^\alpha \hat{U}_\epsilon^{-1} = \Lambda^\beta_\alpha \hat{d}x^\beta.$$ 

(3.43)

These give out the transformation laws of $\hat{P}_\alpha$ and $\hat{d}x^\alpha$ under local gravitational gauge transformations.
Gravitational gauge group (GGG) is a transformation group which consists of all non-singular translation operators $\hat{U}_\epsilon$. We can easily see that gravitational gauge group is indeed a group, for

1. the product of two arbitrary non-singular translation operators is also a non-singular translation operator, which is also an element of the gravitational gauge group. So, the product of the group satisfies closure property which is expressed in eq(3.31);

2. the product of the gravitational gauge group also satisfies the associative law which is expressed in eq(3.36);

3. the gravitational gauge group has its unit element $1$, it satisfies
   \[ 1 \cdot \hat{U}_\epsilon = \hat{U}_\epsilon \cdot 1 = \hat{U}_\epsilon; \quad (3.44) \]

4. every non-singular element $\hat{U}_\epsilon$ has its inverse element which is given by eqs(3.33) and (3.35).

According to gauge principle, the gravitational gauge group is the symmetry of gravitational interactions. The global invariance of gravitational gauge transformation will give out conserved charges which is just the ordinary energy-momentum; the requirement of local gravitational gauge invariance needs introducing gravitational gauge field, and gravitational interactions are completely determined by the local gravitational gauge invariance.

The generators of gravitational gauge group is just the energy-momentum operators $\hat{P}_\alpha$. This is required by gauge principle. It can also be seen from the form of infinitesimal transformations. Suppose that $\epsilon$ is an infinitesimal quantity, then we have

\[ \hat{U}_\epsilon \simeq 1 - i\epsilon^\alpha \hat{P}_\alpha. \quad (3.45) \]

Therefore,

\[ i \frac{\partial \hat{U}_\epsilon}{\partial \epsilon^\alpha} \bigg|_{\epsilon=0} \]

gives out generators $\hat{P}_\alpha$ of gravitational gauge group. It is known that generators of gravitational gauge group commute each other

\[ [\hat{P}_\alpha, \hat{P}_\beta] = 0. \quad (3.47) \]

However, the commutation property of generators does not mean that gravitational gauge group is an Abel group, because two general elements of gravitational gauge group do not commute:

\[ [\hat{U}_{\epsilon_1}, \hat{U}_{\epsilon_2}] \neq 0. \quad (3.48) \]
Gravitational gauge group is a kind of non-Abel gauge group. The non-Abel nature of gravitational gauge group will cause self-interactions of gravitational gauge field.

In order to avoid confusion, we need to pay some attention to some differences between two concepts: space-time translation group and gravitational gauge group. Generally speaking, space-time translation is a kind of coordinates transformation, that is, the objects or fields in space-time are kept fixed while the space-time coordinates that describe the motion of objective matter undergo transformation. But gravitational gauge transformation is a kind of system transformation rather than a kind of coordinates transformation. In system transformation, the space-time coordinate system is kept unchanged while objects or fields undergo transformation. From mathematical point of view, space-time translation and gravitational gauge transformation are essentially the same, and the space-time translation is the inverse transformation of the gravitational gauge transformation; but from physical point of view, space-time translation and gravitational gauge transformation are quite different, especially when we discuss gravitational gauge transformation of gravitational gauge field. For gravitational gauge field, its gravitational gauge transformation is not the inverse transformation of its space-time translation. In a meaning, space-time translation is a kind of mathematical transformation, which contains little dynamical information of interactions; while gravitational gauge transformation is a kind of physical transformation, which contains all dynamical information of interactions and is convenient for us to study physical interactions. Through gravitational gauge symmetry, we can determine the whole gravitational interactions among various kinds of fields. This is the reason why we do not call gravitational gauge transformation space-time translation. This is important for all of our discussions on gravitational gauge transformations of various kinds of fields.

Suppose that $\phi(x)$ is an arbitrary scalar field. Its gravitational gauge transformation is

$$\phi(x) \rightarrow \phi'(x) = (\hat{U}_\epsilon \phi(x)).$$

(3.49)

Similar to ordinary $SU(N)$ non-able gauge field theory, there are two kinds of scalars. For example, in chiral perturbative theory, the ordinary $\pi$ mesons are scalar fields, but they are vector fields in isospin space. Similar case exists in gravitational gauge field theory. A Lorentz scalar can be a scalar or a vector or a tensor in the space of gravitational gauge group. If $\phi(x)$ is a scalar in the space of gravitational gauge group, we just simply denote it as $\phi(x)$ in gauge group space. If it is a vector in the space of gravitational gauge group, it can be expanded in the gravitational gauge group space in the following way:

$$\phi(x) = \phi^{\alpha}(x) \cdot \hat{P}_\alpha.$$  

(3.50)
The transformation of component field is

\[ \phi^\alpha(x) \to \phi'^\alpha(x) = \Lambda^\alpha_\beta \hat{U}_\epsilon \phi^\beta(x) \hat{U}_\epsilon^{-1}. \]  

(3.51)

The important thing that we must remember is that, the \( \alpha \) index is not a Lorentz index, it is just a group index. For gravitation gauge group, it is quite special that a group index looks like a Lorentz index. We must be carefully on this important thing. This will cause some fundamental changes on quantum gravity. Lorentz scalar \( \phi(x) \) can also be a tensor in gauge group space. Suppose that it is a \( n \)th order tensor in gauge group space, then it can be expanded as

\[ \phi(x) = \phi^{\alpha_1 \ldots \alpha_n}(x) \cdot \hat{P}_{\alpha_1} \cdots \hat{P}_{\alpha_n}. \]  

(3.52)

The transformation of component field is

\[ \phi^{\alpha_1 \ldots \alpha_n}(x) \to \phi'^{\alpha_1 \ldots \alpha_n}(x) = \Lambda^{\alpha_1}_\beta_1 \cdots \Lambda^{\alpha_n}_\beta_n \hat{U}_\epsilon \phi^{\beta_1 \ldots \beta_n}(x) \hat{U}_\epsilon^{-1}. \]  

(3.53)

If \( \phi(x) \) is a spinor field, the above discussion is also valid. That is, a spinor can also be a scalar or a vector or a tensor in the space of gravitational gauge group. The gravitational gauge transformations of the component fields are also given by eqs.(3.49-3.53). There is no transformations in spinor space, which is different from that of the Lorentz transformation of a spinor.

Suppose that \( A_\mu(x) \) is an arbitrary vector field. Here, the index \( \mu \) is a Lorentz index. Its gravitational gauge transformation is:

\[ A_\mu(x) \to A'_\mu(x) = (\hat{U}_\epsilon A_\mu(x)). \]  

(3.54)

Please remember that there is no rotation in the space of Lorentz index \( \mu \), while in the general coordinates transformations of general relativity, there is rotation in the space of Lorentz index \( \mu \). The reason is that gravitational gauge transformation is a kind of system transformation, while in general relativity, the general coordinates transformation is a kind of coordinates transformation. If \( A_\mu(x) \) is a scalar in the space of gravitational gauge group, eq(3.54) is all for its gauge transformation. If \( A_\mu(x) \) is a vector in the space of gravitational gauge group, it can be expanded as:

\[ A_\mu(x) = A^{\alpha}_\mu(x) \cdot \hat{P}_\alpha. \]  

(3.55)

The transformation of component field is

\[ A^{\alpha}_\mu(x) \to A'^{\alpha}_\mu(x) = \Lambda^{\alpha}_\beta \hat{U}_\epsilon A^{\beta}_\mu(x) \hat{U}_\epsilon^{-1}. \]  

(3.56)
If $A_\mu(x)$ is a $n$th order tensor in the space of gravitational gauge group, then

$$A_\mu(x) = A^{\alpha_1 \cdots \alpha_n}_\mu(x) \cdot \hat{P}_{\alpha_1} \cdots \hat{P}_{\alpha_n}. \quad (3.57)$$

The transformation of component fields is

$$A^{\alpha_1 \cdots \alpha_n}_\mu(x) \rightarrow A^{\alpha_1 \cdots \alpha_n}_\mu(x) = \Lambda^{\alpha_1}_{\beta_1} \cdots \Lambda^{\alpha_n}_{\beta_n} \hat{U}_\epsilon A^{\beta_1 \cdots \beta_n}_\mu(x) \hat{U}^{-1}_\epsilon. \quad (3.58)$$

Therefore, under gravitational gauge transformations, the behavior of a group index is quite different from that of a Lorentz index. However, they have the same behavior in global Lorentz transformations.

Generally speaking, suppose that $T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m}(x)$ is an arbitrary tensor, its gravitational gauge transformations are:

$$T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m}(x) \rightarrow T'_{\nu_1 \cdots \nu_m}^{\mu_1 \cdots \mu_n}(x) = (\hat{U}_\epsilon T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m}(x)). \quad (3.59)$$

If it is a $p$th order tensor in group space, then

$$T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m}(x) = T^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m}(x) \cdot \hat{P}_{\alpha_1} \cdots \hat{P}_{\alpha_p}. \quad (3.60)$$

The transformation of component fields is

$$T^{\mu_1 \cdots \mu_n;\alpha_1 \cdots \alpha_p}_{\nu_1 \cdots \nu_m}(x) \rightarrow T'_{\nu_1 \cdots \nu_m}^{\mu_1 \cdots \mu_n;\alpha_1 \cdots \alpha_p}(x) = \Lambda^{\alpha_1}_{\beta_1} \cdots \Lambda^{\alpha_p}_{\beta_p} \hat{U}_\epsilon T^{\mu_1 \cdots \mu_n;\beta_1 \cdots \beta_p}_{\nu_1 \cdots \nu_m}(x) \hat{U}^{-1}_\epsilon. \quad (3.61)$$

$\eta^{\mu\nu}$ is a second order Lorentz tensor, but it is a scalar in group space. It is the metric of the coordinate space. A Lorentz index can be raised or descended by this metric tensor. In a special coordinate system, it is selected to be:

$$\begin{align*}
\eta^{0\ 0} &= -1, \\
\eta^{1\ 1} &= 1, \\
\eta^{2\ 2} &= 1, \\
\eta^{3\ 3} &= 1,
\end{align*}$$

and other components of $\eta^{\mu\nu}$ vanish. $\eta^{\mu\nu}$ is the traditional Minkowski metric.

## 4 Physics Picture of Gravity

As we have mentioned above, quantum gauge theory of gravity is logically independent of traditional quantum gravity. It is know that, there are at least two pictures
of gravity. In one picture, gravity is treated as space-time geometry. In this picture, space-time is curved and there is no physical gravitational interactions, for all effects of gravity are represented by space-time metric. In another picture, gravity is treated as a kind of fundamental interactions. In this picture, space-time is always flat and space-time metric is always selected to be the Minkowski metric. For the sake of simplicity, we call the first picture geometry picture of gravity and the second picture physics picture of gravity.

The concepts of "physics picture of gravity" and "geometrical picture of gravity" are key important to understand the present theory. In order to understand these important things, I use quantum mechanics as an example. In quantum mechanics, there are many pictures, such Schrodinger picture, Heisenberg picture, ··· etc. In Schrodinger picture, operators of physical quantities are fixed and do not change with time, but wave functions are evolve with time. On the contrary, in Heisenberg picture, wave functions are fixed and do not change with time, but operators evolve with time. If we want to know whether wave functions is changed with time or not, you must first determine in which picture you study wave functions. If you do not know in which picture you study wave functions, you will not know whether wave functions should be changed with time or not. Now, similar case happens in quantum gauge theory of gravity. If you want to know whether space-time is curved or not, you must first determine in which picture gravity is studied. In physics picture of gravity, space-time is flat, but in geometry picture of gravity, space-time is curved. Quantum gauge theory of gravity is formulated in the physics picture of gravity, classical Newton’s theory of gravity is also formulated in the physics picture of gravity, and general relativity is formulated in the geometry picture of gravity. Please do not discuss any problem simultaneous in two pictures, which is dangerous.

Quantum gauge theory of gravity is formulated in the physics picture of gravity. So, in quantum gauge theory of gravity, space-time is always flat and gravity is treated as a kind of fundamental interactions. In order to avoid confusing, we do not introduce any concept of curved space-time and we do not use any language of geometry in this paper. It is suggest that anyone read this paper do not try to find any geometrical meaning of any physical quantities, do not use the language of geometry to understand anything of this paper and forget everything about the concept of fibre bundles, connections, curved space-time metric, ··· etc, for the present theory is not formulated in the geometry picture of gravity. After we go into the geometry picture of gravity and set up the geometrical picture of quantum gauge theory of gravity, we can use geometry language and study the geometry meaning of the present theory. But at present, we will not use the language of geometry.

There are mainly the following three reasons to introduce the physics picture
of gravity and formulate quantum gauge theory of gravity in the physics picture of gravity:

1. It has a clear interaction picture, so we can use perturbation theory to calculate the amplitudes of physical process.

2. We can use traditional gauge field theory to study quantum behavior of gravitational interactions, and four different kinds of fundamental interactions in Nature can be formulated in the same manner.

3. The perturbatively renormalizability of the theory can be easily proved in the physics picture of gravity.

Gravitational gauge transformation is different from space-time translation. In gravitational gauge transformation, space-time is fixed, space-time coordinates are not changed, only fields and objects undergo some translation. In a meaning, gravitational gauge transformation is a kind of physical transformation on objects and fields. The traditional space-time translation is a kind of transformation in which objects and fields are kept unchanged while space-time coordinates undergo some translation. In a meaning, space-time translation is a kind of geometrical transformation on space-time. Because quantum gauge theory of gravity is set up in the physics picture of gravity, we have to use gravitational gauge transformation in our discussion, for physics picture needs physical transformation. I do not discuss translation transformation of space-time and gauge translations in physics picture of gravity. In a meaning, space-time translation is a kind of geometrical transformation, which contains geometrical information of space-time structure and is convenient for us to study space-time geometry; while gravitational gauge transformation is a kind of physical transformation, which contains all dynamical information of gravitational interactions and is convenient for us to study physical interactions. Though from mathematical point of view, for global transformations, space-time translation is the inverse transformation of the gravitational gauge transformation. But from physical point of view and for local transformation, they are not the same. In the new theory, I do not gauge translation group, but gauge gravitational gauge group, for translation group is different from gravitational gauge group. Translation group is the symmetry of space-time, but gravitational gauge group is the symmetry group of physical fields and objects. They have essential difference from physical point of view. However, in geometry picture of gravity, the space-time transformation is used.
5 Pure Gravitational Gauge Fields

Before we study gravitational field, we must determine which field represents gravitational field. In the traditional gravitational gauge theory, gravitational field is represented by space-time metric tensor. If there is gravitational field in space-time, the space-time metric will not be equivalent to Minkowski metric, and space-time will become curved. In other words, in the traditional gravitational gauge theory, quantum gravity is formulated in curved space-time. In this paper, we will not follow this way. The underlying point of view of this new quantum gauge theory of gravity is that it is formulated in the framework of traditional quantum field theory, gravity is treated as a kind of physical interactions in flat space-time and the gravitational field is represented by gauge potential. In other words, if we put gravity into the structure of space-time, the space-time will become curved and there will be no physical gravity in space-time, because all gravitational effects are put into space-time metric and gravity is geometrized. But if we study physical gravitational interactions, it is better to rescue gravity from space-time metric and treat gravity as a kind of physical interactions. In this case, space-time is flat and there is physical gravity in Minkowski space-time. For this reason, we will not introduce the concept of curved space-time to study quantum gravity in first twelve chapters of this paper. So, in the first twelve chapters of this paper, the space-time is always flat, gravitational field is represented by gauge potential and gravitational interactions are always treated as physical interactions. In fact, what gravitational field is represented by gauge potential is required by gauge principle.

Now, let’s begin to construct the Lagrangian of gravitational gauge theory. For the sake of simplicity, let’s suppose that \( \phi(x) \) is a Lorentz scalar and gauge group scalar. According to above discussions, its gravitational gauge transformation is:

\[
\phi(x) \to \phi'(x) = (\hat{U}_\epsilon \phi(x)).
\] (5.1)

Because

\[
(\partial_\mu \hat{U}_\epsilon) \neq 0,
\] (5.2)

partial differential of \( \phi(x) \) does not transform covariantly under gravitational gauge transformation:

\[
\partial_\mu \phi(x) \to \partial_\mu \phi'(x) \neq (\hat{U}_\epsilon \partial_\mu \phi(x)).
\] (5.3)

In order to construct an action which is invariant under local gravitational gauge transformation, gravitational gauge covariant derivative is highly necessary. The gravitational gauge covariant derivative is defined by

\[
D_\mu = \partial_\mu - igC_\mu(x),
\] (5.4)
where $C_\mu(x)$ is the gravitational gauge field. It is a Lorentz vector. Under gravitational gauge transformations, it transforms as

$$C_\mu(x) \to C'_\mu(x) = \hat{U}_\epsilon(x) C_\mu(x) \hat{U}_\epsilon^{-1}(x) + \frac{i}{g} \hat{U}_\epsilon(x) (\partial_\mu \hat{U}_\epsilon^{-1}(x)).$$  \hspace{1cm} (5.5)

Using the original definition of $\hat{U}_\epsilon$, we can strictly proved that

$$[\partial_\mu , \hat{U}_\epsilon] = (\partial_\mu \hat{U}_\epsilon).$$  \hspace{1cm} (5.6)

Therefore, we have

$$\hat{U}_\epsilon \partial_\mu \hat{U}_\epsilon^{-1} = \partial_\mu + \hat{U}_\epsilon (\partial_\mu \hat{U}_\epsilon^{-1}),$$  \hspace{1cm} (5.7)

$$\hat{U}_\epsilon D_\mu \hat{U}_\epsilon^{-1} = \partial_\mu - ig C'_\mu(x).$$  \hspace{1cm} (5.8)

So, under local gravitational gauge transformations,

$$D_\mu \phi(x) \to D'_\mu \phi'(x) = (\hat{U}_\epsilon D_\mu \phi(x)), \hspace{1cm} (5.9)$$

$$D_\mu(x) \to D'_\mu(x) = \hat{U}_\epsilon D_\mu(x) \hat{U}_\epsilon^{-1}. \hspace{1cm} (5.10)$$

Gravitational gauge field $C_\mu(x)$ is vector field, it is a Lorentz vector. It is also a vector in gauge group space, so it can be expanded as linear combinations of generators of gravitational gauge group:

$$C_\mu(x) = C_\mu^\alpha(x) \cdot \hat{P}_\alpha. \hspace{1cm} (5.11)$$

$C_\mu^\alpha$ are component fields of gravitational gauge field. It looks like a second rank tensor. But according to our previous discussion, it is not a tensor field, it is a vector field. The index $\alpha$ is not a Lorentz index, it is just a gauge group index. Gravitational gauge field $C_\mu^\alpha$ has only one Lorentz index, so it is a kind of vector field. This is a result of gauge principle. The gravitational gauge transformation of component field is:

$$C_\mu^\alpha(x) \to C'_\mu^\alpha(x) = \Lambda^\alpha_\beta (\hat{U}_\epsilon C_\mu^\beta(x)) - \frac{1}{g} (\hat{U}_\epsilon \partial_\mu \epsilon^\alpha_\beta(y)), \hspace{1cm} (5.12)$$

where $y$ is a function of space-time coordinates which satisfy:

$$(\hat{U}_\epsilon y(x)) = x.$$.  \hspace{1cm} (5.13)
Define matrix $G$ as
\[ G = (G^\alpha_\mu) = (\delta^\alpha_\mu - gC^\alpha_\mu), \] (5.14)
where $C^\alpha_\mu$ is the gravitational gauge field which will be introduced below. A simple form for matrix $G$ is
\[ G = I - gC, \] (5.15)
where $I$ is a unit matrix and $C = (C^\alpha_\mu)$. Therefore,
\[ G^{-1} = \frac{1}{I - gC}. \] (5.16)
$G^{-1}$ is the inverse matrix of $G$, it satisfies
\[ (G^{-1})^\beta_\mu G^\alpha_\mu = \delta_\beta^\alpha, \] (5.17)
\[ G^\alpha_\mu (G^{-1})^\nu_\alpha = \delta^\nu_\mu. \] (5.18)
Define
\[ g^{\alpha\beta} \triangleq \eta^{\mu\nu} G^\alpha_\mu G^\beta_\nu. \] (5.19)
\[ g_{\alpha\beta} \triangleq \eta_{\mu\nu} (G^{-1})^\mu_\alpha (G^{-1})^\nu_\beta. \] (5.20)
It can be easily proved that
\[ g_{\alpha\beta} g^{\beta\gamma} = \delta^\gamma_\alpha, \] (5.21)
\[ g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_\gamma. \] (5.22)
Under gravitational gauge transformations, they transform as
\[ g_{\alpha\beta}(x) \rightarrow g'_{\alpha\beta}(x) = \Lambda^{\alpha}_{\alpha_1} \Lambda^{\beta}_{\beta_1} (\hat{U}_{\epsilon} g_{\alpha_1\beta_1}(x)), \] (5.23)
\[ g^{\alpha\beta}(x) \rightarrow g'^{\alpha\beta}(x) = \Lambda^{\alpha}_{\alpha_1} \Lambda^{\beta}_{\beta_1} (\hat{U}_{\epsilon} g^{\alpha_1\beta_1}(x)), \] (5.24)
The strength of gravitational gauge field is defined by
\[ F_{\mu\nu} = \frac{1}{-ig} [D_\mu, D_\nu], \] (5.25)
or
\[ F_{\mu\nu} = \partial_\mu C_\nu(x) - \partial_\nu C_\mu(x) - igC_\mu(x)C_\nu(x) + igC_\nu(x)C_\mu(x). \] (5.26)
$F_{\mu\nu}$ is a second order Lorentz tensor. It is a vector in group space, so it can be expanded in group space,
\[ F_{\mu\nu}(x) = F^{\alpha}_{\mu\nu}(x) \cdot \hat{P}_\alpha. \] (5.27)
The explicit form of component field strengths is

\[ F_{\mu\nu}^\alpha = \partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha - gC_\mu^\beta \partial_\beta C_\nu^\alpha + gC_\nu^\beta \partial_\beta C_\mu^\alpha \]  \hspace{1cm} (5.28)

The strength of gravitational gauge fields transforms covariantly under gravitational gauge transformation:

\[ F_{\mu\nu} \rightarrow F'_{\mu\nu} = \hat{U}_\epsilon F_{\mu\nu} \hat{U}_\epsilon^{-1}. \] \hspace{1cm} (5.29)

The gravitational gauge transformation of the component field strength is

\[ F_{\mu\nu}^\alpha \rightarrow F'_{\mu\nu}^\alpha = \Lambda^\alpha_\beta (\hat{U}_\epsilon F_{\mu\nu}^\beta). \] \hspace{1cm} (5.30)

Similar to traditional gauge field theory, the kinematical term for gravitational gauge field can be selected as

\[ L_0 = -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} g_{\alpha\beta} F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta. \] \hspace{1cm} (5.31)

We can easily prove that this Lagrangian does not invariant under gravitational gauge transformation, it transforms covariantly

\[ L_0 \rightarrow L'_0 = (\hat{U}_\epsilon L_0). \] \hspace{1cm} (5.32)

In order to resume the gravitational gauge symmetry of the action, we introduce an extremely important factor, whose form is

\[ J(C) = \sqrt{-\det g_{\alpha\beta}}, \] \hspace{1cm} (5.33)

where \( g_{\alpha\beta} \) is given by eq.(5.20). The gravitational gauge transformations of \( g_{\alpha\beta} \) is given by eq.(5.23). Then \( J(C) \) transforms as

\[ J(C) \rightarrow J'(C') = J \cdot (\hat{U}_\epsilon J(C)), \] \hspace{1cm} (5.34)

where \( J \) is the Jacobian of the transformation,

\[ J = \det \left( \frac{\partial (x - \epsilon)^\mu}{\partial x^\nu} \right). \] \hspace{1cm} (5.35)

The Lagrangian for gravitational gauge field is selected as

\[ \mathcal{L} = J(C) \mathcal{L}_0 = \sqrt{-\det g_{\alpha\beta}} \cdot \mathcal{L}_0, \] \hspace{1cm} (5.36)
and the action for gravitational gauge field is

\[ S = \int d^4 x \mathcal{L}. \] (5.37)

It can be proved that this action has gravitational gauge symmetry. In other words, it is invariant under gravitational gauge transformation,

\[ S \rightarrow S' = S. \] (5.38)

In order to prove the gravitational gauge symmetry of the action, the following relation is important,

\[ \int d^4 x J(\hat{U}_\epsilon f(x)) = \int d^4 x f(x), \] (5.39)

where \( f(x) \) is an arbitrary function of space-time coordinate.

According to gauge principle, the global gauge symmetry will give out conserved charges. Now, let’s discuss the conserved charges of global gravitational gauge transformation. Suppose that \( \epsilon^\alpha \) is an infinitesimal constant 4-vector. Then, in the first order approximation, we have

\[ \hat{U}_\epsilon = 1 - \epsilon^\alpha \partial_\alpha + O(\epsilon^2). \] (5.40)

The first order variation of the gravitational gauge field is

\[ \delta C^\alpha_\mu(x) = -\epsilon^\nu \partial_\nu C^\alpha_\mu, \] (5.41)

and the first order variation of action is:

\[ \delta S = \int d^4 x \epsilon^\alpha \partial_\mu T^\mu_{i\alpha}, \] (5.42)

where \( T^\mu_{i\alpha} \) is the inertial energy-momentum tensor, whose definition is

\[ T^\mu_{i\alpha} \equiv J(C) \left( -\frac{\partial \mathcal{L}_0}{\partial \partial_\mu C_\beta} \partial_\alpha C^\beta_\nu + \delta^\mu_\alpha \mathcal{L}_0 \right). \] (5.43)

It is a conserved current,

\[ \partial_\mu T^\mu_{i\alpha} = 0. \] (5.44)

Except for the factor \( J(C) \), the form of the inertial energy-momentum tensor is almost completely the same as that in the traditional quantum field theory. It means that gravitational interactions will change energy-momentum of matter fields, which is what we expected in general relativity.
The Euler-Lagrange equation for gravitational gauge field is

\[ \partial_\mu \frac{\partial L}{\partial \partial_\mu C^\alpha_\nu} = \frac{\partial L}{\partial C^\alpha_\nu}. \]  

This form is completely the same as what we have ever seen in quantum field theory. But if we insert eq.(5.36) into it, we will get

\[ \partial_\mu \frac{\partial L}{\partial C^\alpha_\nu} + g G_\alpha^{-1} L_0 - g G_\alpha^{-1} (\partial_\mu C^\alpha_\nu) \frac{\partial L}{\partial \partial_\mu C^\alpha_\nu}. \]  

Eq.(5.28) can be changed into

\[ F^\alpha_\mu_\nu = (D_\mu C^\alpha_\nu) - (D_\nu C^\alpha_\mu), \]  

so the Lagrangian \( L_0 \) depends on gravitational gauge fields completely through its covariant derivative. Therefore,

\[ \partial \frac{\partial L_0}{\partial C^\alpha_\nu} = -g \partial \frac{\partial L_0}{\partial D_\nu C^\beta_\mu} \partial_\alpha C^\beta_\mu + \frac{1}{2} \eta^{\mu_\rho} \eta^{\lambda_\sigma} g_{\alpha_\beta} G_{\gamma}^{-1} F^\gamma_{\mu_\lambda} F^\beta_{\rho_\sigma}. \]

Using the above relations and

\[ \partial \frac{\partial L_0}{\partial \partial_\mu C^\alpha_\nu} = -\eta^{\mu_\lambda} \eta^{\nu_\rho} \eta_{2\alpha_\beta} F^\beta_{\lambda_\tau} \eta^{\nu_\lambda} \eta^{\sigma_\tau} \eta_{2\alpha_\beta} F^\beta_{\lambda_\tau} C^\mu_\sigma, \]

the above equation of motion of gravitational gauge fields are changed into

\[ \partial_\mu (\eta^{\mu_\lambda} \eta^{\nu_\tau} g_{\alpha_\beta} F^\beta_{\lambda_\tau}) = -g T^\nu_{\alpha_\lambda}, \]

where

\[ T^\nu_{\alpha_\lambda} = -\partial \frac{\partial L_0}{\partial \partial_\mu C^\alpha_\nu} \partial_\alpha C^\beta_\mu + G_\alpha^{-1} L_0 - G_\sigma^{-1} \partial \frac{\partial L_0}{\partial \partial_\mu C^\alpha_\nu} \partial_\alpha C^\beta_\mu \]

\[ -\frac{1}{2} \eta^{\mu_\rho} \eta^{\lambda_\sigma} g_{\alpha_\beta} G_{\gamma}^{-1} F^\gamma_{\mu_\lambda} F^\beta_{\rho_\sigma} + \partial_\mu (\eta^{\nu_\lambda} \eta^{\sigma_\tau} g_{\alpha_\beta} F^\beta_{\lambda_\tau} C^\mu_\sigma). \]

\( T^\nu_{\alpha_\lambda} \) is also a conserved current, that is

\[ \partial_\nu T^\nu_{\alpha_\lambda} = 0, \]

because of the following identity

\[ \partial_\nu \partial_\mu (\eta^{\mu_\lambda} \eta^{\nu_\tau} g_{\alpha_\beta} F^\beta_{\lambda_\tau}) = 0. \]

\( T^\nu_{\alpha_\lambda} \) is called gravitational energy-momentum tensor, which is the source of gravitational gauge field. Now we get two different energy-momentum tensors, one is
the inertial energy-momentum tensor $T^{\nu}_{\alpha \sigma}$ and another is the gravitational energy-momentum tensor $T^{\nu}_{g \alpha}$. They are similar, but they are different. The inertial energy-momentum tensor $T^{\nu}_{\alpha \sigma}$ is given by conservation law which is associate with global gravitational gauge symmetry, it gives out an energy-momentum 4-vector:

$$P_{i\alpha} = \int d^3 \vec{x} \ T^{0}_{i\alpha}. \quad (5.54)$$

It is a conserved charges,

$$\frac{d}{dt} P_{i\alpha} = 0. \quad (5.55)$$

The time component of $P_{i\alpha}$, that is $P_{i0}$, gives out the Hamiltonian $H$ of the system,

$$H = -P_{i0} = \int d^3 \vec{x} \ J(C)(\pi^\mu_{\alpha} C^\alpha_\mu - \mathcal{L}_0). \quad (5.56)$$

According to our conventional belief, $H$ should be the inertial energy of the system, therefore $P_{\alpha \sigma}$ is the inertial energy-momentum of the system. The gravitational energy-momentum is given by equation of motion of gravitational gauge field, it is also a conserved current. The space integration of the time component of it gives out a conserved energy-momentum 4-vector,

$$P_{g\alpha} = \int d^3 \vec{x} \ T^{0}_{g\alpha}. \quad (5.57)$$

It is also a conserved charge,

$$\frac{d}{dt} P_{g\alpha} = 0. \quad (5.58)$$

The time component of it just gives out the gravitational energy of the system. This can be easily seen. Set $\nu$ and $\alpha$ in eq.(5.50) to 0, we get

$$\partial^i F^{0}_{i0} = -g T^{0}_{g0}. \quad (5.59)$$

The field strength of gravitational field is defined by

$$E^i = -F^0_{i0}. \quad (5.60)$$

The space integration of eq.(5.59) gives out

$$\int d^3 \vec{\sigma} \cdot \vec{E} = g \int d^3 \vec{x} \ T^{0}_{g0}. \quad (5.61)$$

According to Newton’s classical theory of gravity, $\int d^3 \vec{\sigma} \cdot \vec{E}$ in the right hand term is just the gravitational mass of the system. Denote the gravitational mass of the system as $M_g$, that is

$$M_g = - \int d^3 \vec{x} \ T^{0}_{g0}. \quad (5.62)$$
Then eq(5.61) is changed into

\[ \oint d\hat{s} \cdot \vec{E} = -gM. \]

(5.63)

This is just the classical Newton’s law of universal gravitation. It can be strictly proved that gravitational mass is different from inertial mass. They are not equivalent. But their difference is at least first order infinitesimal quantity if the gravitational field \( gC^\alpha_\mu \), for this difference is proportional to \( gC^\alpha_\mu \). So, this difference is too small to be detected in experiments. But in the environment of strong gravitational field, the difference will become relatively larger and will be easier to be detected. Much more highly precise measurement of this difference is strongly needed to test this prediction and to test the validity of the equivalence principle. In the chapter of classical limit of quantum gauge theory of gravity, we will return to discuss this problem again.

Now, let’s discuss self-coupling of gravitational field. The Lagrangian of gravitational gauge field is given by eq(5.36). Because

\[ J(C) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left( \sum_{n=1}^{\infty} \frac{g^n}{n} \text{tr}(C^n) \right)^m \]

(5.64)

there are vertexes of \( n \) gravitational gauge fields in tree diagram where \( n \) can be arbitrary integer number that is greater than 3. This property is important for renormalization of the theory. Because the coupling constant of the gravitational gauge interactions has negative mass dimension, any kind of regular vertex exists divergence. In order to cancel these divergences, we need to introduce the corresponding counterterms. Because of the existence of the vertex of \( n \) gravitational gauge fields in tree diagram in the non-renormalized Lagrangian, we need not introduce any new counterterm which does not exist in the non-renormalized Lagrangian, what we need to do is to redefine gravitational coupling constant \( g \) and gravitational gauge field \( C^\alpha_\mu \) in renormalization. If there is no \( J(C) \) term in the original Lagrangian, then we will have to introduce infinite counterterms in renormalization, and therefore the theory is non-renormalizable. Because of the existence of the factor \( J(C) \), though quantum gauge theory of gravity looks like a non-renormalizable theory according to power counting law, it is indeed renormalizable. In a word, the factor \( J(C) \) is highly important for the quantum gauge theory of gravity.

6 Gravitational Interactions of Scalar Fields

Now, let’s start to discuss gravitational interactions of matter fields. First, we discuss gravitational interactions of scalar fields. For the sake of simplicity, we first
discuss real scalar field. Suppose that \( \phi(x) \) is a real scalar field. The traditional Lagrangian for the real scalar field is

\[
-\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - \frac{m^2}{2} \phi^2(x),
\]

(6.1)

where \( m \) is the mass of scalar field. This is the Lagrangian for a free real scalar field. The Euler-Lagrangian equation of motion of it is

\[
(\eta^{\mu\nu} \partial_\mu \partial_\nu - m^2) \phi(x) = 0,
\]

(6.2)

which is the famous Klein-Gordon equation.

Now, replace the ordinary partial derivative \( \partial_\mu \) with gauge covariant derivative \( D_\mu \), and add into the Lagrangian of pure gravitational gauge field, we get

\[
\mathcal{L}_0 = -\frac{1}{2} \eta^{\mu\nu} (D_\mu \phi)(D_\nu \phi) - \frac{m^2}{2} \phi^2 - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} g_{\alpha\beta} F^{\alpha}_{\mu\nu} F^{\beta}_{\rho\sigma}.
\]

(6.3)

The full Lagrangian is selected to be

\[
\mathcal{L} = J(C) \mathcal{L}_0,
\]

(6.4)

and the action \( S \) is defined by

\[
S = \int d^4x \, \mathcal{L}.
\]

(6.5)

Using our previous definitions of gauge covariant derivative \( D_\mu \) and strength of gravitational gauge field \( F^\alpha_{\mu\nu} \), we can obtain an explicit form of Lagrangian \( \mathcal{L} \),

\[
\mathcal{L} = \mathcal{L}_F + \mathcal{L}_I,
\]

(6.6)

with \( \mathcal{L}_F \) the free Lagrangian and \( \mathcal{L}_I \) the interaction Lagrangian. Their explicit expressions are

\[
\mathcal{L}_F = -\frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - \frac{m^2}{2} \phi^2(x) - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} g_{\alpha\beta} F^{\alpha}_{0\mu\nu} F^{\beta}_{0\rho\sigma},
\]

(6.7)

\[
\mathcal{L}_I = \mathcal{L}_F \cdot (J(C) - 1) - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} (J(C) g_{\alpha\beta} - \eta_{\alpha\beta}) F^{\alpha}_{0\mu\nu} F^{\beta}_{0\rho\sigma}
\]

\[+ g J(C) \eta^{\mu\rho} C^{\alpha}_{\mu} (\partial_\rho \phi) (\partial_\nu \phi) - \frac{g^2}{2} J(C) \eta^{\mu\rho} C^{\alpha}_{\mu} C^{\beta}_{\nu} (\partial_\alpha \phi) (\partial_\beta \phi)
\]

\[+ g J(C) \eta^{\mu\rho} \eta^{\nu\sigma} g_{\alpha\beta} (\partial_\mu C^{\alpha}_{\nu} - \partial_\nu C^{\alpha}_{\mu}) C^{\delta}_{\rho} \partial_\delta C^{\beta}_{\sigma}
\]

\[+ \frac{1}{2} g^2 J(C) \eta^{\mu\rho} \eta^{\nu\sigma} g_{\alpha\beta} (C^{\delta}_{\mu} \partial_\delta C^{\alpha}_{\nu} - C^{\delta}_{\nu} \partial_\delta C^{\alpha}_{\mu}) C^{\epsilon}_{\rho} \partial_\epsilon C^{\beta}_{\sigma},
\]

(6.8)
where,
\[ F^\alpha_{\mu\nu} = \partial_\mu C^\alpha_\nu - \partial_\nu C^\alpha_\mu. \]  

(6.9)

From eq.(6.8), we can see that scalar field can directly couples to any number of gravitational gauge fields. This is one of the most important interaction properties of gravity. Other kinds of interactions, such as strong interactions, weak interactions and electromagnetic interactions do not have this kind of interaction properties. Because the gravitational coupling constant has negative mass dimension, renormalization of theory needs this kind of interaction properties. In other words, if matter field can not directly couple to any number of gravitational gauge fields, the theory will be non-renormalizable.

The symmetries of the theory can be easily seen from eq.(6.3). First, let’s discuss Lorentz symmetry. In eq.(6.3), some indexes are Lorentz indexes and some are group indexes. Lorentz indexes and group indexes have different transformation law under gravitational gauge transformation, but they have the same transformation law under Lorentz transformation. Therefor, it can be easily seen that both \( \mathcal{L}_0 \) and \( J(C) \) are Lorentz scalars, the Lagrangian \( \mathcal{L} \) and action \( S \) are invariant under global Lorentz transformation.

Under gravitational gauge transformations, real scalar field \( \phi(x) \) transforms as

\[ \phi(x) \rightarrow \phi'(x) = (\hat{U}_\epsilon \phi(x)), \]  

therefore,

\[ D_\mu \phi(x) \rightarrow D'_\mu \phi'(x) = (\hat{U}_\epsilon D_\mu \phi(x)). \]  

(6.11)

It can be easily proved that \( \mathcal{L}_0 \) transforms covariantly

\[ \mathcal{L}_0 \rightarrow \mathcal{L}'_0 = (\hat{U}_\epsilon \mathcal{L}_0), \]  

(6.12)

and the action eq.(6.5) of the system is invariant,

\[ S \rightarrow S' = S. \]  

(6.13)

Please remember that eq.(5.39) is an important relation to be used in the proof of the gravitational gauge symmetry of the action.

Global gravitational gauge symmetry gives out conserved charges. Suppose that \( \hat{U}_\epsilon \) is an infinitesimal gravitational gauge transformation, it will have the form of eq.(5.40). The first order variations of fields are

\[ \delta C^\alpha_\mu(x) = -\epsilon^\nu (\partial_\nu C^\alpha_\mu(x)), \]  

\[ \delta \phi(x) = -\epsilon^\nu (\partial_\nu \phi(x)), \]  

(6.14)

(6.15)
Using Euler-Lagrange equation of motions for scalar fields and gravitational gauge fields, we can obtain that

\[ \delta S = \int d^4 x \epsilon^\alpha \partial_\mu T_{i\alpha}^\mu, \quad (6.16) \]

where

\[ T_{i\alpha}^\mu = J(C)(-\frac{\partial L_0}{\partial \partial_\mu \phi} \partial_\alpha \phi - \frac{\partial L_0}{\partial \partial_\mu C_\mu^\beta} \partial_\alpha C_\mu^\beta + \delta_\alpha^\mu L_0). \quad (6.17) \]

Because action is invariant under global gravitational gauge transformation,

\[ \delta S = 0, \quad (6.18) \]

and \( \epsilon^\alpha \) is an arbitrary infinitesimal constant 4-vector, we obtain,

\[ \partial_\mu T_{i\alpha}^\mu = 0. \quad (6.19) \]

This is the conservation equation for inertial energy-momentum tensor. \( T_{i\alpha}^\mu \) is the conserved current which corresponds to the global gravitational gauge symmetry. The space integration of the time component of inertial energy-momentum tensor gives out the conserved charge, which is just the inertial energy-momentum of the system. The time component of the conserved charge is the Hamilton of the system, which is

\[ H = -P_{i\alpha} = \int d^3 \vec{x} \ J(C)(\pi_\phi \dot{\phi} + \pi_\alpha^\mu \dot{C}_\mu^\alpha - L_0), \quad (6.20) \]

where \( \pi_\phi \) and \( \pi_\alpha^\mu \) are canonical conjugate momenta of the real scalar field and gravitational field

\[ \pi_\phi = \frac{\partial L_0}{\partial \dot{\phi}}, \quad (6.21) \]

\[ \pi_\alpha^\mu = \frac{\partial L_0}{\partial \dot{C}_\mu^\alpha}. \quad (6.22) \]

The inertial space momentum of the system is given by

\[ P^i = P_i^i = \int d^3 \vec{x} \ J(C)(-\pi_\phi \partial_i \phi - \pi_\alpha^\mu \partial_i C_\mu^\alpha). \quad (6.23) \]

According to gauge principle, after quantization, they will become generators of quantum gravitational gauge transformation.

Using the definition (5.19), we can change the Lagrangian given by eq.(6.3) into

\[ L_0 = -\frac{1}{2} g^{\alpha\beta}(\partial_\alpha \phi)(\partial_\beta \phi) - \frac{m^2}{2} \phi^2 - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} g_{\alpha\beta} F_\mu^\alpha F_{\nu\sigma}^\beta. \quad (6.24) \]
\( g^{\alpha \beta} \) is the metric tensor of curved group space-time. We can easily see that, when there is no gravitational field in space-time, that is,

\[
C_\mu^\alpha = 0, \tag{6.25}
\]

the group space-time will be flat

\[
g^{\alpha \beta} = \eta^{\alpha \beta}. \tag{6.26}
\]

This is what we expected in general relativity. We do not talk to much on this problem here, for we will discuss this problem again in details in the chapter on Einstein-like field equation with cosmological term.

Euler-Lagrange equations of motion can be easily deduced from action principle. Keep gravitational gauge field \( C_\mu^\alpha \) fixed and let real scalar field vary infinitesimally, then the first order infinitesimal variation of action is

\[
\delta S = \int d^4x J(C) \left( \frac{\partial L_0}{\partial \phi} - \frac{\partial \mathcal{L}_0}{\partial \partial_\phi} - gG^{-1\nu}_\alpha (\partial_\mu C^\alpha_\nu) \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \phi} \right) \delta \phi. \tag{6.27}
\]

Because \( \delta \phi \) is an arbitrary variation of scalar field, according to action principle, we get

\[
\frac{\partial \mathcal{L}_0}{\partial \partial_\phi} - \frac{\partial \mathcal{L}_0}{\partial \partial_\phi} - gG^{-1\nu}_\alpha (\partial_\mu C^\alpha_\nu) \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \phi} = 0. \tag{6.28}
\]

Because of the existence of the factor \( J(C) \), the equation of motion for scalar field is quite different from the traditional form in quantum field theory. But the difference is a second order infinitesimal quantity if we suppose that both gravitational coupling constant and gravitational gauge field are first order infinitesimal quantities. Because

\[
\frac{\partial \mathcal{L}_0}{\partial \partial_\phi} = -g^{\alpha \beta} \partial_\beta \phi, \tag{6.29}
\]

\[
\frac{\partial \mathcal{L}_0}{\partial \phi} = -m^2 \phi, \tag{6.30}
\]

the explicit form of the equation of motion of scalar field is

\[
g^{\alpha \beta} \partial_\alpha \partial_\beta \phi - m^2 \phi + (\partial_\alpha g^{\alpha \beta}) \partial_\beta \phi + g g^{\alpha \beta} (\partial_\beta \phi) G^{-1\nu}_\gamma (\partial_\alpha C^\gamma_\nu) = 0. \tag{6.31}
\]

The equation of motion for gravitational gauge field is:

\[
\partial_\mu (\eta^{\nu \lambda} \eta^{\rho \tau} g_{\alpha \beta} F^\beta_{\lambda \tau}) = -g T^\mu_{ga}, \tag{6.32}
\]

where \( T^\nu_{ga} \) is the gravitational energy-momentum tensor, whose definition is:

\[
T^\nu_{ga} = -\frac{\partial \mathcal{L}_0}{\partial \partial_\nu C_\mu^\alpha} \partial_\alpha C^\beta_\mu - \frac{\partial \mathcal{L}_0}{\partial \partial_\phi} \partial_\phi + G^{-1\nu}_\alpha \mathcal{L}_0 - G^{-1\gamma}_\beta (\partial_\mu C^\beta_\lambda) \frac{\partial \mathcal{L}_0}{\partial \partial_\mu C^\gamma_\nu} - \frac{1}{2} \eta^{\mu \nu} \eta^{\lambda \sigma} g_{\alpha \beta} G^{-1\gamma}_\mu F^\beta_{\lambda \sigma} + \partial_\mu (\eta^{\nu \lambda} \eta^{\gamma \tau} g_{\alpha \beta} F^\beta_{\lambda \tau} C^\gamma_\nu). \tag{6.33}
\]

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We can see again that, for matter field, its inertial energy-momentum tensor is also different from the gravitational energy-momentum tensor, this difference completely originate from the influences of gravitational gauge field. Compare eq.(6.33) with eq.(6.17), and set gravitational gauge field to zero, that is

\[ D_\mu \phi = \partial_\mu \phi, \]  \tag{6.34} \\
\[ J(C) = 1, \]  \tag{6.35} \\
then we find that two energy-momentum tensors are completely the same:

\[ T^\mu_{\ i\alpha} = T^\mu_{\ g\alpha}. \]  \tag{6.36} \\

It means that the equivalence principle only strictly hold in a space-time where there is no gravitational field. In the environment of strong gravitational field, such as in black hole, the equivalence principle will be strongly violated.

Define

\[ L = \int d^3 x \ L \rightarrow \int d^3 x J(C) L_0. \]  \tag{6.37} \\
Then, we can easily prove that

\[ \frac{\delta L}{\delta \phi} = J(C) \left( \frac{\partial L_0}{\partial \phi} - \partial_\mu \frac{\partial L_0}{\partial C^\alpha_\mu} - g G^{-1}_\alpha \left( \partial_\beta C^\alpha_\mu \frac{\partial L_0}{\partial \phi} \right) \right), \]  \tag{6.38} \\
\[ \frac{\delta L}{\delta \phi} = J(C) \frac{\partial L_0}{\partial \phi}, \]  \tag{6.39} \\
\[ \frac{\delta L}{\delta C^\alpha_\nu} = J(C) \left( \frac{\partial L_0}{\partial C^\alpha_\nu} - \partial_\mu \frac{\partial L_0}{\partial C^\beta_\mu} + g G^{-1}_\nu \frac{\partial L_0}{\partial \phi} - g G^{-1}_\mu \left( \partial_\beta C^\beta_\nu \frac{\partial L_0}{\partial \phi} \right) \right), \]  \tag{6.40} \\
\[ \frac{\delta L}{\delta C^\alpha_\nu} = J(C) \frac{\partial L_0}{\partial C^\alpha_\nu}. \]  \tag{6.41} \\

Then, Hamilton’s action principle gives out the following equations of motion:

\[ \frac{\delta L}{\delta \phi} - \frac{d}{dt} \frac{\delta L}{\delta \dot{\phi}} = 0, \]  \tag{6.42} \\
\[ \frac{\delta L}{\delta C^\alpha_\nu} - \frac{d}{dt} \frac{\delta L}{\delta \dot{C}^\alpha_\nu} = 0. \]  \tag{6.43} \\
These two equations of motion are essentially the same as the Euler-Lagrange equations of motion which we have obtained before. But these two equations have more beautiful forms.
The Hamiltonian of the system is given by a Legendre transformation,

\[ H = \int d^3 \vec{x} \left( \frac{\delta L}{\delta \dot{\phi}} \phi + \frac{\delta L}{\delta C_\mu^\alpha} C_\mu^\alpha - L \right) \]

\[ = \int d^3 \vec{x} \ J(C)(\pi_\phi \dot{\phi} + \pi_\mu^\alpha C_\mu^\alpha - L_0), \tag{6.44} \]

where \( \pi_\phi \) and \( \pi_\mu^\alpha \) are canonical conjugate momenta whose definitions are given by (6.21) and (6.22). It can be easily seen that the Hamiltonian given by Legendre transformation is completely the same as that given by inertial energy-momentum tensor. After Legendre transformation, \( \phi, C_\mu^\alpha, J(C)\pi_\phi \) and \( J(C)\pi_\mu^\alpha \) are canonical independent variables. Let these variables vary infinitesimally, we can get

\[ \frac{\delta H}{\delta \phi} = -\frac{\delta L}{\delta \phi}, \tag{6.45} \]

\[ \frac{\delta H}{\delta (J(C)\pi_\phi)} = \dot{\phi}, \tag{6.46} \]

\[ \frac{\delta H}{\delta C_\alpha^\nu} = -\frac{\delta L}{\delta C_\alpha^\nu}, \tag{6.47} \]

\[ \frac{\delta H}{\delta (J(C)\pi_\mu^\alpha)} = \dot{C}_\mu^\alpha. \tag{6.48} \]

Then, Hamilton’s equations of motion read:

\[ \frac{d}{dt} \phi = \frac{\delta H}{\delta (J(C)\pi_\phi)}, \tag{6.49} \]

\[ \frac{d}{dt} (J(C)\pi_\phi) = -\frac{\delta H}{\delta \phi}, \tag{6.50} \]

\[ \frac{d}{dt} C_\mu^\alpha = \frac{\delta H}{\delta (J(C)\pi_\mu^\alpha)}, \tag{6.51} \]

\[ \frac{d}{dt} (J(C)\pi_\mu^\alpha) = -\frac{\delta H}{\delta C_\mu^\alpha}. \tag{6.52} \]

The forms of the Hamilton’s equations of motion are completely the same as those appears in usual quantum field theory and usual classical analytical mechanics. Therefore, the introduction of the factor \( J(C) \) does not affect the forms of Lagrange equations of motion and Hamilton’s equations of motion.

The Poisson brackets of two general functional of canonical arguments can be defined by

\[ \{ A, B \} = \int d^3 \vec{x} \left( \frac{\delta A}{\delta \phi} \frac{\delta B}{\delta (J(C)\pi_\phi)} - \frac{\delta A}{\delta (J(C)\pi_\phi)} \frac{\delta B}{\delta \phi} \right) \]

\[ + \frac{\delta A}{\delta C_\mu^\alpha} \frac{\delta B}{\delta (J(C)\pi_\mu^\alpha)} - \frac{\delta A}{\delta (J(C)\pi_\mu^\alpha)} \frac{\delta B}{\delta C_\mu^\alpha} \right), \tag{6.53} \]
According to this definition, we have
\[
\{ \phi(\vec{x}, t) , (J(C)\pi_\phi)(\vec{y}, t) \} = \delta^3(\vec{x} - \vec{y}), \quad (6.54)
\]
\[
\{ C^\alpha_\nu(\vec{x}, t) , (J(C)\pi_\nu^\alpha)(\vec{y}, t) \} = \delta^\nu_\mu \delta^\alpha_\beta \delta^3(\vec{x} - \vec{y}). \quad (6.55)
\]
These two relations can be used as the starting point of canonical quantization of quantum gravity.

Using Poisson brackets, the Hamilton’s equations of motion can be expressed in other forms,
\[
\frac{d}{dt}\phi(\vec{x}, t) = \{ \phi(\vec{x}, t) , H \}, \quad (6.56)
\]
\[
\frac{d}{dt}(J(C)\pi_\phi)(\vec{x}, t) = \{(J(C)\pi_\phi)(\vec{x}, t) , H \}, \quad (6.57)
\]
\[
\frac{d}{dt}C^\alpha_\nu(\vec{x}, t) = \{ C^\alpha_\nu(\vec{x}, t) , H \}, \quad (6.58)
\]
\[
\frac{d}{dt}(J(C)\pi_\nu^\alpha)(\vec{x}, t) = \{(J(C)\pi_\nu^\alpha)(\vec{x}, t) , H \}. \quad (6.59)
\]
Therefore, if \( A \) is an arbitrary functional of the canonical arguments \( \phi, C^\alpha_\mu, J(C)\pi_\phi \) and \( J(C)\pi_\nu^\alpha \), then we have
\[
\dot{A} = \{ A , H \}. \quad (6.60)
\]
After quantization, this equation will become the Heisenberg equation.

If \( \phi(x) \) is a complex scalar field, its traditional Lagrangian is
\[
-\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi^*(x) - m^2\phi(x)\phi^*(x). \quad (6.61)
\]
Replace ordinary partial derivative with gauge covariant derivative, and add into the Lagrangian for pure gravitational gauge field, we get,
\[
\mathcal{L}_0 = -\eta^{\mu\nu}(D_\mu\phi)(D_\nu\phi)^* - m^2\phi\phi^* - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}g_{\alpha\beta}F^\alpha_{\mu\nu}F^\beta_{\rho\sigma}. \quad (6.62)
\]
Repeating all above discussions, we can get the whole theory for gravitational interactions of complex scalar fields. We will not repeat this discussion here.
7 Gravitational Interactions of Dirac Field

In the usual quantum field theory, the Lagrangian for Dirac field is

\[-\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi. \tag{7.1}\]

Replace ordinary partial derivative with gauge covariant derivative, and add into the Lagrangian of pure gravitational gauge field, we get,

\[\mathcal{L}_0 = -\bar{\psi}(\gamma^\mu D_\mu + m)\psi - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}g_{\alpha\beta}F^\alpha_{\mu\nu}F^\beta_{\rho\sigma}. \tag{7.2}\]

The full Lagrangian of the system is

\[\mathcal{L} = J(C)\mathcal{L}_0, \tag{7.3}\]

and the corresponding action is

\[S = \int d^4x \mathcal{L} = \int d^4x J(C)\mathcal{L}_0. \tag{7.4}\]

This Lagrangian can be separated into two parts,

\[\mathcal{L} = \mathcal{L}_F + \mathcal{L}_I, \tag{7.5}\]

with \(\mathcal{L}_F\) the free Lagrangian and \(\mathcal{L}_I\) the interaction Lagrangian. Their explicit forms are

\[\mathcal{L}_F = -\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{\alpha\beta}F^\alpha_{\mu\nu}F^\beta_{\rho\sigma}, \tag{7.6}\]

\[\mathcal{L}_I = \mathcal{L}_F \cdot (J(C) - 1) - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}(J(C)g_{\alpha\beta} - \eta_{\alpha\beta})F^\alpha_{0\mu\nu}F^\beta_{0\rho\sigma}
+ gJ(C)\bar{\psi}\gamma^\mu(\partial_\alpha\psi)C^\alpha_{\mu} + gJ(C)\eta^{\mu\rho}\eta^{\nu\sigma}g_{\alpha\beta}(\partial_\mu C^\alpha_{\nu} - \partial_\nu C^\alpha_{\mu})C^\beta_{\rho\sigma} \tag{7.7}\]

From \(\mathcal{L}_I\), we can see that Dirac field can directly couple to any number of gravitational gauge fields, the mass term of Dirac field also take part in gravitational interactions. All these interactions are completely determined by the requirement of gravitational gauge symmetry. The Lagrangian function before renormalization almost contains all kind of divergent vertex, which is important in the renormalization of the theory. Besides, from eq.(7.7), we can directly write out Feynman rules of the corresponding interaction vertexes.

Because the traditional Lagrangian function eq.(7.1) is invariant under global Lorentz transformation, which is already proved in the traditional quantum field
theory, and the covariant derivative has the same behavior as partial derivative under global Lorentz transformation, the first two terms of Lagrangian \( \mathcal{L} \) are global Lorentz invariant. We have already prove that the Lagrangian function for pure gravitational gauge field is invariant under global Lorentz transformation. Therefor, \( \mathcal{L} \) has global Lorentz symmetry.

The gravitational gauge transformation of Dirac field is
\[
\psi(x) \rightarrow \psi'(x) = (\hat{U}_\epsilon \psi(x)).
\] (7.8)
\( \bar{\psi} \) transforms similarly,
\[
\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = (\hat{U}_\epsilon \bar{\psi}(x)).
\] (7.9)
Dirac \( \gamma \)-matrices is not a physical field, so it keeps unchanged under gravitational gauge transformation,
\[
\gamma^\mu \rightarrow \gamma^\mu.
\] (7.10)
It can be proved that, under gravitational gauge transformation, \( \mathcal{L}_0 \) transforms as
\[
\mathcal{L}_0 \rightarrow \mathcal{L}'_0 = (\hat{U}_\epsilon \mathcal{L}_0).
\] (7.11)
So,
\[
\mathcal{L} \rightarrow \mathcal{L}' = J(\hat{U}_\epsilon \mathcal{L}_0),
\] (7.12)
where \( J \) is the Jacobi of the corresponding space-time translation. Then using eq.(5.39), we can prove that the action \( S \) has gravitational gauge symmetry.

Suppose that \( \hat{U}_\epsilon \) is an infinitesimal global transformation, then the first order infinitesimal variations of Dirac field are
\[
\delta \psi = -\epsilon^\nu \partial_\nu \psi,
\] (7.13)
\[
\delta \bar{\psi} = -\epsilon^\nu \partial_\nu \bar{\psi}.
\] (7.14)
The first order variation of action is
\[
\delta S = \int d^4x \epsilon^\alpha \partial_\mu T^\mu_{i\alpha},
\] (7.15)
where \( T^\mu_{i\alpha} \) is the inertial energy-momentum tensor whose definition is,
\[
T^\mu_{i\alpha} \equiv J(C) \left( -\frac{\partial L_0}{\partial \partial_\mu \psi} \partial_\alpha \psi - \frac{\partial L_0}{\partial \partial_\mu C^\alpha} \partial_\alpha C^\beta + \delta^\mu_\alpha \mathcal{L}_0 \right).
\] (7.16)
The global gravitational gauge symmetry of action gives out conservation equation of the inertial energy-momentum tensor,
\[
\partial_\mu T^\mu_{i\alpha} = 0.
\] (7.17)
The inertial energy-momentum tensor is the conserved current which expected by
gauge principle. The space integration of its time component gives out the conserved
energy-momentum of the system,

\[ H = -P_{i\ 0} = \int d^3 \vec{x} \ J(C)(\pi_\psi \ \dot{\psi} + \pi^\mu_\alpha C^{\alpha}_\mu - \mathcal{L}_0), \]  

(7.18)

\[ P^i = P_{i\ i} = \int d^3 \vec{x} \ J(C)(-\pi_\psi \partial_\psi - \pi^\mu_\alpha \partial_\mu C^{\alpha}_\mu), \]  

(7.19)

where

\[ \pi_\psi = \frac{\partial \mathcal{L}_0}{\partial \dot{\psi}}. \]  

(7.20)

The equation of motion for Dirac field is

\[ (\gamma^\mu D_\mu + m)\psi = 0. \]  

(7.21)

From this expression, we can see that the factor \( J(C) \) does not affect the equation of
motion of Dirac field. This is caused by the asymmetric form of the Lagrangian. If
we use a symmetric form of Lagrangian, the factor \( J(C) \) will also affect the equation
of motion of Dirac field, which will be discussed later.

The equation of motion of gravitational gauge field can be easily deduced,

\[ \partial_\nu (\eta_\mu^\nu \eta_\sigma^\tau g_{\alpha\beta} F^{\beta}_\lambda \tau) = -g T^\nu_{g\alpha}, \]  

(7.22)

where \( T^\nu_{g\alpha} \) is the gravitational energy-momentum tensor, whose definition is:

\[ T^\nu_{g\alpha} = -\frac{\partial \mathcal{L}_0}{\partial \partial_\nu C^{\alpha}_\mu} \partial_\alpha C^{\beta}_\mu - \frac{\partial \mathcal{L}_0}{\partial \partial_\nu \psi} \partial_\alpha \psi + G^{-1\lambda}_{\beta}(\partial_\mu C^{\beta}_{\lambda}) \frac{\partial \mathcal{L}_0}{\partial D_\nu C_{\psi}^{\alpha}} + \partial_\mu (\eta_\nu^\lambda \eta_\sigma^\tau g_{\alpha\beta} F^{\beta}_\lambda \tau) - \frac{1}{2} \eta_\mu^\rho \eta_\lambda^\sigma g_{\alpha\beta} G^{-1\nu}_{\lambda\tau} F^{\rho}_{\mu\tau} \]  

(7.23)

We see again that the gravitational energy-momentum tensor is different from the
inertial energy-momentum tensor.

In usual quantum field theory, the Lagrangian for Dirac field has a more sym-
metric form, which is

\[ -\bar{\psi}(\gamma^\mu \ \bar{\partial}_\mu + m)\psi, \]  

(7.24)

where

\[ \bar{\partial}_\mu = \frac{\partial_\mu - \bar{\partial}_\mu}{2}. \]  

(7.25)

The Euler-Lagrange equation of motion of eq.(7.24) also gives out the conventional
Dirac equation.
Now replace ordinary space-time partial derivative with covariant derivative, and
add into the Lagrangian of pure gravitational gauge field, we get,
\[ \mathcal{L}_0 = -\bar{\psi}(\gamma^\mu \overrightarrow{D}_\mu + m)\psi - \frac{1}{4}\eta^{\mu\nu}\eta^{\rho\sigma}g_{\alpha\beta}F^\alpha_{\mu\nu}F^\beta_{\rho\sigma}, \]  
(7.26)
where \( \overrightarrow{D}_\mu \) is defined by
\[ \overrightarrow{D}_\mu = \frac{D_\mu - \leftarrow{D}_\mu}{2}. \]  
(7.27)
Operator \( \leftarrow{D}_\mu \) is understood in the following way
\[ f(x) \leftarrow{D}_\mu g(x) = (D_\mu f(x))g(x), \]  
(7.28)
with \( f(x) \) and \( g(x) \) two arbitrary functions. The Lagrangian density \( \mathcal{L} \) and action \( S \) are also defined by eqs.(7.3-7.4). In this case, the free Lagrangian \( \mathcal{L}_F \) and interaction Lagrangian \( \mathcal{L}_I \) are given by
\[ \mathcal{L}_F = -\bar{\psi}(\gamma^\mu \overrightarrow{D}_\mu + m)\psi - \frac{1}{4}\eta^{\mu\nu}\eta^{\rho\sigma}g_{\alpha\beta}F^\alpha_{\mu\nu}F^\beta_{\rho\sigma}, \]  
(7.29)
\[ \mathcal{L}_I = \mathcal{L}_F \cdot (J(C) - 1) - \frac{1}{4}\eta^{\mu\nu}\eta^{\rho\sigma}(J(C)g_{\alpha\beta} - \eta_{\alpha\beta})F^\alpha_{\mu\nu}F^\beta_{\rho\sigma} \]
\[ + gJ(C)(\bar{\psi}\gamma^\mu \overrightarrow{\partial}_\alpha \psi)C^\alpha_\mu + gJ(C)\eta^{\mu\nu}\eta^{\rho\sigma}g_{\alpha\beta}(\partial_\mu C^\alpha_\nu - \partial_\nu C^\alpha_\mu)C^\delta_\rho \partial_\sigma C^\beta_\delta \]  
(7.30)
\[ - \frac{1}{2}g^2J(C)\eta^{\mu\nu}\eta^{\rho\sigma}g_{\alpha\beta}(C^\delta_\mu \partial_\delta C^\alpha_\nu - C^\delta_\nu \partial_\delta C^\alpha_\mu)C^\epsilon_\rho \partial_\epsilon C^\beta_\sigma. \]
(7.31)
The Euler-Lagrange equation of motion for Dirac field is
\[ \frac{\partial \mathcal{L}_0}{\partial \dot{\psi}} - \partial_{\mu} \frac{\partial \mathcal{L}_0}{\partial \partial_{\mu} \psi} - gG^{-1\nu}(\partial_{\mu} C^\alpha_\nu) \frac{\partial \mathcal{L}_0}{\partial \partial_{\mu} \psi} = 0. \]  
(7.32)
Because
\[ \frac{\partial \mathcal{L}_0}{\partial \dot{\psi}} = -\frac{1}{2}\gamma^\mu D_\mu \psi - m\psi, \]  
(7.33)
eq(7.31) will be changed into
\[ (\gamma^\mu D_\mu + m)\psi = -\frac{1}{2}\gamma^\mu(\partial_\alpha C^\alpha_\mu)\psi - \frac{1}{2}g\gamma^\mu \psi G^{-1\nu}(D_\mu C^\beta_\nu). \]  
(7.34)
If gravitational gauge field vanishes, this equation of motion will return to the traditional Dirac equation.

The inertial energy-momentum tensor now becomes
\[ T_{\mu i\alpha} = \frac{\partial L_0}{\partial \mu \psi} \partial_\alpha \psi - (\partial_\alpha \bar{\psi}) \frac{\partial L_0}{\partial \mu C_\nu^\beta} \partial_\nu C_\alpha^\beta + \delta_\alpha^\mu L_0, \]
(7.35)
and the gravitational energy-momentum tensor becomes
\[ T_{g\nu} = -\frac{\partial \xi_0}{\partial D_\nu C_\mu^\alpha} \partial_\alpha C_\mu^\beta - \frac{\partial \xi_0}{\partial D_\nu \psi} \partial_\nu \psi - (\partial_\alpha \bar{\psi}) \frac{\partial \xi_0}{\partial D_\nu \psi} + C_\gamma^{-\gamma^\nu} L_0 \]
\[ - G_\beta^{-\nu \lambda} (\partial_\mu C_\lambda^\beta) \frac{\partial L_0}{\partial \mu C_\nu^\alpha} + \partial_\mu (\eta^{\mu \lambda} \eta^{\sigma \tau} g_{\alpha \beta} F_{\lambda \tau}^\beta C_\mu^\alpha) \]
\[ - \frac{1}{2} \eta^{\mu \rho} \eta^{\nu \sigma} g_{\alpha \beta} G^{-1 \gamma^\nu}_{\gamma \lambda} F_{\mu \nu}^\alpha F_{\rho \sigma}^\beta. \]
(7.36)
Both of them are conserved energy-momentum tensor. But they are not equivalent.

### 8 Gravitational Interactions of Vector Field

The traditional Lagrangian for vector field is
\[ -\frac{1}{4} \eta^{\mu \rho} \eta^{\nu \sigma} A_{\mu \nu} A_{\rho \sigma} - \frac{m^2}{2} \eta^{\mu \nu} A_{\mu} A_{\nu}, \]
(8.1)
where \( A_{\mu \nu} \) is the strength of vector field which is given by
\[ \partial_\mu A_{\nu} = \partial_\nu A_{\mu}. \]
(8.2)
The Lagrangian \( L_0 \) that describes gravitational interactions between vector field and gravitational fields is
\[ L_0 = -\frac{1}{4} \eta^{\mu \rho} \eta^{\nu \sigma} A_{\mu \nu} A_{\rho \sigma} - \frac{m^2}{2} \eta^{\mu \nu} A_{\mu} A_{\nu} - \frac{1}{4} \eta^{\mu \rho} \eta^{\nu \sigma} g_{\alpha \beta} F_{\mu \rho}^\alpha F_{\nu \sigma}^\beta. \]
(8.3)
In eq.(8.3), the definition of strength \( A_{\mu \nu} \) is not given by eq.(8.2), it is given by
\[ A_{\mu \nu} = D_\mu A_\nu - D_\nu A_\mu \]
\[ = \partial_\mu A_\nu - \partial_\nu A_\mu - gC_\alpha^\mu \partial_\alpha A_\nu + gC_\nu^\alpha \partial_\alpha A_\mu, \]
(8.4)
where \( D_\mu \) is the gravitational gauge covariant derivative, whose definition is given by eq.(5.4). The full Lagrangian \( L \) is given by,
\[ L = J(C)L_0. \]
(8.5)
The action $S$ is defined by

$$S = \int d^4x \mathcal{L}.$$  \hfill (8.6)

The Lagrangian $\mathcal{L}$ can be separated into two parts: the free Lagrangian $\mathcal{L}_F$ and interaction Lagrangian $\mathcal{L}_I$. The explicit forms of them are

$$\mathcal{L}_F = -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} A_{0\mu\nu} A_{0\rho\sigma} - \frac{m^2}{2} \eta^{\mu\nu} A_{\mu} A_{\nu} - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{\alpha\beta} F_{0\mu\nu}^\alpha F_{0\rho\sigma}^\beta, \hfill (8.7)$$

$$\mathcal{L}_I = \mathcal{L}_F \cdot (J(C) - 1) - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} (J(C) g_{\alpha\beta} - \eta_{\alpha\beta}) F_{0\mu\nu}^\alpha F_{0\rho\sigma}^\beta$$

$$+ g J(C) \eta^{\mu\rho} \eta^{\nu\sigma} A_{0\mu\nu} C_\alpha^\rho \partial_\alpha A_\sigma$$

$$- \frac{1}{2} g J(C) \eta^{\mu\rho} \eta^{\nu\sigma} (C_\mu^\alpha C_\rho^\beta (\partial_\alpha A_\nu)(\partial_\beta A_\sigma) - C_\nu^\alpha C_\rho^\beta (\partial_\alpha A_\mu)(\partial_\beta A_\sigma))$$

$$+ g J(C) \eta^{\mu\rho} \eta^{\nu\sigma} g_{\alpha\beta} (\partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha) C_\rho^\delta \partial_\delta C_\sigma^\beta$$

$$- \frac{1}{2} g^2 J(C) \eta^{\mu\rho} \eta^{\nu\sigma} g_{\alpha\beta} (C_\delta^\mu \partial_\delta C_\rho^\alpha - C_\delta^\nu \partial_\delta C_\rho^\alpha) C_\sigma^\epsilon \partial_\epsilon C_\beta^\gamma,$$

where $A_{0\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The first three lines of $\mathcal{L}_I$ contain interactions between vector field and gravitational gauge fields. It can be seen that the vector field can also directly couple to arbitrary number of gravitational gauge fields, which is one of the most important properties of gravitational gauge interactions. This interaction property is required and determined by local gravitational gauge symmetry.

Under Lorentz transformations, group index and Lorentz index have the same behavior. Therefore every term in the Lagrangian $\mathcal{L}$ are Lorentz scalar, and the whole Lagrangian $\mathcal{L}$ and action $S$ have Lorentz symmetry.

Under gravitational gauge transformations, vector field $A_\mu$ transforms as

$$A_\mu(x) \rightarrow A'_\mu(x) = (\hat{U}_\epsilon A_\mu(x)). \hfill (8.9)$$

$D_\mu A_\nu$ and $A_{\mu\nu}$ transform covariantly,

$$D_\mu A_\nu \rightarrow D'_\mu A'_\nu = (\hat{U}_\epsilon D_\mu A_\nu), \hfill (8.10)$$

$$A_{\mu\nu} \rightarrow A'_{\mu\nu} = (\hat{U}_\epsilon A_{\mu\nu}). \hfill (8.11)$$

So, the gravitational gauge transformations of $\mathcal{L}_0$ and $\mathcal{L}$ respectively are

$$\mathcal{L}_0 \rightarrow \mathcal{L}'_0 = (\hat{U}_\epsilon \mathcal{L}_0), \hfill (8.12)$$
The action of the system is gravitational gauge invariant.

The global gravitational gauge transformation gives out conserved current of gravitational gauge symmetry. Under infinitesimal global gravitational gauge transformation, the vector field $A_\mu$ transforms as

$$\delta A_\mu = -\epsilon^\alpha \partial_\alpha A_\mu. \quad (8.14)$$

The first order variation of action is

$$\delta S = \int d^4x \epsilon^\alpha \partial_\mu T^\mu_{i\alpha}, \quad (8.15)$$

where $T^\mu_{i\alpha}$ is the inertial energy-momentum tensor whose definition is,

$$T^\mu_{i\alpha} = J(C) \left( -\frac{\partial L_0}{\partial A_\mu} \partial_\alpha A_\nu - \frac{\partial L_0}{\partial C_\nu} \partial_\alpha C_\nu + \delta^\mu_\alpha L_0 \right). \quad (8.16)$$

$T^\mu_{i\alpha}$ is a conserved current. The space integration of its time component gives out inertial energy-momentum of the system,

$$H = -P_i = \int d^3\vec{x} J(C)(\pi^\mu \dot{A}_\mu + \pi^\mu_\alpha \dot{C}_\mu - L_0), \quad (8.17)$$

$$P^i = P_{i\alpha} = \int d^3\vec{x} J(C)(-\pi^\mu \partial_i A_\mu - \pi^\mu_\alpha \partial_i C_\mu), \quad (8.18)$$

where

$$\pi^\mu = \frac{\partial L_0}{\partial A_\mu}. \quad (8.19)$$

The equation of motion for vector field is

$$\frac{\partial L_0}{\partial A_\nu} - \partial_\mu \frac{\partial L_0}{\partial A_\nu} - gG^{-1\lambda}_{\mu} \left( \partial_\mu C_\lambda^\alpha \right) \frac{\partial L_0}{\partial C_\lambda^\alpha} = 0. \quad (8.20)$$

From eq.(8.3), we can obtain

$$\frac{\partial L_0}{\partial A_\nu} = -\eta^{\rho\nu} \eta^{\sigma} C_\lambda^\rho A_\sigma, \quad (8.21)$$

$$\frac{\partial L_0}{\partial A_\nu} = -m^2 \eta^{\nu A_\lambda}. \quad (8.22)$$
Then, eq. (8.20) is changed into
\[
\eta^{\mu\rho} \eta^{\nu\sigma} D_\mu A_{\rho\sigma} - m^2 \eta^{\mu\nu} A_\mu = -\eta^{\lambda\rho} \eta^{\nu\sigma} (\partial_\mu C_\lambda) A_{\rho\sigma} - g \eta^{\mu\rho} \eta^{\nu\sigma} A_{\rho\sigma} C^{-1}_\alpha (D_\mu C_\alpha). \quad (8.23)
\]

The equation of motion of gravitational gauge field is
\[
\partial_\mu (\eta^{\mu\lambda} \eta^{\nu\tau} g_{\alpha\beta} F^\beta_\nu) = -g T^\nu_{ga}, \quad (8.24)
\]
where \( T^\nu_{ga} \) is the gravitational energy-momentum tensor,
\[
T^\nu_{ga} = -\frac{\partial L_0}{\partial D_\nu C_\mu} \partial_\alpha C^\beta_\mu - \frac{\partial L_0}{\partial \partial_\nu A_\mu} \partial_\alpha A_\mu + G^{-1}_\alpha \mathcal{L}_0 - G^{-1}_\alpha \gamma (\partial_\mu C_\gamma) \eta^{\nu\sigma} g_{\alpha\beta} F^\beta_\mu F^\gamma_\nu + \partial_\mu (\eta^{\nu\lambda} \eta^{\sigma\tau} g_{\alpha\beta} F^\beta_\nu C_\mu) \quad (8.25)
\]
\[
-\frac{1}{2} \eta^{\mu\rho} \eta^{\lambda\sigma} g_{\alpha\beta} G^{-1}_\nu F^\gamma_\mu F^\beta_\rho.
\]

\( T^\nu_{ga} \) is also a conserved current. The space integration of its time component gives out the gravitational energy-momentum which is the source of gravitational interactions. It can be also seen that inertial energy-momentum tensor and gravitational energy-momentum tensor are not equivalent.

### 9 Gravitational Interactions of Gauge Fields

It is known that QED, QCD and unified electroweak theory are all gauge theories. In this chapter, we will discuss how to unify these gauge theories with gravitational gauge theory, and how to unify four different kinds of fundamental interactions formally.

First, let’s discuss QED theory. As an example, let’s discuss electromagnetic interactions of Dirac field. The traditional electromagnetic interactions between Dirac field \( \psi \) and electromagnetic field \( A_\mu \) is
\[
-\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} A_{\mu\nu} A_{\rho\sigma} - \bar{\psi} (\gamma^\mu (\partial_\mu - ieA_\mu) + m) \psi. \quad (9.1)
\]

The Lagrangian that describes gravitational gauge interactions between gravitational gauge field and Dirac field or electromagnetic field and describes electromagnetic interactions between Dirac field and electromagnetic field is
\[
\mathcal{L}_0 = -\bar{\psi} (\gamma^\mu (D_\mu - ieA_\mu) + m) \psi - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} A_{\mu\nu} A_{\rho\sigma} - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} g_{\alpha\beta} F^\alpha_\mu F^\beta_\rho. \quad (9.2)
\]
where $D_\mu$ is the gravitational gauge covariant derivative which is given by eq.(5.4) and the strength of electromagnetic field $A_\mu$ is

$$A_{\mu\nu} = A_{\mu\nu} + gG^{-1}_{\alpha}A_\lambda F^\alpha_{\mu\nu},$$

(9.3)

where $A_{\mu\nu}$ is given by eq.(8.4) and $G^{-1}$ is given by eq.(5.16). The full Lagrangian density and the action of the system are respectively given by,

$$\mathcal{L} = J(C)\mathcal{L}_0,$$

(9.4)

$$S = \int d^4x \mathcal{L}.$$  

(9.5)

The system given by above Lagrangian has both $U(1)$ gauge symmetry and gravitational gauge symmetry. Under $U(1)$ gauge transformations,

$$\psi(x) \rightarrow \psi'(x) = e^{-i\alpha(x)}\psi(x),$$

(9.6)

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{e}D_\mu\alpha(x),$$

(9.7)

$$C^\alpha_\mu(x) \rightarrow C'^\alpha_\mu(x) = C^\alpha_\mu(x).$$

(9.8)

It can be proved that the Lagrangian $\mathcal{L}$ is invariant under $U(1)$ gauge transformation. Under gravitational gauge transformations,

$$\psi(x) \rightarrow \psi'(x) = (\hat{U}_\epsilon\psi(x)),$$

(9.9)

$$A_\mu(x) \rightarrow A'_\mu(x) = (\hat{U}_\epsilon A_\mu(x)), $$

(9.10)

$$C^\mu_\mu(x) \rightarrow C'_\mu(x) = \hat{U}_\epsilon C^\mu_\mu(x)\hat{U}_{\epsilon}^{-1}(x) + \frac{i}{g}\hat{U}_\epsilon(x)(\partial_\mu\hat{U}_{\epsilon}^{-1}(x)).$$

(9.11)

The action $S$ given by eq.(9.4) is invariant under gravitational gauge transformation.

Lagrangian $\mathcal{L}$ can be separated into free Lagrangian $\mathcal{L}_F$ and interaction Lagrangian $\mathcal{L}_I$,

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_I,$$

(9.12)

where

$$\mathcal{L}_F = -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma} A_{0\mu\nu}A_{0\rho\sigma} - \bar{\psi}(\gamma^\mu\partial_\mu + m)\psi - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{\alpha\beta}F_{0\mu\nu}F_{0\rho\sigma},$$

(9.13)
The traditional Lagrangian for QCD is
\[ L = L_F \cdot (J(C) - 1) + ie \cdot J(C) \bar{\psi} \gamma^\mu \psi A_\mu - \frac{1}{4} \eta^{\mu\rho\eta^{\nu\sigma}} (J(C) g_{\alpha\beta} - \eta_{\alpha\beta}) F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta + g J(C) \bar{\psi} \gamma^\mu \partial_\mu \psi C^\alpha_\mu + g J(C) \eta^{\mu\nu} \psi A_{0\mu\nu} C^\alpha_\mu \partial_\sigma A_\sigma \]
\[ - \frac{1}{2} J(C) \eta^{\mu\rho} \eta^{\nu\sigma} A_{\mu\nu} G^{-1\lambda}_\alpha A_\lambda F_{\rho\sigma}^\alpha \]
\[ - \frac{1}{4} g J(C) \eta^{\mu\rho} \psi_\mu A_\nu \bar{\psi}_\nu - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} A_{\mu\nu} \bar{A}_{\rho\sigma} \]
\[ = L_0 = - \sum \bar{\psi}_n \left[ \gamma^\mu (\partial_\mu - ig_c A^{\lambda}_\mu \frac{\lambda_i}{2}) + m_n \right] \psi_n - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} A_{\mu\nu} \bar{A}_{\rho\sigma}, \]
gluons, gravitons and intermediate gauge bosons $W^\pm$ and $Z^0$), and possible Higgs bosons. According to the Standard Model, leptons form left-hand doublets and right-hand singlets. Let’s denote

$$
\psi_L^{(1)} = \left( \begin{array}{c} \nu_e \\ e \end{array} \right)_L, \quad \psi_L^{(2)} = \left( \begin{array}{c} \nu_\mu \\ \mu \end{array} \right)_L, \quad \psi_L^{(3)} = \left( \begin{array}{c} \nu_\tau \\ \tau \end{array} \right)_L, (9.19)
$$

$$
\psi_R^{(1)} = e_R, \quad \psi_R^{(2)} = \mu_R, \quad \psi_R^{(3)} = \tau_R. (9.20)
$$

Neutrinos have no right-hand singlets. The weak hypercharge for left-hand doublets $\psi_L^{(i)}$ is $-1$ and for right-hand singlet $\psi_R^{(i)}$ is $-2$. All leptons carry no color charge.

In order to define the wave function for quarks, we have to introduce Kabayashi-Maskawa mixing matrix first, whose general form is,

$$
K = \begin{pmatrix}
c_1 & s_1 c_3 & s_1 s_3 \\
-s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i \delta} & c_1 c_2 s_3 + s_2 c_3 e^{i \delta} \\
s_1 s_2 & -c_1 s_2 c_3 - c_2 s_3 e^{i \delta} & -c_1 s_2 s_3 + c_2 c_3 e^{i \delta}
\end{pmatrix} (9.21)
$$

where

$$
c_i = \cos \theta_i, \quad s_i = \sin \theta_i \quad (i = 1, 2, 3) (9.22)
$$

and $\theta_i$ are generalized Cabibbo angles. The mixing between three different quarks $d, s$ and $b$ is given by

$$
\begin{pmatrix}
d \\
s \\
b
\end{pmatrix} = K \begin{pmatrix}
d \\
s \\
b
\end{pmatrix}. (9.23)
$$

Quarks also form left-hand doublets and right-hand singlets,

$$
q_L^{(1)a} = \begin{pmatrix} u_L^a \\ d_L^a \end{pmatrix}, \quad q_L^{(2)a} = \begin{pmatrix} c_L^a \\ s_L^a \end{pmatrix}, \quad q_L^{(3)a} = \begin{pmatrix} t_L^a \\ b_L^a \end{pmatrix}, (9.24)
$$

$$
q_R^{(1)a} = u_R^a, \quad q_R^{(2)a} = c_R^a, \quad q_R^{(3)a} = t_R^a, (9.25)
$$

where index $a$ is color index. It is known that left-hand doublets have weak isospin $\frac{1}{2}$ and weak hypercharge $\frac{4}{3}$, right-hand singlets have no weak isospin, $q_u^{(j)a}$s have weak hypercharge $\frac{1}{3}$ and $q_{bd}^{(j)a}$s have weak hypercharge $-\frac{2}{3}$.

For gauge bosons, gravitational gauge field is also denoted by $C_\mu^\alpha$. The gluon field is denoted $A_\mu$,

$$
A_\mu = A_\mu^i \frac{\lambda^i}{2}. (9.26)
$$

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The color gauge covariant field strength tensor is also given by eq.(9.18). The $U(1)_Y$ gauge field is denoted by $B_\mu$ and $SU(2)$ gauge field is denoted by $F_\mu$

$$F_\mu = F_\mu^a \sigma_a \frac{2}{2},$$  
(9.27)

where $\sigma_a$ is the Pauli matrix. The $U(1)_Y$ gauge field strength tensor is given by

$$B_{\mu \nu} = B_{\mu \nu} + g G^{-1}_\alpha (B_{\lambda \mu}) F^\alpha_{\lambda \nu},$$  
(9.28)

where

$$B_{\mu \nu} = D_{\mu} B_{\nu} - D_{\nu} B_{\mu},$$  
(9.29)

and the $SU(2)$ gauge field strength tensor is given by

$$F^n_{\mu \nu} = F^n_{\mu \nu} + g G^{-1}_\alpha F^n_{\lambda} F^\alpha_{\mu \nu},$$  
(9.30)

$$F^n_{\mu \nu} = D_{\mu} F^n_{\nu} - D_{\nu} F^n_{\mu} + g_w \epsilon_{lmn} F^l_{\mu} F^m_{\nu},$$  
(9.31)

where $g_w$ is the coupling constant for $SU(2)$ gauge interactions and the coupling constant for $U(1)_Y$ gauge interactions is $g'_w$.

If there exist Higgs particles in Nature, the Higgs fields is represented by a complex scalar $SU(2)$ doublet,

$$\phi = \left( \begin{array}{c} \phi^\dagger \\ \phi^0 \end{array} \right).$$  
(9.32)

The hypercharge of Higgs field $\phi$ is 1.

The Lagrangian $\mathcal{L}_0$ that describes four kinds of fundamental interactions is given
by

\[ \mathcal{L}_0 = -\sum_{j=1}^3 \bar{\psi}_L^{(j)} \gamma^\mu (D_\mu + i \frac{2}{3} g'_w B_\mu - ig_w F_\mu) \psi_L^{(j)} \]

\[ - \sum_{j=1}^3 \bar{e}_R^{(j)} \gamma^\mu (D_\mu + ig'_w B_\mu) e_R^{(j)} \]

\[ - \sum_{j=1}^3 \bar{q}_L^{(j) a} \gamma^\mu \left( (D_\mu - ig_w F_\mu - i \frac{2}{3} g'_w B_\mu) \delta_{ab} - ig_c A_\mu^k \left( \frac{\lambda^k}{2} \right)_{ab} \right) q_L^{(j)b} \]

\[ - \sum_{j=1}^3 \sum_{a=1}^3 \bar{q}_u^{(j)a} \gamma^\mu \left( (D_\mu + i \frac{1}{3} g'_w B_\mu) \delta_{ab} - ig_c A_\mu^k \left( \frac{\lambda^k}{2} \right)_{ab} \right) q_u^{(j)b} \]

\[ - \frac{1}{4} \eta^{\mu \nu} \eta^{\rho \sigma} F_{\mu \nu}^{a} F_{\rho \sigma}^a - \frac{1}{4} \eta^{\mu \rho} \eta^{\nu \sigma} B_{\mu \nu}^{a} B_{\rho \sigma}^a \]

\[ - \left[ (D_\mu - i \frac{2}{3} g'_w B_\mu - ig_w F_\mu) \phi \right]^\dagger \left[ (D_\mu - i \frac{2}{3} g'_w B_\mu - ig_w F_\mu) \phi \right] \]

\[ - \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \]

\[ - \sum_{j=1}^3 \sum_{k=1}^3 \left[ f_j^{(j)} \bar{\phi} q_L^{(j)k} + \bar{\psi}_L^{(j)} \phi e_R^{(j)} \right] \]

\[ = \sum_{j=1}^3 \sum_{a=1}^3 \sum_{b=1}^3 \left[ f_{ja}^{(j)} \bar{\phi} q_L^{(j)k} + \bar{f}_j^{(j)} \phi q_L^{(j)k} \right] \]

where

\[ \bar{\phi} = i \sigma_2 \phi^* \]

\[ = \left( \begin{array}{c} \phi^\dagger \\ -\phi \end{array} \right) . \]

The full Lagrangian is given by

\[ \mathcal{L} = J(C) \mathcal{L}_0. \]
10 Classical Limit of Quantum Gravity

In this chapter, we mainly discuss leading order approximation of quantum gauge theory of gravity, which will give out classical Newton’s theory of gravity.

First, we discuss an important problem qualitatively. It is know that, in usual gauge theory, such as QED, the coulomb force between two objects which carry like electric charges is always mutual repulsive. Gravitational gauge theory is also a kind of gauge theory, is the force between two static massive objects attractive or repulsive? For the sake of simplicity, we use Dirac field as an example to discuss this problem. The discussions for other kinds of fields can be proceeded similarly.

Suppose that the gravitational field is very weak, so both the gravitational field and the gravitational coupling constant are first order infinitesimal quantities. Then in leading order approximation, both inertial energy-momentum tensor and gravitational energy-momentum tensor give the same results, which we denoted as

$$ T_\alpha^\mu = \bar{\psi} \gamma^\mu \partial_\alpha \psi. \quad (10.1) $$

The time component of the current is

$$ T_\alpha^0 = -i \bar{\psi} \partial_\alpha \psi = \bar{\psi} \gamma^0 \partial_\alpha \psi. \quad (10.2) $$

Its space integration gives out the energy-momentum of the system. The interaction Lagrangian between Dirac field and gravitational field is given by eq.(7.7). After considering the equation of motion of Dirac field, the coupling between Dirac field and Gravitational gauge field in the leading order is:

$$ \mathcal{L}_I \approx g T_\alpha^\mu C^\alpha_{\mu}. \quad (10.3) $$

The leading order interaction Hamiltonian density is given by

$$ \mathcal{H}_I \approx -\mathcal{L}_I \approx -g T_\alpha^\mu C^\alpha_{\mu}. \quad (10.4) $$

The equation of motion of gravitational gauge field in the leading order is:

$$ \partial_\lambda \partial^\lambda (\eta^{\nu\tau} \eta_{\alpha\beta} C^\beta_{\tau}) - \partial^\nu (\eta_{\alpha\beta} C^\beta_{\lambda}) = -g T_\alpha^{\nu}. \quad (10.5) $$

As a classical limit approximation, let’s consider static gravitational interactions between two static objects. In this case, the leading order component of energy-momentum tensor is \( T_0^0 \), other components of energy-momentum tensor is a first order infinitesimal quantity. So, we only need to consider the equation of motion of \( \nu = \alpha = 0 \) of eq.(10.5), which now becomes

$$ \partial_\lambda \partial^\lambda C^0_{\lambda} = -g T_0^0. \quad (10.6) $$
For static problems, all time derivatives vanish. Therefore, the above equation is changed into
\[ \nabla^2 C_0^0 = -gT_0^0. \] (10.7)
This is just the Newton’s equation of gravitational field. Suppose that there is only one point object at the origin of the coordinate system. Because \( T_0^0 \) is the negative value of energy density, we can let
\[ T_0^0 = -M \delta(\vec{x}). \] (10.8)
Applying
\[ \nabla^2 \frac{1}{r} = -4\pi \delta(\vec{x}), \] (10.9)
with \( r = |\vec{x}| \), we get
\[ C_0^0 = -\frac{gM}{4\pi r}. \] (10.10)
This is just the gravitational potential which is expected in Newton’s theory of gravity.

Suppose that there is another point object at the position of point \( \vec{x} \) with mass \( m \). The gravitational potential energy between these two objects is that
\[ V(r) = \int d^3 \vec{y} \mathcal{H} = -g \int d^3 \vec{y} T_{20}^0(\vec{x})C_0^0, \] (10.11)
with \( C_0^0 \) is the gravitational potential generated by the first point object, and \( T_{20}^0 \) is the \((0,0)\) component of the energy-momentum tensor of the second object,
\[ T_{20}^0(\vec{y}) = -m \delta(\vec{y} - \vec{x}). \] (10.12)
The final result for gravitational potential energy between two point objects is
\[ V(r) = -\frac{g^2 Mm}{4\pi r}. \] (10.13)
The gravitational potential energy between two point objects is always negative, which is what expected by Newton’s theory of gravity and is the inevitable result of the attractive nature of gravitational interactions.

The gravitational force that the first point object acts on the second point object is
\[ \vec{f} = -\nabla V(r) = -\frac{g^2 Mm}{4\pi r^2} \hat{r}, \] (10.14)
where \( \hat{r} = \vec{r} / r \). This is the famous formula of Newton’s gravitational force. Therefore, in the classical limit, the gravitational gauge theory can return to Newton’s
theory of gravity. Besides, from eq.(10.14), we can clearly see that the gravitational interaction force between two point objects is attractive.

Now, we want to ask a problem: why in QED, the force between two like electric charges is always repulsive, while in gravitational gauge theory, the force between two like gravitational charges can be attractive? A simple answer to this fundamental problem is that the attractive nature of the gravitational force is an inevitable result of the global Lorentz symmetry of the system. Because of the requirement of global Lorentz symmetry, the Lagrangian function given by eq.(5.31) must use \( g_{\alpha\beta} \), can not use the ordinary \( \delta \)-function \( \delta_{\alpha\beta} \). It can be easily prove that, if we use \( \delta_{\alpha\beta} \) instead of \( g_{\alpha\beta} \) in eq.(5.31), the Lagrangian of pure gravitational gauge field is not invariant under global Lorentz transformation. On the other hand, if we use \( \delta_{\alpha\beta} \) instead of \( \eta_{\alpha\beta} \) in eq.(5.31), the gravitational force will be repulsive which obviously contradicts with experiment results. In QED, \( \delta_{ab} \) is used to construct the Lagrangian for electromagnetic fields, therefore, the interaction force between two like electric charges is always repulsive.

One fundamental influence of using the metric \( g_{\alpha\beta} \) in the Lagrangian of pure gravitational field is that the kinematic energy term of gravitation field \( C_{\mu}^0 \) is always negative. According to eq.(5.31), the free lagrangian of pure gravitational gauge field is

\[
\mathcal{L}_0 F = -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{\alpha\beta} F_{\mu\nu}^{\alpha} F_{\rho\sigma}^{\beta}.
\] (10.15)

In above relation, \( \eta^{00} \) is negative which causes that the kinematic energy of gravitation field \( C_{\mu}^0 \) is negative. This result is novels, but it is not surprising, for gravitational interaction energy is always negative. In a meaning, it is the reflection of the negative nature of graviton’s kinematic energy. Though the kinematic energy term of gravitation field \( C_{\mu}^0 \) is always negative, the kinematic energy term of gravitation field \( C_{\mu}^0 \) is always positive. The negative energy problem of graviton does not cause any trouble in quantum gauge theory of gravity. Contrarily, it will help us to understand some puzzle phenomena of Nature. From theoretical point of view, the negative nature of graviton’s kinematic energy is essentially an inevitable result of global Lorentz symmetry. Global Lorentz symmetry of the system, attractive nature of gravitational interaction force and negative nature of graviton’s kinematical energy are essentially related to each other, and they have the same origin in nature.

In general relativity, gravitational field obeys Einstein field equation, which is usually written in the following form,

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu},
\] (10.16)
where $R_{\mu\nu}$ is Ricci tensor, $R$ is curvature, $G$ is Newton gravitational constant and $\lambda$ is cosmology constant. The classical limit of Einstein field equation is

$$\nabla^2 g_{00} = -8\pi G T_{00}. \quad (10.17)$$

Compare this equation with eq.(10.7) and use eq.(5.20), we get

$$g^2 = 4\pi G. \quad (10.18)$$

In order to get eq.(10.18), the following relations are used

$$T_{00} = -T^0_0, \quad g_{00} \simeq -(1 + 2gC_0^0). \quad (10.19)$$

In general relativity, Einstein field equation transforms covariantly under general coordinates transformation, in other words, it is a general covariant equation. In gravitational gauge theory, the system has local gravitational gauge symmetry. From mathematical point of view, general coordinates transformation is equivalent to local gravitational gauge transformation. Therefore, it seems that two theories have the same symmetry. On the other hand, both theories have global Lorentz symmetry.

### 11 Path Integral Quantization of Gravitational Gauge Fields

For the sake of simplicity, in this chapter and the next chapter, we only discuss pure gravitational gauge field. For pure gravitational gauge field, its Lagrangian function is

$$\mathcal{L} = -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} g_{\alpha\beta} J(C) F_{\mu\nu} F_{\rho\sigma}. \quad (11.1)$$

Its space-time integration gives out the action of the system

$$S = \int d^4 x \mathcal{L}. \quad (11.2)$$

This action has local gravitational gauge symmetry. Gravitational gauge field $C^\alpha_\mu$ has $4 \times 4 = 16$ degrees of freedom. But, if gravitons are massless, the system has only $2 \times 4 = 8$ degrees of freedom. There are gauge degrees of freedom in the theory. Because only physical degrees of freedom can be quantized, in order to quantize the system, we have to introduce gauge conditions to eliminate un-physical degrees of freedom. For the sake of convenience, we take temporal gauge conditions

$$C^\alpha_0 = 0, \quad (\alpha = 0, 1, 2, 3). \quad (11.3)$$
In temporal gauge, the generating functional $W[J]$ is given by

$$W[J] = N \int [DC] \left( \prod_{\alpha,x} \delta(C^\alpha(x)) \right) \exp \left\{ i \int d^4x (\mathcal{L} + J^\mu_\alpha C^\alpha_\mu) \right\},$$

(11.4)

where $N$ is the normalization constant, $J^\mu_\alpha$ is a fixed external source and $[DC]$ is the integration measure,

$$[DC] = \prod_{\mu=0}^3 \prod_{\alpha=0}^3 \prod_j \left( \varepsilon dC^\alpha_\mu(\tau_j)/\sqrt{2\pi i\hbar} \right).$$

(11.5)

We use this generation functional as our starting point of the path integral quantization of gravitational gauge field.

Generally speaking, the action of the system has local gravitational gauge symmetry, but the gauge condition has no local gravitational gauge symmetry. If we make a local gravitational gauge transformations, the action of the system is kept unchanged while gauge condition will be changed. Therefore, through local gravitational gauge transformation, we can change one gauge condition to another gauge condition. The most general gauge condition is

$$f^\alpha(C(x)) - \varphi^\alpha(x) = 0,$$

(11.6)

where $\varphi^\alpha(x)$ is an arbitrary space-time function. The Fadeev-Popov determinant $\Delta_f(C)$ [32] is defined by

$$\Delta_f^{-1}(C) \equiv \int [Dg] \prod_{x,\alpha} \delta \left( f^\alpha(g C(x)) - \varphi^\alpha(x) \right),$$

(11.7)

where $g$ is an element of gravitational gauge group, $g C$ is the gravitational gauge field after gauge transformation $g$ and $[Dg]$ is the integration measure on gravitational gauge group

$$[Dg] = \prod_x d^4\epsilon(x),$$

(11.8)

where $\epsilon(x)$ is the transformation parameter of $\hat{U}_\epsilon$. Both $[Dg]$ and $[DC]$ are not invariant under gravitational gauge transformation. Suppose that,

$$[D(g g')] = J_1(g')[Dg],$$

$$[D g C] = J_2(g)[DC].$$

(11.9)

(11.10)
$J_1(g)$ and $J_2(g)$ satisfy the following relations

\[ J_1(g) \cdot J_1(g^{-1}) = 1, \]  
(11.11)

\[ J_2(g) \cdot J_2(g^{-1}) = 1. \]  
(11.12)

It can be proved that, under gravitational gauge transformations, the Fadeev-Popov determinant transforms as

\[ \Delta_f^{-1}(g') = J_1^{-1}(g') \Delta_f^{-1}(C). \]  
(11.13)

Insert eq.(11.7) into eq.(11.4), we get

\[ W[J] = N \int [\mathcal{D}g] \left[ \prod_{\alpha,y} \delta(C_\alpha^\alpha(y)) \right] \cdot \Delta_f(C) \]

\[ \cdot \left[ \prod_{\beta,z} \delta(f^\beta(C(z)) - \varphi^\beta(z)) \right] \cdot \exp \left\{ i \int d^4x (\mathcal{L} + J_\alpha^\mu C_\mu^\alpha) \right\}. \]  
(11.14)

Make a gravitational gauge transformation,

\[ C(x) \rightarrow g^{-1} C(x), \]  
(11.15)

then,

\[ gC(x) \rightarrow gg^{-1} C(x). \]  
(11.16)

After this transformation, the generating functional is changed into

\[ W[J] = N \int [\mathcal{D}g] \int [\mathcal{D}C] \cdot J_1(g) J_2(g^{-1}) \cdot \left[ \prod_{\alpha,y} \delta(g^{-1} C_\alpha^\alpha(y)) \right] \cdot \Delta_f(C) \]

\[ \cdot \left[ \prod_{\beta,z} \delta(f^\beta(C(z)) - \varphi^\beta(z)) \right] \cdot \exp \left\{ i \int d^4x (\mathcal{L} + J_\alpha^\mu g^{-1} C_\mu^\alpha) \right\}. \]  
(11.17)

Suppose that the gauge transformation $g_0(C)$ transforms general gauge condition $f^\beta(C) - \varphi^\beta = 0$ to temporal gauge condition $C_\alpha^\alpha = 0$, and suppose that this transformation $g_0(C)$ is unique. Then two $\delta$-functions in eq.(11.17) require that the integration on gravitational gauge group must be in the neighborhood of $g_0^{-1}(C)$. Therefore eq.(11.17) is changed into

\[ W[J] = N \int [\mathcal{D}C] \Delta_f(C) \cdot \left[ \prod_{\beta,z} \delta(f^\beta(C(z)) - \varphi^\beta(z)) \right] \]

\[ \cdot \exp \left\{ i \int d^4x (\mathcal{L} + J_\alpha^\mu g_0 \cdot C_\mu^\alpha) \right\} \]

\[ \cdot J_1(g_0^{-1}) J_2(g_0) \cdot \int [\mathcal{D}g] \left[ \prod_{\alpha,y} \delta(g^{-1} C_\alpha^\alpha(y)) \right]. \]  
(11.18)
The last line in eq.(11.18) will cause no trouble in renormalization, and if we consider the contribution from ghost fields which will be introduced below, it will becomes a quantity which is independent of gravitational gauge field. So, we put it into normalization constant \( N \) and still denote the new normalization constant as \( N \).

We also change \( J^\mu_\alpha \delta C^\alpha_\mu \) into \( J^\mu_\alpha C^\alpha_\mu \), this will cause no trouble in renormalization. Then we get

\[
W[J] = N \int [DC] \Delta_f(C) \cdot [\Pi_{\beta,z} \delta(f^\beta(C(z)) - \varphi^\beta(z))] 
\cdot \exp \{i \int d^4x(L + J^\mu_\alpha C^\alpha_\mu)\}. 
\tag{11.19}
\]

In fact, we can use this formula as our start-point of path integral quantization of gravitational gauge field, so we need not worried about the influences of the third lines of eq.(11.18).

Use another functional

\[
\exp \left\{ -\frac{i}{2\alpha} \int d^4x \eta_{\alpha\beta} \varphi^\alpha(x) \varphi^\beta(x) \right\}, \tag{11.20}
\]

times both sides of eq.(11.19) and then make functional integration \( \int [D\varphi] \), we get

\[
W[J] = N \int [DC] \Delta_f(C) \cdot \exp \left\{ i \int d^4x(L - \frac{1}{2\alpha} \eta_{\alpha\beta} f^\alpha f^\beta + J^\alpha_\mu C^\alpha_\mu) \right\}. \tag{11.21}
\]

Now, let’s discuss the contribution from \( \Delta_f(C) \) which is related to the ghost fields. Suppose that \( g = \hat{U}_\epsilon \) is an infinitesimal gravitational gauge transformation. Then eq.(5.12) gives out

\[
\delta C^\alpha_\mu(x) = C^\alpha_\mu(x) - \frac{1}{g} D^\alpha_{\mu \sigma} \epsilon^\sigma, \tag{11.22}
\]

where

\[
D^\alpha_{\mu \sigma} = \delta^\alpha_\sigma \partial_\mu - g \delta^\alpha_\sigma C^\beta_\mu \partial_\beta + g \partial_\sigma C^\alpha_\mu. \tag{11.23}
\]

In order to deduce eq.(11.22), the following relation is used

\[
\Lambda^\alpha_\beta = \delta^\alpha_\beta + \partial_\beta \epsilon^\alpha + o(\epsilon^2). \tag{11.24}
\]

\( D_\mu \) can be regarded as the covariant derivate in adjoint representation, for

\[
D_\mu \epsilon = [D_\mu, \epsilon], \tag{11.25}
\]

\[
(D_\mu \epsilon)^\alpha = D^\alpha_{\mu \sigma} \epsilon^\sigma. \tag{11.26}
\]

Using all these relations, we have,

\[
f^\alpha(\delta C(x)) = f^\alpha(C) - \frac{1}{g} \int d^4y \frac{\delta f^\alpha(C(x))}{\delta C^\beta_\mu(y)} D^\beta_{\mu \sigma}(y) \epsilon^\sigma(y) + o(\epsilon^2). \tag{11.27}
\]
Therefore, according to eq.(11.7) and eq.(11.6), we get

\[\Delta f^{-1}(C) = \int [D\epsilon] \prod_{x,\alpha} \delta \left( -\frac{1}{g} \int d^4 y \frac{\delta f^\alpha(C(x))}{\delta C^\beta(y)} D^\beta_{\mu \sigma}(y) \epsilon^\sigma(y) \right). \]  

(11.28)

Define

\[M^\alpha_\sigma(x, y) = -g \frac{\delta}{\delta \epsilon(y)} f^\alpha(gC(x)) = \int d^4 z \frac{\delta f^\alpha(C(x))}{\delta C^\beta(z)} D^\beta_{\mu \sigma}(z) \delta(z - y). \]  

(11.29)

Then eq.(11.28) is changed into

\[\Delta f^{-1}(C) = \int [D\epsilon] \prod_{x,\alpha} \delta \left( -\frac{1}{g} \int d^4 y M^\alpha_\sigma(x, y) \epsilon^\sigma(y) \right) \]  

(11.30)

Therefore,

\[\Delta f(C) = \text{const.} \times (\text{det}M)^{-1}. \]  

(11.31)

Put this constant into normalization constant, then generating functional eq.(11.21) is changed into

\[W[J] = N \int [DC] \text{det}M \cdot \exp \left\{ i \int d^4 x \left( L - \frac{1}{2\alpha} \eta_{\alpha\beta} f^\alpha f^\beta + J^\alpha_{\mu} C^\alpha_{\mu} \right) \right\}. \]  

(11.32)

In order to evaluate the contribution from \(\text{det}M\), we introduce ghost fields \(\eta^\alpha(x)\) and \(\bar{\eta}_\alpha(x)\). Using the following relation

\[\int [D\eta][D\bar{\eta}] \exp \left\{ i \int d^4 x d^4 y \bar{\eta}_\alpha(x) M^\alpha_\beta(x, y) \eta^\beta(y) \right\} = \text{const.} \times \text{det}M \]  

(11.33)

and put the constant into the normalization constant, we can get

\[W[J] = N \int [DC][D\eta][D\bar{\eta}] \exp \left\{ i \int d^4 x \left( L - \frac{1}{2\alpha} \eta_{\alpha\beta} f^\alpha f^\beta + \bar{\eta} \mathbf{M} \eta + J^\alpha_{\mu} C^\alpha_{\mu} \right) \right\}, \]  

(11.34)

where \(\int d^4 x \bar{\eta} \mathbf{M} \eta\) is a simplified notation, whose explicit expression is

\[\int d^4 x \bar{\eta} \mathbf{M} \eta = \int d^4 x d^4 y \bar{\eta}_\alpha(x) M^\alpha_\beta(x, y) \eta^\beta(y). \]  

(11.35)

The appearance of the non-trivial ghost fields is an inevitable result of the non-Able nature of the gravitational gauge group.
Now, let’s take Lorentz covariant gauge condition,

\[ f^\alpha(C) = \partial^\mu C^\alpha_\mu. \]  \tag{11.36}

Then

\[
\int d^4 x \bar{\eta} M \eta = - \int d^4 x (\partial^\mu \bar{\eta}_\alpha(x)) D_\mu^\alpha \beta(x) \eta^\beta(x).
\]  \tag{11.37}

And eq.(11.34) is changed into

\[
W[J, \beta, \bar{\beta}] = N \int [D C][D \eta][D \bar{\eta}] \exp \left\{ i \int d^4 x \left( L - \frac{1}{2\alpha} \eta_{\alpha \beta} f^\alpha f^\beta - (\partial^\mu \bar{\eta}_\alpha) D_\mu^\alpha \beta \eta^\beta + J^\alpha_\alpha C^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha \right) \right\},
\]  \tag{11.38}

where we have introduced external sources \( \eta^\alpha(x) \) and \( \bar{\eta}_\alpha(x) \) of ghost fields.

The effective Lagrangian \( L_{eff} \) is defined by

\[
L_{eff} \equiv L - \frac{1}{2\alpha} \eta_{\alpha \beta} f^\alpha f^\beta - (\partial^\mu \bar{\eta}_\alpha) D_\mu^\alpha \beta \eta^\beta.
\]  \tag{11.39}

\( L_{eff} \) can separate into free Lagrangian \( L_F \) and interaction Lagrangian \( L_I \),

\[
L_{eff} = L_F + L_I,
\]  \tag{11.40}

where

\[
L_F = -\frac{1}{2} \eta^\mu \eta^\nu \eta_{\rho \sigma} g_{\alpha \beta} \left[ (\partial_\mu C^\alpha_\nu)(\partial_\rho C^\beta_\sigma) - (\partial_\mu C^\alpha_\nu)(\partial_\sigma C^\beta_\rho) \right] - \frac{1}{2\alpha} \eta_{\alpha \beta} (\partial^\mu C^\alpha_\mu)(\partial^\nu C^\beta_\nu) - (\partial^\mu \bar{\eta}_\alpha)(\partial^\nu \eta^\alpha),
\]  \tag{11.41}

\[
L_I = -\frac{1}{2} (J(C) g_{\alpha \beta} - \eta_{\alpha \beta}) \eta^\mu \eta^\nu \eta_{\rho \sigma} \left[ (\partial_\mu C^\alpha_\nu)(\partial_\rho C^\beta_\sigma) - (\partial_\mu C^\alpha_\nu)(\partial_\sigma C^\beta_\rho) \right] + g J(C) \eta^\mu \eta^\nu \eta_{\rho \sigma} g_{\alpha \beta} (\partial_\mu C^\alpha_\nu - \partial_\nu C^\alpha_\mu - \partial_\rho C^\beta_\sigma)(\partial_\sigma C^\beta_\rho) - \frac{1}{2} g^2 J(C) \eta^\mu \eta^\nu \eta_{\rho \sigma} g_{\alpha \beta} (C^\beta_\mu \partial_\rho C^\alpha_\nu - C^\beta_\nu \partial_\rho C^\alpha_\mu)(\partial_\sigma C^\beta_\rho) + g (\partial^\mu \bar{\eta}_\alpha) C^\beta_\mu (\partial^\nu \eta^\alpha) - g (\partial^\mu \bar{\eta}_\alpha)(\partial^\nu \eta^\alpha).
\]  \tag{11.42}

From the interaction Lagrangian, we can see that ghost fields do not couple to \( J(C) \). This is the reflection of the fact that ghost fields are not physical fields, they are virtual fields. Besides, the gauge fixing term does not couple to \( J(C) \) either. Using effective Lagrangian \( L_{eff} \), the generating functional \( W[J, \beta, \bar{\beta}] \) can be simplified to

\[
W[J, \beta, \bar{\beta}] = N \int [D C][D \eta][D \bar{\eta}] \exp \left\{ i \int d^4 x (L_{eff} + J^\alpha_\alpha C^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\},
\]  \tag{11.43}
Use eq.(11.41), we can deduce propagator of gravitational gauge fields and ghost fields. First, we change its form to
\[
\int d^4x L_F = \int d^4x \left\{ \frac{1}{2} C_\mu^\alpha \left[ \eta_{\alpha\beta} \left( \eta^{\mu\nu} \partial^2 - \left( 1 - \frac{1}{\alpha} \right) \partial^\mu \partial^\nu \right) \right] C_\nu^\beta + \bar{\eta}_\alpha \partial^2 \eta^\alpha \right\}. \tag{11.44}
\]
Denote the propagator of gravitational gauge field as
\[
-i \Delta^{\alpha\beta}_{F\mu\nu}(x), \tag{11.45}
\]
and denote the propagator of ghost field as
\[
-i \Delta^\alpha_{F\beta}(x). \tag{11.46}
\]
They satisfy the following equation,
\[
- \left[ \eta_{\alpha\beta} \left( \eta^{\mu\nu} \partial^2 - \left( 1 - \frac{1}{\alpha} \right) \partial^\mu \partial^\nu \right) \right] \Delta^{\beta\gamma}_{F\nu\rho}(x) = \delta(x) \delta^\gamma_\alpha \delta^\mu_\rho, \tag{11.47}
\]
\[
- \partial^2 \Delta^\alpha_{F\beta}(x) = \delta^\alpha_\beta \delta(x). \tag{11.48}
\]
Make Fourier transformations to momentum space
\[
\Delta^{\alpha\beta}_{F\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} \widetilde{\Delta}^{\alpha\beta}_{F\mu\nu}(k) \cdot e^{ikx}, \tag{11.49}
\]
\[
\Delta^\alpha_{F\beta}(x) = \int \frac{d^4k}{(2\pi)^4} \widetilde{\Delta}^\alpha_{F\beta}(k) \cdot e^{ikx}, \tag{11.50}
\]
where \(\widetilde{\Delta}^{\alpha\beta}_{F\mu\nu}(k)\) and \(\widetilde{\Delta}^\alpha_{F\beta}(k)\) are corresponding propagators in momentum space. They satisfy the following equations,
\[
\eta_{\alpha\beta} \left[ k^2 \eta^{\mu\nu} - \left( 1 - \frac{1}{\alpha} \right) k^\mu k^\nu \right] \widetilde{\Delta}^{\beta\gamma}_{F\nu\rho}(k) = \delta^\gamma_\alpha \delta^\mu_\rho, \tag{11.51}
\]
\[
k^2 \widetilde{\Delta}^\alpha_{F\beta}(k) = \delta^\alpha_\beta. \tag{11.52}
\]
The solutions to these two equations give out the propagators in momentum space,
\[
-i \widetilde{\Delta}^{\alpha\beta}_{F\mu\nu}(k) = \frac{-i}{k^2 - i\epsilon} \eta^{\alpha\beta} \left[ \eta_{\mu\nu} - \left( 1 - \alpha \right) \frac{k_\mu k_\nu}{k^2 - i\epsilon} \right], \tag{11.53}
\]
\[
-i \widetilde{\Delta}^\alpha_{F\beta}(k) = \frac{-i}{k^2 - i\epsilon} \delta^\alpha_\beta. \tag{11.54}
\]
It can be seen that the forms of these propagators are quite similar to those in traditional non-Able gauge theory. The only difference is that the metric is different.

The interaction Lagrangian $L_I$ is a function of gravitational gauge field $C^\alpha_\mu$ and ghost fields $\eta^\alpha$ and $\bar{\eta}_\alpha$, 

$$L_I = L_I(C, \eta, \bar{\eta}).$$

Then eq.(11.43) is changed into,

$$W[J, \beta, \bar{\beta}] = N \int [DC][D\eta][D\bar{\eta}] \exp \left\{ i \int d^4 x L_I(C, \eta, \bar{\eta}) \right\}$$

$$= \exp \left\{ i \int d^4 x \left( \frac{1}{i} \delta^{\alpha}_\beta \Delta^{\alpha \beta} F_{\mu \nu} (x-y) J_\mu^\beta (y) \right) 
+ \bar{\eta}_\alpha (x) \Delta^{\alpha \beta} (x-y) \eta^\beta (y) \right\}$$

Finally, let’s discuss Feynman rules. Here, we only give out the lowest order interactions in gravitational gauge theory. It is known that, a vertex can involve arbitrary number of gravitational gauge fields. Therefore, it is impossible to list all Feynman rules for all kinds of vertex.

The interaction Lagrangian between gravitational gauge field and ghost field is

$$g (\partial^\mu \bar{\eta}_\alpha) C^\beta_\nu (\partial^\beta \eta^\alpha) - g (\partial^\mu \bar{\eta}_\alpha) (\partial_\sigma C^\alpha_\mu) \eta^\sigma.$$  

This vertex belongs to $C^\alpha_\mu (k) \bar{\eta}_\beta (q) \eta^\delta (p)$ three body interactions, its Feynman rule is

$$-ig\delta^\beta_\delta q^\mu p_\alpha + ig\delta^\beta_\delta q^\mu k_\delta.$$  

The lowest order interaction Lagrangian between gravitational gauge field and Dirac field is

$$g \bar{\psi} \gamma^\mu \partial_\alpha \psi C^\alpha_\mu - g n^\mu_\alpha \bar{\psi} \gamma^\nu \partial_\nu \psi C^\alpha_\mu - gm \delta^\mu_\alpha \bar{\psi} \psi C^\alpha_\mu.$$  

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This vertex belongs to $C_\mu^\alpha(k)\bar{\psi}(q)\psi(p)$ three body interactions, its Feynman rule is

$$-g\gamma^\mu p_\alpha + g\eta_1^{\mu\nu}\gamma^\nu p_\nu - ig\delta^{\mu}_\alpha. \quad (11.61)$$

The lowest order interaction Lagrangian between gravitational gauge field and real scalar field is

$$g\eta^{\mu\nu}C_\mu^\alpha(\partial_\nu\phi)(\partial_\alpha\phi) - \frac{1}{2}g\delta^{\mu}_\alpha C_\mu^\alpha((\partial^\nu\phi)(\partial_\nu\phi) + m^2\phi^2). \quad (11.62)$$

This vertex belongs to $C_\mu^\alpha(k)\phi(q)\phi(p)$ three body interactions, its Feynman rule is

$$-ig\eta^{\mu\nu}(p_\nu q_\alpha + q_\nu p_\alpha) - ig\delta^{\mu}_\alpha(-p^\nu q_\nu + m^2). \quad (11.63)$$

The lowest order interaction Lagrangian between gravitational gauge field and complex scalar field is

$$g\eta^{\mu\nu}C_\mu^\alpha((\partial_\nu\phi)(\partial_\alpha\phi^*) + (\partial_\alpha\phi^*)(\partial_\nu\phi)) - g\delta^{\mu}_\alpha C_\mu^\alpha((\partial^\nu\phi)(\partial_\nu\phi^*) + m^2\phi\phi^*). \quad (11.64)$$

This vertex belongs to $C_\mu^\alpha(k)\phi^*(-q)\phi(p)$ three body interactions, its Feynman rule is

$$ig\eta^{\mu\nu}(p_\nu q_\alpha + q_\nu p_\alpha) - ig\delta^{\mu}_\alpha(p^\nu q_\nu + m^2). \quad (11.65)$$

The lowest order coupling between vector field and gravitational gauge field is

$$g\eta^{\mu\nu}\eta^{\rho\sigma}A_{0\mu\nu}C_\rho^\alpha \partial_\alpha A_\sigma$$

$$+ (g\delta^{\lambda}_\chi C_\chi^\tau(-\frac{1}{4}\eta^{\mu\rho}\eta^{\tau\sigma}A_{0\mu\rho}A_{0\tau\sigma} - \frac{m^2}{2}\eta^{\mu\nu}A_\mu A_\nu), \quad (11.66)$$

where

$$A_{0\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (11.67)$$

This vertex belongs to $C_\mu^\alpha(k)A_\rho(p)A_\sigma(q)$ three body interactions. Its Feynman rule is

$$-ig\eta^{\mu\beta}\eta^{\rho\sigma}(p_\beta q_\alpha + p_\alpha q_\beta) + ig\eta^{\mu\rho}\eta^{\sigma\beta}p_\beta q_\alpha + ig\eta^{\mu\sigma}\eta^{\rho\beta}q_\beta p_\alpha$$

$$+ \frac{i}{2}g\delta^{\lambda}_\alpha \eta^{\rho\beta}\eta^{\sigma\rho}(p_\lambda q_\beta + q_\lambda p_\beta) \quad (11.68)$$

$$- \frac{i}{2}g\delta^{\lambda}_\alpha \eta^{\rho\beta}\eta^{\sigma\rho}p_\nu q_\beta - \frac{i}{2}g\delta^{\mu}_\alpha \eta^{\rho\nu}\eta^{\beta\sigma}q_\nu p_\beta - igm^2\delta^{\mu}_\alpha \eta^{\nu\sigma}. \quad (11.69)$$

The lowest order self coupling of gravitational gauge fields is

$$g\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{\alpha\beta}[(\partial_\mu C_\nu^\alpha)C_\rho^{\beta\gamma}(\partial_\gamma C_\sigma^{\beta\delta}) - (\partial_\mu C_\nu^\alpha)C_\sigma^{\beta\gamma}(\partial_\gamma C_\rho^{\beta\delta})] - \frac{1}{4}(g\delta^{\lambda}_\chi C_\chi^\tau)\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{\alpha\beta}F^{\alpha}_{0\mu\nu}F^{\beta}_{0\rho\sigma} - \frac{3}{2}\eta^{\mu\rho}\eta^{\nu\sigma}\eta^{\tau\rho}C^{\alpha}_{\mu\nu}F^{\alpha}_{0\mu\nu}F^{\beta}_{0\rho\sigma}. \quad (11.69)$$
This vertex belongs to $C^\alpha_\nu(p)C^\beta_\sigma(q)C^\gamma_\rho(r)$ three body interactions. Its Feynman rule is

$$-ig[\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{\alpha\beta}(p_\mu q_\gamma + q_\mu p_\gamma) + \eta^{\mu\sigma}\eta^{\nu\rho}\eta_{\alpha\gamma}(p_\mu r_\beta + r_\mu p_\beta)$$

$$+\eta^{\mu\nu}\eta^{\rho\sigma}\eta_{\gamma\beta}(q_\mu r_\alpha + r_\mu q_\alpha)]$$

$$+ig[\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{\alpha\beta}p_\mu q_\gamma + \eta^{\mu\sigma}\eta^{\nu\rho}\eta_{\alpha\beta}q_\mu p_\gamma + \eta^{\mu\sigma}\eta^{\nu\rho}\eta_{\alpha\gamma}P_\mu r_\beta$$

$$+\eta^{\mu\nu}\eta^{\rho\sigma}\eta_{\alpha\gamma}r_\mu p_\beta + \eta^{\mu\rho}\eta^{\nu\sigma}\eta_{\beta\gamma}q_\mu r_\alpha + \eta^{\mu\sigma}\eta^{\nu\rho}\eta_{\beta\gamma}r_\mu q_\alpha]$$

$$+ig\delta^\gamma_\alpha\eta_{\alpha\beta}(p_\mu q_\mu^{\nu\sigma} - p^\nu q^\sigma) + ig\delta^\nu_\alpha\eta_{\beta\gamma}(q_\mu r_\mu^{\nu\rho} - r^\nu q^\rho)$$

$$+ig\delta^\rho_\beta\eta_{\alpha\gamma}(r_\mu P^{\mu\nu}p^{\nu\sigma} - p^\nu r^\sigma)$$

(11.70)

...
of the original theory. Because most of counterterms come from the factor \( J(C) \), this factor is key important for renormalization. Without this factor, the theory is non-renormalizable. In a word, the gravitational gauge theory is a renormalizable gauge theory. Now, let’s start our discussion on renormalization from the generalized BRST transformations. Our proof is quite similar to the proof of the renormalizability of non-Able gauge field theory.[33, 34, 35, 36, 37, 38]

The generalized BRST transformations are

\[
\delta C_\mu^\alpha = -D_\mu^\alpha \beta \eta^\beta \delta \lambda, \quad (12.1)
\]

\[
\delta \eta^\alpha = g \eta^\sigma (\partial_\sigma \eta^\alpha) \delta \lambda, \quad (12.2)
\]

\[
\delta \bar{\eta}_\alpha = \frac{1}{\alpha} \eta_{\alpha \beta} f^\beta \delta \lambda, \quad (12.3)
\]

\[
\delta \eta^{\mu \nu} = 0, \quad (12.4)
\]

\[
\delta g_{\alpha \beta} = g (g_{\alpha \sigma} (\partial_\beta \eta^\sigma) + g_{\sigma \beta} (\partial_\alpha \eta^\sigma) + \eta^\sigma (\partial_\sigma g_{\alpha \beta})) \delta \lambda, \quad (12.5)
\]

where \( \delta \lambda \) is an infinitesimal Grassman constant. It can be strict proved that the generalized BRST transformations for fields \( C_\mu^\alpha \) and \( \eta^\alpha \) are nilpotent:

\[
\delta (D_\mu^\alpha \beta \eta^\beta) = 0, \quad (12.6)
\]

\[
\delta (\eta^\sigma (\partial_\sigma \eta^\alpha)) = 0. \quad (12.7)
\]

It means that all second order variations of fields vanish.

Using the above transformation rules, it can be strictly proved the generalized BRST transformation for gauge field strength tensor \( F^\alpha_{\mu \nu} \) is

\[
\delta F^\alpha_{\mu \nu} = g \left( -(\partial_\sigma \eta^\alpha) F^\sigma_{\mu \nu} + \eta^\sigma (\partial_\sigma F^\alpha_{\mu \nu}) \right) \delta \lambda, \quad (12.8)
\]

and the transformation for the factor \( J(C) \) is

\[
\delta J(C) = g \left( (\partial_\alpha \eta^\alpha) J(C) + \eta^\alpha (\partial_\alpha J(C)) \right) \delta \lambda. \quad (12.9)
\]

Therefore, under generalized BRST transformations, the Lagrangian \( \mathcal{L} \) given by eq.(11.1) transforms as

\[
\delta \mathcal{L} = g (\partial_\alpha (\eta^\alpha \mathcal{L})) \delta \lambda. \quad (12.10)
\]

It is a total derivative term, its space-time integration vanish, i.e., the action of eq.(11.2) is invariant under generalized BRST transformations,

\[
\delta (\int d^4 x \mathcal{L}) = \delta S = 0. \quad (12.11)
\]
On the other hand, it can be strict proved that

\[
\delta \left( -\frac{1}{2\alpha} \eta_{\alpha\beta} f^\alpha f^\beta + \bar{\eta}_\alpha \partial^\mu D_\mu^{\alpha \sigma} \eta^\sigma \right) = 0.
\] (12.12)

The non-renormalized effective Lagrangian is denoted as \( \mathcal{L}_{\text{eff}}^{[0]} \). It is given by

\[
\mathcal{L}_{\text{eff}}^{[0]} = \mathcal{L} - \frac{1}{2\alpha} \eta_{\alpha\beta} f^\alpha f^\beta + \bar{\eta}_\alpha \partial^\mu D_\mu^{\alpha \sigma} \eta^\sigma.
\] (12.13)

The effective action is defined by

\[
S_{\text{eff}}^{[0]} = \int d^4 x \mathcal{L}_{\text{eff}}^{[0]}.
\] (12.14)

Using eqs.(11.11 - 12.12), we can prove that this effective action is invariant under generalized BRST transformations,

\[
\delta S_{\text{eff}}^{[0]} = 0.
\] (12.15)

This is a strict relation without any approximation. It is known that BRST symmetry plays key role in the renormalization of gauge theory, for it ensures the validity of the Ward-Takahashi identity.

Before we go any further, we have to do another important work, i.e., to prove that the functional integration measure \([\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}]\) is also generalized BRST invariant. We have said before that the functional integration measure \([\mathcal{D}C]\) is not a gauge invariant measure, therefore, it is highly important to prove that \([\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}]\) is a generalized BRST invariant measure. BRST transformation is a kind of transformation which involves both bosonic fields and fermionic fields. For the sake of simplicity, let’s formally denote all bosonic fields as \(B = \{B_i\}\) and denote all fermionic fields as \(F = \{F_i\}\). All fields that are involved in generalized BRST transformation are simply denoted by \((B, F)\). Then, generalized BRST transformation is formally expressed as

\[
(B, F) \rightarrow (B', F').
\] (12.16)

The transformation matrix of this transformation is

\[
J = \begin{pmatrix}
\frac{\partial B_i}{\partial B'_j} & \frac{\partial B_i}{\partial F'_j} \\
\frac{\partial F_i}{\partial B'_j} & \frac{\partial F_i}{\partial F'_j}
\end{pmatrix} = \begin{pmatrix}
a & \alpha \\
\beta & b
\end{pmatrix},
\] (12.17)

where

\[
a = \left( \frac{\partial B_i}{\partial B'_j} \right),
\] (12.18)
\( b = \left( \frac{\partial F_k}{\partial F'_i} \right), \) 
(12.19) 
\( \alpha = \left( \frac{\partial B_i}{\partial F'_j} \right), \) 
(12.20) 
\( \beta = \left( \frac{\partial F_i}{\partial B'_j} \right). \) 
(12.21) 

Matrixes \( a \) and \( b \) are bosonic square matrix while \( \alpha \) and \( \beta \) generally are not square matrix. In order to calculate the Jacobian \( \det(J) \), we realize the transformation (12.16) in two steps. The first step is a bosonic transformation 
\[
(B, F) \rightarrow (B', F).
\]
(12.22)

The transformation matrix of this transformation is denoted as \( J_1 \), 
\[
J_1 = \begin{pmatrix}
   a - \alpha b^{-1} \beta & \alpha b^{-1} \\
   0 & 1
\end{pmatrix}.
\]
(12.23)

Its Jacobian is 
\[
\det J_1 = \det(a - \alpha b^{-1} \beta).
\]
(12.24)

Therefore, 
\[
\int \prod_i dB_i \prod_k dF_k = \int \prod_i dB'_i \prod_k dF'_k \cdot \det(a - \alpha b^{-1} \beta).
\]
(12.25)

The second step is a fermionic transformation, 
\[
(B', F) \rightarrow (B', F').
\]
(12.26)

Its transformation matrix is denoted as \( J_2 \), 
\[
J_2 = \begin{pmatrix}
   1 & 0 \\
   \beta & b
\end{pmatrix}.
\]
(12.27)

Its Jacobian is the inverse of the determinant of the transformation matrix, 
\[
(det \ J_2)^{-1} = (det \ b)^{-1}.
\]
(12.28)

Using this relation, eq.(12.25) is changed into 
\[
\int \prod_i dB_i \prod_k dF_k = \int \prod_i dB'_i \prod_k dF'_k \cdot \det(a - \alpha b^{-1} \beta)(det \ b)^{-1}.
\]
(12.29)

For generalized BRST transformation, all non-diagonal matrix elements are proportional to Grassman constant \( \delta \lambda \). Non-diagonal matrix \( \alpha \) and \( \beta \) contains only non-diagonal matrix elements, so, 
\[
\alpha b^{-1} \beta \propto (\delta \lambda)^2 = 0.
\]
(12.30)
It means that

\[
\int \prod_i dB_i \prod_k dF_k = \int \prod_i dB'_i \prod_k dF'_k \cdot \det(a) \cdot (\det b)^{-1}. \tag{12.31}
\]

Generally speaking, \(C_\mu^\alpha\) and \(\partial_\nu C_\mu^\alpha\) are independent degrees of freedom, so are \(\eta^\alpha\) and \(\partial_\nu \eta^\alpha\). Using eqs. (12.1 - 12.3), we obtain

\[
(det a^{-1}) = \det \left[ (\delta^\alpha_\beta + g(\partial_\beta \eta^\alpha)\delta_\nu^\mu) \right]
\]

\[
= \prod_{\mu, \alpha, x} \left[ (\delta^\alpha_\alpha + g(\partial_\alpha \eta^\alpha)\delta_\lambda^\mu) \right]
\]

\[
= \prod_x (1 + g(\partial_\alpha \eta^\alpha)\delta_\lambda).
\] \hspace{1cm} (12.32)

\[
(det b^{-1}) = \det \left( \delta^\alpha_\beta + g(\partial_\beta \eta^\alpha)\delta_\lambda \right)
\]

\[
= \prod_x (1 + g(\partial_\alpha \eta^\alpha)\delta_\lambda).
\] \hspace{1cm} (12.33)

In the second line of eq.(12.32), there is no summation over the repeated \(\alpha\) index. Using these two relations, we have

\[
\det(a) \cdot (\det b)^{-1} = \prod_x 1 = 1. \tag{12.34}
\]

Therefore, under generalized BRST transformation, functional integrational measure \([D C][D \eta][D \bar{\eta}]\) is invariant,

\[
[D C][D \eta][D \bar{\eta}] = [D C'][D \eta'][D \bar{\eta}']. \tag{12.35}
\]

Though both \([D C]\) and \([D \eta]\) are not invariant under generalized BRST transformation, their product is invariant under generalized BRST transformation. This result is interesting and important.

The generating functional \(W^{[0]}[J]\) is

\[
W^{[0]}[J] = N \int [D C][D \eta][D \bar{\eta}] exp \left\{ i \int d^4 x (L^{[0]}_{eff} + J_\alpha^\mu C_\mu^\alpha) \right\}. \tag{12.36}
\]

Because

\[
\int d\eta^\beta \bar{\eta}^\sigma \cdot \bar{\eta}^\alpha \cdot f(\eta, \bar{\eta}) = 0, \tag{12.37}
\]

where \(f(\eta, \bar{\eta})\) is a bilinear function of \(\eta\) and \(\bar{\eta}\), we have

\[
\int [D C][D \eta][D \bar{\eta}] \cdot \bar{\eta}^\alpha(x) \cdot \exp \left\{ i \int d^4 y (L^{[0]}_{eff}(y) + J_\alpha^\mu(y) C_\mu^\alpha(y)) \right\} = 0. \tag{12.38}
\]
If all fields are the fields after generalized BRST transformation, eq. (12.38) still holds, i.e.

$$\int [DC][D\eta][D\bar{\eta}] \cdot \bar{\eta}^\alpha(x) \cdot \exp \left\{ i \int d^4y (L^{[0]}_{eff}(y) + J_\alpha^\mu(y) C_\mu^\alpha(y)) \right\} = 0,$$  \hspace{1cm} (12.39)

where $L^{[0]}_{eff}$ is the effective Lagrangian after generalized BRST transformation. Both functional integration measure and effective action $\int d^4y L^{[0]}_{eff}(y)$ are generalized BRST invariant, so, using eqs. (12.1 - 12.3), we get

$$\int [DC][D\eta][D\bar{\eta}] \left[ \frac{1}{\alpha} f^\alpha(C(x)) \delta \lambda - i \bar{\eta}^\alpha(C(x)) \int d^4z (J_\beta^\mu(z) D_\mu^\beta(z) \eta^\sigma(z) \delta \lambda) \right]$$

$$\cdot \exp \left\{ i \int d^4y (L^{[0]}_{eff}(y) + J_\alpha^\mu(y) C_\mu^\alpha(y)) \right\} = 0.$$

(12.40)

This equation will lead to

$$\frac{1}{\alpha} f^\alpha \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right) W^{[0]}[J] = \int d^4y \ J_\beta^\mu(y) D_\mu^\beta \left( \frac{1}{i} \frac{\delta}{\delta J(x)} \right) W^{[0]DG}[y, x, J] = 0,$$  \hspace{1cm} (12.41)

where

$$W^{[0]DG}[y, x, J] = Ni \int [DC][D\eta][D\bar{\eta}] \bar{\eta}^\alpha(x) \eta^\sigma(y) \exp \left\{ i \int d^4z (L^{[0]}_{eff} + J_\alpha^\mu C_\mu^\alpha) \right\}.$$

(12.42)

This is the generalized Ward-Takahashi identity for generating functional $W^{[0]}[J]$.

Now, let's introduce the external sources of ghost fields, then the generation functional becomes

$$W^{[0]}[J, \beta, \bar{\beta}] = N \int [DC][D\eta][D\bar{\eta}] \exp \left\{ i \int d^4x (L^{[0]}_{eff} + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta} \eta^\alpha) \right\},$$

(12.43)

In renormalization of the theory, we have to introduce external sources $K_\alpha^\mu$ and $L_\alpha$ of the following composite operators,

$$D_{\mu\beta}^\alpha \eta^\beta, \ g \eta^\sigma (\partial_\sigma \eta^\alpha).$$

(12.44)

Then the effective Lagrangian becomes

$$\tilde{L}^{[0]}(C, \eta, \bar{\eta}, K, L) = L - \frac{1}{2\alpha} \eta_\alpha f^\alpha f^\beta + \bar{\eta}_\alpha \partial^\mu D_{\mu\beta}^\alpha \eta^\beta + K_\alpha^\mu D_{\mu\beta}^\alpha \eta^\beta + g L_\alpha \eta^\sigma (\partial_\sigma \eta^\alpha)$$

$$= L^{[0]}_{eff} + K_\alpha^\mu D_{\mu\beta}^\alpha \eta^\beta + g L_\alpha \eta^\sigma (\partial_\sigma \eta^\alpha).$$

(12.45)
Then,

\[ \tilde{S}^{[0]}[C, \eta, \bar{\eta}, K, L] = \int d^4x \tilde{\mathcal{L}}^{[0]}(C, \eta, \bar{\eta}, K, L). \]  

(12.46)

It is easy to deduce that

\[ \frac{\delta \tilde{S}^{[0]}}{\delta K_\mu^\alpha} = \mathbf{D}_\mu^\alpha \eta^\beta, \]  

(12.47)

\[ \frac{\delta \tilde{S}^{[0]}}{\delta L_\alpha} = g \eta^\sigma (\partial_\sigma \eta^\alpha). \]  

(12.48)

The generating functional now becomes,

\[ W^{[0]}[J, \beta, \bar{\beta}, K, L] = N \int [\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}] \exp \left\{ i \int d^4x (\tilde{\mathcal{L}}^{[0]} + J^{\mu}_\alpha C^\alpha_\mu + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\}. \]  

(12.49)

In previous discussion, we have already proved that \( S^{[0]}_{\text{eff}} \) is generalized BRST invariant. External sources \( K_\mu^\alpha \) and \( L_\alpha \) keep unchanged under generalized BRST transformation. Using nilpotent property of generalized BRST transformation, it is easy to prove that the two new terms \( K_\mu^\alpha \mathbf{D}_\mu^\alpha \eta^\beta \) and \( gL_\alpha \eta^\sigma (\partial_\sigma \eta^\alpha) \) in \( \tilde{\mathcal{L}}^{[0]} \) are also generalized BRST invariant,

\[ \delta (K_\mu^\alpha \mathbf{D}_\mu^\alpha \eta^\beta) = 0, \]  

(12.50)

\[ \delta (gL_\alpha \eta^\sigma (\partial_\sigma \eta^\alpha)) = 0. \]  

(12.51)

Therefore, the action given by (12.46) are generalized BRST invariant,

\[ \delta \tilde{S}^{[0]} = 0. \]  

(12.52)

It gives out

\[ \int d^4x \left\{ - (\mathbf{D}_\mu^\alpha \eta^\beta(x)) \delta \lambda \frac{\delta}{\delta \eta^\beta(x)} + g \eta^\sigma(x) (\partial_\sigma \eta^\alpha(x)) \delta \lambda \frac{\delta}{\delta \eta^\alpha(x)} \right\} \tilde{S}^{[0]} = 0. \]  

(12.53)

Using relations (12.47 - 12.48), we can get

\[ \int d^4x \left\{ \frac{\delta \tilde{S}^{[0]}}{\delta K^\mu_\alpha(x)} \frac{\delta \tilde{S}^{[0]}}{\delta C^\alpha_\mu(x)} + \frac{\delta \tilde{S}^{[0]}}{\delta L_\alpha(x)} \frac{\delta \tilde{S}^{[0]}}{\delta \eta^\alpha(x)} + \frac{1}{\alpha} f^\alpha(x) \frac{\delta \tilde{S}^{[0]}}{\delta \bar{\eta}_\alpha(x)} \right\} \tilde{S}^{[0]} = 0. \]  

(12.54)

On the other hand, from (12.45 - 12.46), we can obtain that

\[ \frac{\delta \tilde{S}^{[0]}}{\delta \bar{\eta}_\alpha(x)} = \partial^\mu \left( \mathbf{D}_\mu^\alpha \eta^\beta(x) \right). \]  

(12.55)
Combine (12.47) with (12.55), we get
\[
\frac{\delta \tilde{S}^{[0]}}{\delta \bar{\eta}_\alpha(x)} = \partial^\mu \left( \frac{\delta \tilde{S}^{[0]}}{\delta K_\alpha^\mu(x)} \right). 
\] (12.56)

In generation functional \( W^{[0]}[J, \beta, \bar{\beta}, K, L] \), all fields are integrated, so, if we set all fields to the fields after generalized BRST transformations, the final result should not be changed, i.e.
\[
\tilde{W}^{[0]}[J, \beta, \bar{\beta}, K, L] = N \int [DC'][D\eta'][D\bar{\eta}']
\cdot \exp \left\{ i \int d^4x \left( \mathcal{L}^{[0]}(C', \eta', \bar{\eta}', K, L) + J^\mu C^\alpha_\mu + \bar{\eta}'_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha \right) \right\}. 
\] (12.57)

Both action (12.46) and functional integration measure \([DC][D\eta][D\bar{\eta}]\) are generalized BRST invariant, so, the above relation gives out
\[
\int [DC][D\eta][D\bar{\eta}] \left\{ i \int d^4x \left( J^\mu \frac{\delta \tilde{S}^{[0]}}{\delta K_\alpha^\mu(x)} - \bar{\beta}_\alpha \frac{\delta \tilde{S}^{[0]}}{\delta L_\alpha(x)} + \frac{1}{\alpha} \bar{\eta}_\alpha \eta f^\alpha \beta^\alpha \right) \right\} 
\cdot \exp \left\{ i \int d^4y (\mathcal{L}^{[0]}(C, \eta, \bar{\eta}, K, L) + J^\mu C^\alpha_\mu + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\} = 0. 
\] (12.58)

On the other hand, because the ghost field \( \bar{\eta}_\alpha \) was integrated in \( W^{[0]}[J, \beta, \bar{\beta}, K, L] \), if we use \( \bar{\eta}'_\alpha \) in the in functional integration, it will not change the generating functional. That is
\[
\tilde{W}^{[0]}[J, \beta, \bar{\beta}, K, L] = N \int [DC][D\eta][D\bar{\eta}] 
\cdot \exp \left\{ i \int d^4x (\mathcal{L}^{[0]}(C, \eta, \bar{\eta}', K, L) + J^\mu C^\alpha_\mu + \bar{\eta}'_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\}. 
\] (12.59)

Suppose that
\[
\bar{\eta}'_\alpha = \bar{\eta}_\alpha + \delta \bar{\eta}_\alpha. 
\] (12.60)

Then (12.59) and (12.49) will gives out
\[
\int [DC][D\eta][D\bar{\eta}] \left\{ \int d^4x \delta \bar{\eta}_\alpha \left( \frac{\delta \tilde{S}^{[0]}}{\delta \bar{\eta}_\alpha(x)} + \beta^\alpha(x) \right) \right\} 
\cdot \exp \left\{ i \int d^4y (\mathcal{L}^{[0]}(C, \eta, \bar{\eta}, K, L) + J^\mu C^\alpha_\mu + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\} = 0. 
\] (12.61)
Because $\delta \bar{\eta}_\alpha$ is an arbitrary variation, from (12.61), we will get
\[
\int [\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}](\frac{\delta \tilde{Z}}{\delta \eta_\alpha(x)}) + \beta^\alpha(x)) \exp \left\{ i \int d^4 y (\mathcal{L}^0(C, \eta, \bar{\eta}, K, L) + J_\mu^\alpha C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\bar{\beta}}_\alpha \eta^\alpha) \right\} = 0.
\] (12.62)

The generating functional of connected Green function is given by
\[
\tilde{Z}^0[J, \beta, \bar{\beta}, K, L] = -i \ln \tilde{W}^0[J, \beta, \bar{\beta}, K, L].
\] (12.63)

After Legendre transformation, we will get the generating functional of irreducible vertex $\tilde{\Gamma}^0[C, \bar{\eta}, \bar{\eta}, K, L]$,
\[
\tilde{\Gamma}^0[C, \bar{\eta}, \bar{\eta}, K, L] = \tilde{Z}^0[J, \beta, \bar{\beta}, K, L] - \int d^4 x \left( J_\mu^\alpha C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\bar{\beta}}_\alpha \eta^\alpha \right).
\] (12.64)

Functional derivative of the generating functional $\tilde{Z}^0$ gives out the classical fields $C_\mu^\alpha, \eta^\alpha$ and $\bar{\eta}_\alpha$,
\[
C_\mu^\alpha = \frac{\delta \tilde{Z}^0}{\delta J_\mu^\alpha},
\] (12.65)
\[
\eta^\alpha = \frac{\delta \tilde{Z}^0}{\delta \beta_\alpha},
\] (12.66)
\[
\bar{\eta}_\alpha = -\frac{\delta \tilde{Z}^0}{\delta \bar{\beta}_\alpha}.
\] (12.67)

Then, functional derivative of the generating functional $\tilde{\Gamma}^0$ gives out external sources $J_\mu^\alpha, \beta_\alpha$ and $\bar{\beta}_\alpha$,
\[
\frac{\delta \tilde{\Gamma}^0}{\delta C_\mu^\alpha} = -J_\mu^\alpha,
\] (12.68)
\[
\frac{\delta \tilde{\Gamma}^0}{\delta \eta^\alpha} = \beta_\alpha,
\] (12.69)
\[
\frac{\delta \tilde{\Gamma}^0}{\delta \bar{\eta}_\alpha} = -\beta^\alpha.
\] (12.70)
Besides, there are two other relations which can be strictly deduced from (12.64),

\[
\frac{\delta \tilde{\Gamma}^{[0]}}{\delta K^\mu_\alpha} = \frac{\delta \tilde{Z}^{[0]}}{\delta K^\mu_\alpha}, \quad (12.71)
\]

\[
\frac{\delta \tilde{\Gamma}^{[0]}}{\delta L_\alpha} = \frac{\delta \tilde{Z}^{[0]}}{\delta L_\alpha}. \quad (12.72)
\]

It is easy to prove that

\[
\frac{i \delta \tilde{S}^{[0]}_{\alpha}(x)}{\delta K^\mu_\alpha} \exp \left\{ i \int d^4y (\mathcal{L}^{[0]}(C, \eta, \bar{\eta}, K, L) + J^\mu_\alpha C^\alpha_\mu + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\}
\]

\[
= \frac{\delta}{\delta K^\mu_\alpha(x)} \exp \left\{ i \int d^4y (\mathcal{L}^{[0]}(C, \eta, \bar{\eta}, K, L) + J^\mu_\alpha C^\alpha_\mu + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\},
\]

\[
\frac{i \delta \tilde{S}^{[0]}_{\alpha}(x)}{\delta L_\alpha} \exp \left\{ i \int d^4y (\mathcal{L}^{[0]}(C, \eta, \bar{\eta}, K, L) + J^\mu_\alpha C^\alpha_\mu + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\}
\]

\[
= \frac{\delta}{\delta L_\alpha(x)} \exp \left\{ i \int d^4y (\mathcal{L}^{[0]}(C, \eta, \bar{\eta}, K, L) + J^\mu_\alpha C^\alpha_\mu + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\}. \quad (12.73)
\]

Using these two relations, we can change (12.58) into

\[
\int [D\bar{\eta}][D\eta] \left\{ \int d^4x \left( J^\mu_\alpha(x) \frac{\delta}{\delta K^\mu_\alpha(x)} - \frac{\bar{\beta}_\alpha(x)}{\delta L_\alpha(x)} + i \frac{\bar{\eta}_\alpha(x)}{\delta L_\alpha(x)} J^\mu_\alpha(x) \right) \right\} \cdot \exp \left\{ i \int d^4y (\mathcal{L}^{[0]}(C, \eta, \bar{\eta}, K, L) + J^\mu_\alpha C^\alpha_\mu + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\} = 0. \quad (12.75)
\]

Using relations (12.68 - 12.70) and definition of generating functional (12.57), we can rewrite this equation into

\[
\int d^4x \left\{ \frac{\delta \tilde{W}^{[0]}}{\delta K^\mu_\alpha(x)} \frac{\delta \tilde{\Gamma}^{[0]}}{\delta C^\alpha_\mu(x)} + \frac{\delta \tilde{W}^{[0]}}{\delta L_\alpha(x)} \frac{\delta \tilde{\Gamma}^{[0]}}{\delta \bar{\eta}_\sigma(x)} + i \frac{\eta_\alpha(x)}{\delta L_\alpha(x)} f^\alpha \left( \frac{1}{i \delta J^\mu_\rho(x)} \right) \frac{\delta \tilde{W}^{[0]}}{\delta \bar{\eta}_\sigma(x)} \right\} = 0. \quad (12.76)
\]

Using (12.63), we can obtain that

\[
\frac{\delta \tilde{W}^{[0]}}{\delta K^\mu_\alpha(x)} = i \frac{\delta \tilde{\Gamma}^{[0]}}{\delta K^\mu_\alpha(x)} \tilde{W}^{[0]}; \quad (12.77)
\]

\[
\frac{\delta \tilde{W}^{[0]}}{\delta L_\alpha(x)} = i \frac{\delta \tilde{\Gamma}^{[0]}}{\delta L_\alpha(x)} \tilde{W}^{[0]}. \quad (12.78)
\]
Then (12.76) is changed into
\[
\int d^4x \left\{ \frac{\delta \tilde{\Gamma}^{[0]}_{\mu}}{\delta K^\mu_\alpha(x)} \frac{\delta \tilde{\Gamma}^{[0]}_\alpha}{\delta C^\alpha_\mu(x)} + \frac{\delta \tilde{\Gamma}^{[0]}_\alpha}{\delta L_\alpha(x)} \frac{\delta \tilde{\Gamma}^{[0]}_\alpha}{\delta \eta^\alpha(x)} + \frac{i}{\alpha} \eta_{\alpha\sigma} f^\alpha \frac{\delta \tilde{\Gamma}^{[0]}_\alpha}{\delta \bar{\eta}^\sigma(x)} \right\} = 0. \tag{12.79}
\]

Using (12.56) and (12.73), (12.62) becomes
\[
\int [D\tilde{C}] [D\eta] [D\bar{\eta}] \left\{ -i \partial^\mu \frac{\delta \tilde{\Gamma}^{[0]}_\alpha}{\delta K^\mu_\alpha(x)} + \beta^\alpha(x) \right\} 
\cdot \exp \left\{ i \int d^4y (\mathcal{L}^{[0]}(C, \eta, \bar{\eta}, K, L) + J_\alpha^\mu C^\alpha_\mu + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\} = 0. \tag{12.80}
\]

In above equation, the factor \(-i \partial^\mu \frac{\delta \tilde{\Gamma}^{[0]}_\alpha}{\delta K^\mu_\alpha(x)} + \beta^\alpha(x)\) can move out of functional integration, then (12.80) gives out
\[
\partial^\mu \frac{\delta \tilde{\Gamma}^{[0]}_\alpha}{\delta K^\mu_\alpha(x)} = \frac{\delta \tilde{\Gamma}^{[0]}_\alpha}{\delta \bar{\eta}^\alpha(x)}. \tag{12.81}
\]

In order to obtain this relation, (12.57), (12.77) and (12.70) are used.

Define
\[
\bar{\Gamma}^{[0]}[C, \bar{\eta}, \eta, K, L] = \tilde{\Gamma}^{[0]}[C, \bar{\eta}, \eta, K, L] + \frac{1}{2\alpha} \int d^4x \eta_{\alpha\beta} f^\alpha f^\beta \tag{12.82}
\]

It is easy to prove that
\[
\frac{\delta \tilde{\Gamma}^{[0]}_\alpha}{\delta K^\mu_\alpha(x)} = \frac{\delta \bar{\Gamma}^{[0]}_\alpha}{\delta K^\mu_\alpha(x)}, \tag{12.83}
\]
\[
\frac{\delta \tilde{\Gamma}^{[0]}_\alpha}{\delta L_\alpha(x)} = \frac{\delta \bar{\Gamma}^{[0]}_\alpha}{\delta L_\alpha(x)}, \tag{12.84}
\]
\[
\frac{\delta \tilde{\Gamma}^{[0]}_\alpha}{\delta \eta^\alpha(x)} = \frac{\delta \bar{\Gamma}^{[0]}_\alpha}{\delta \eta^\alpha(x)}, \tag{12.85}
\]
\[
\frac{\delta \tilde{\Gamma}^{[0]}_\alpha}{\delta \bar{\eta}^\alpha(x)} = \frac{\delta \bar{\Gamma}^{[0]}_\alpha}{\delta \bar{\eta}^\alpha(x)}, \tag{12.86}
\]
\[
\frac{\delta \bar{\Gamma}^{[0]}_\alpha}{\delta C^\alpha_\mu(x)} = \frac{\delta \tilde{\Gamma}^{[0]}_\alpha}{\delta C^\alpha_\mu(x)} - \frac{1}{\alpha} \eta_{\alpha\beta} \partial^\mu f^\beta. \tag{12.87}
\]

Using these relations, (12.81) and (12.79) are changed into
\[
\partial^\mu \frac{\delta \bar{\Gamma}^{[0]}_\alpha}{\delta K^\mu_\alpha(x)} = \frac{\delta \bar{\Gamma}^{[0]}_\alpha}{\delta \bar{\eta}^\alpha(x)}. \tag{12.88}
\]
\[ \int d^4x \left\{ \frac{\delta \Gamma^{[0]}}{\delta K_\alpha^\mu(x)} \frac{\delta \Gamma^{[0]}}{\delta C_\alpha^\mu(x)} + \frac{\delta \Gamma^{[0]}}{\delta L_\alpha(x)} \frac{\delta \Gamma^{[0]}}{\delta \eta^\alpha(x)} \right\} = 0. \quad (12.89) \]

Eqs. (12.88 - 12.89) are generalized Ward-Takahashi identities of generating functional of regular vertex. It is the foundations of the renormalization of the gravitational gauge theory.

Generating functional \( \sim \Gamma^{[0]} \) is the generating functional of regular vertex with external sources, which is constructed from the Lagrangian \( \sim L_{\text{eff}} \). It is a functional of physical field, therefore, we can make a functional expansion

\[ \sim \Gamma^{[0]} = \sum_n \int \frac{\delta^n \sim \Gamma^{[0]}}{\delta C_{\mu_1}^\alpha(x_1) \cdots \delta C_{\mu_n}^\alpha(x_n)} \Big|_{C=\eta=\bar{\eta}=0} C^{\alpha_1}(x_1) \cdots C^{\alpha_n}(x_n) d^4x_1 \cdots d^4x_n \]

\[ + \sum_n \int \frac{\delta^2 \sim \Gamma^{[0]}}{\delta \eta_{\beta_1}(y_1) \delta \eta^2_{\beta_2}(y_2) \delta C_{\mu_1}^\alpha(x_1) \cdots \delta C_{\mu_n}^\alpha(x_n)} \Big|_{C=\eta=\bar{\eta}=0} \eta_{\beta_1}(y_1) \eta^2_{\beta_2}(y_2) C^{\alpha_1}(x_1) \cdots C^{\alpha_n}(x_n) d^4y_1 d^4y_2 d^4x_1 \cdots d^4x_n \]

\[ + \cdots. \quad (12.90) \]

In this functional expansion, the expansion coefficients are regular vertexes with external sources. Before renormalization, these coefficients contain divergences. If we calculate these divergences in the methods of dimensional regularization, the form of these divergence will not violate gauge symmetry of the theory [40, 41]. In other words, in the method of dimensional regularization, gravitational gauge symmetry is not violated and the generating functional of regular vertex satisfies Ward-Takahashi identities (12.88 - 12.89). In order to eliminate the ultraviolet divergences of the theory, we need to introduce counterterms into Lagrangian. All these counterterms are formally denoted by \( \delta L \). Then the renormalized Lagrangian is

\[ \sim L_{\text{eff}} = \sim L_{\text{eff}}^{[0]} + \delta L. \quad (12.91) \]

Because \( \delta L \) contains all counterterms, \( \sim L_{\text{eff}} \) is the Lagrangian density after complete renormalization. The generating functional of regular vertex which is calculate from \( \sim L_{\text{eff}} \) is denoted by \( \sim \Gamma \). The regular vertexes calculated from this generating functional \( \sim \Gamma \) contain no ultraviolet divergence anymore. Then let external sources \( K_\alpha^\mu \) and \( L_\alpha \) vanish, we will get generating functional \( \Gamma \) of regular vertex without external sources,

\[ \Gamma = \sim \Gamma \big|_{K=L=0}. \quad (12.92) \]

The regular vertexes which are generated from \( \Gamma \) will contain no ultraviolet divergence either. Therefore, the S-matrix for all physical process are finite. For
a renormalizable theory, the counterterm $\delta \mathcal{L}$ only contain finite unknown parameters which are needed to be determined by experiments. If counterterm $\delta \mathcal{L}$ contains infinite unknown parameters, the theory will lost its predictive power and it is conventionally regarded as a non-renormalizable theory. Now, the main task for us is to prove that the counterterm $\delta \mathcal{L}$ for the gravitational gauge theory only contains a few unknown parameters. If we do this, we will have proved that the gravitational gauge theory is renormalizable.

Now, we use inductive method to prove the renormalizability of the gravitational gauge theory. In the previous discussion, we have proved that the generating functional of regular vertex before renormalization satisfies Ward-Takahashi identities (12.88 - 12.89). The effective Lagrangian density that contains all counterterms which cancel all divergences of $l$-loops ($0 \leq l \leq L$) is denoted by $\tilde{\mathcal{L}}^{[L]}$. $\Gamma^{[L]}$ is the generating functional of regular vertex which is calculated from $\tilde{\mathcal{L}}^{[L]}$. The regular vertex which is generated by $\tilde{\Gamma}^{[L]}$ will contain no divergence if the number of the loops of the diagram is not greater than $L$. We have proved that the generating functional $\tilde{\Gamma}^{[L]}$ satisfies Ward-Takahashi identities if $L = 0$. Hypothesis that Ward-Takahashi identities are also satisfied when $L = N$, that is

$$\partial^\mu \frac{\delta \tilde{\Gamma}^{[N]}}{\delta K^{\mu}_{\alpha}(x)} = \frac{\delta \tilde{\Gamma}^{[N]}}{\delta \tilde{\eta}_{\alpha}(x)}, \quad (12.93)$$

$$\int d^4x \left\{ \frac{\delta \tilde{\Gamma}^{[N]}}{\delta K^{\mu}_{\alpha}(x)} \frac{\delta \Gamma^{[N]}}{\delta C^{\rho}_{\mu}(x)} + \frac{\delta \Gamma^{[N]}}{\delta L^{\alpha}(x)} \frac{\delta \Gamma^{[N]}}{\delta \eta^{\rho}(x)} \right\} = 0. \quad (12.94)$$

Our goal is to prove that Ward-Takahashi identities are also satisfied when $L = N + 1$. Now, let’s introduce a special product which is defined by

$$A \ast B \equiv \int d^4x \left\{ \frac{\delta A}{\delta K^{\mu}_{\alpha}(x)} \frac{\delta B}{\delta C^{\rho}_{\mu}(x)} + \frac{\delta A}{\delta L^{\alpha}(x)} \frac{\delta B}{\delta \eta^{\rho}(x)} \right\}. \quad (12.95)$$

Then (12.94) will be simplified to

$$\tilde{\Gamma}^{[N]} \ast \tilde{\Gamma}^{[N]} = 0. \quad (12.96)$$

$\tilde{\Gamma}^{[N]}$ contains all contributions from all possible diagram with arbitrary loops. The contribution from $l$-loop diagram is proportional to $h^l$. We can expand $\tilde{\Gamma}^{[N]}$ as a power serials of $h^l$,

$$\tilde{\Gamma}^{[N]} = \sum_M h^M \tilde{\Gamma}^{[N]}_M, \quad (12.97)$$

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where $\bar{\Gamma}^{[N]}_M$ is the contribution from all $M$-loop diagrams. According to our inductive hypothesis, all $\bar{\Gamma}^{[N]}_M$ are finite is $M \leq N$. Therefore, divergence first appear in $\bar{\Gamma}^{[N]}_{N+1}$. Substitute (12.97) into (12.96), we will get
\[
\sum_{M,L} h^{M+L} \bar{\Gamma}^{[N]}_M * \bar{\Gamma}^{[N]}_L = 0. \tag{12.98}
\]

The $(L + 1)$-loop contribution of (12.98) is
\[
\sum_{M=0}^{N+1} \bar{\Gamma}^{[N]}_M * \bar{\Gamma}^{[N]}_{N-M+1} = 0. \tag{12.99}
\]

$\bar{\Gamma}^{[N]}_{N+1}$ can separate into two parts: finite part $\bar{\Gamma}^{[N]}_{N+1,F}$ and divergent part $\bar{\Gamma}^{[N]}_{N+1,div}$, that is
\[
\bar{\Gamma}^{[N]}_{N+1} = \bar{\Gamma}^{[N]}_{N+1,F} + \bar{\Gamma}^{[N]}_{N+1,div}. \tag{12.100}
\]

$\bar{\Gamma}^{[N]}_{N+1,div}$ is a divergent function of $(4-D)$ if we calculate loop diagrams in dimensional regularization. In other words, all terms in $\bar{\Gamma}^{[N]}_{N+1,div}$ are divergent terms when $(4-D)$ approaches zero. Substitute (12.100) into (12.99), if we only concern divergent terms, we will get
\[
\bar{\Gamma}^{[N]}_{N+1,div} * \bar{\Gamma}^{[N]}_0 + \bar{\Gamma}^{[N]}_0 * \bar{\Gamma}^{[N]}_{N+1,div} = 0. \tag{12.101}
\]

$\bar{\Gamma}^{[N]}_{N+1,F}$ has no contribution to the divergent part. Because $\bar{\Gamma}^{[N]}_0$ represents contribution from tree diagram and counterterms has no contribution to tree diagram, we have
\[
\bar{\Gamma}^{[N]}_0 = \bar{\Gamma}^{[0]}_0. \tag{12.102}
\]

Denote
\[
\bar{\Gamma}_0 = \bar{\Gamma}^{[N]}_0 = \bar{S}^{[0]} + \frac{1}{2\alpha} \int d^4 x \eta_{\alpha\beta} f^\alpha f^\beta. \tag{12.103}
\]

Then (12.101) is changed into
\[
\bar{\Gamma}^{[N]}_{N+1,div} * \bar{\Gamma}_0 + \bar{\Gamma}_0 * \bar{\Gamma}^{[N]}_{N+1,div} = 0. \tag{12.104}
\]

Substitute (12.97) into (12.93), we get
\[
\partial^\mu \frac{\delta \bar{\Gamma}^{[N]}_{N+1}}{\delta K^\mu_\alpha(x)} = \frac{\delta \bar{\Gamma}^{[N]}_{N+1}}{\delta \eta_\alpha(x)}. \tag{12.105}
\]

The finite part $\bar{\Gamma}^{[N]}_{N+1,F}$ has no contribution to the divergent part, so we have
\[
\partial^\mu \frac{\delta \bar{\Gamma}^{[N]}_{N+1,div}}{\delta K^\mu_\alpha(x)} = \frac{\delta \bar{\Gamma}^{[N]}_{N+1,div}}{\delta \eta_\alpha(x)}. \tag{12.106}
\]
The operator $\hat{g}$ is defined by

$$\hat{g} \equiv \int d^4x \left\{ \frac{\delta f_0}{\delta C_{\alpha}^{\mu}(x)} \frac{\delta}{\delta K_{\alpha}^{\mu}(x)} + \frac{\delta f_0}{\delta L_{\alpha}(x)} \frac{\delta}{\delta \eta^\alpha(x)} \right\}.$$  \hspace{1cm} (12.107)

Using this definition, (12.104) simplifies to

$$\hat{g} \bar{\Gamma}_{N+1,\text{div}}^{[N]} = 0.$$  \hspace{1cm} (12.108)

It can be strictly proved that operator $\hat{g}$ is a nilpotent operator, i.e.

$$\hat{g}^2 = 0.$$  \hspace{1cm} (12.109)

Suppose that $f[C]$ is an arbitrary functional of gravitational gauge field $C_{\mu}^{\alpha}$ which is invariant under local gravitational gauge transformation. $f[C]$ is invariant under generalized BRST transformation. The generalized BRST transformation of $f[C]$ is

$$\delta f[C] = - \int d^4x \frac{\delta f[C]}{\delta C_{\mu}^{\alpha}(x)} D_{\mu\beta}^{\alpha} \eta^\beta \delta \lambda.$$  \hspace{1cm} (12.110)

Because $\delta \lambda$ is an arbitrary Grassman variable, (12.110) gives out

$$\delta f[C] = - \int d^4x \frac{\delta f[C]}{\delta C_{\mu}^{\alpha}(x)} D_{\mu\beta}^{\alpha} \eta^\beta.$$  \hspace{1cm} (12.111)

Because $f[C]$ is a functional of only gravitational gauge fields, its functional derivatives to other fields vanish

$$\frac{\delta f[C]}{\delta K_{\alpha}^{\mu}(x)} = 0,$$  \hspace{1cm} (12.112)

$$\frac{\delta f[C]}{\delta L_{\alpha}(x)} = 0,$$  \hspace{1cm} (12.113)

$$\frac{\delta f[C]}{\delta \eta^\alpha(x)} = 0.$$  \hspace{1cm} (12.114)

Using these relations, (12.111) is changed into

$$\delta f[C] = \hat{g} f[C].$$  \hspace{1cm} (12.115)
The generalized BRST symmetry of $f[C]$ gives out the following important property of operator $\hat{g}$, 
\[ \hat{g}f[C] = 0. \tag{12.116} \]

Using two important properties of operator $\hat{g}$ which are shown in eq.(12.109) and eq.(12.116), we could see that the solution of eq.(12.108) can be written in the following form 
\[ \hat{\Gamma}_N^{[N]}_{N+1,\text{div}} = f[C] + \hat{g}f'[C, \eta, \bar{\eta}, K, L], \tag{12.117} \]
$f[C]$ is a gauge invariant functional and $f'[C, \eta, \bar{\eta}, K, L]$ is an arbitrary functional of fields $C_\mu^\alpha(x), \eta^\alpha(x), \bar{\eta}_{\alpha}(x)$ and external sources $K_\mu^\alpha(x)$ and $L_{\alpha}(x)$.

Now, let’s consider constrain from eq.(12.105). Using eq.(12.112) and eq.(12.114), we can see that $f[C]$ satisfies eq.(12.105), so eq.(12.105) has no constrain on $f[C]$. Define a new variable 
\[ B_\mu^\alpha = K_\mu^\alpha - \partial_\mu \bar{\eta}_\alpha. \tag{12.118} \]
$f_1[B]$ is an arbitrary functional of $B$. It can be proved that 
\[ \frac{\delta f_1[B]}{\delta B_\mu^\alpha(x)} = \frac{\delta f_1[B]}{\delta K_\mu^\alpha(x)}, \tag{12.119} \]
\[ \frac{\delta f_1[B]}{\delta \bar{\eta}_\alpha(x)} = \partial_\mu \frac{\delta f_1[B]}{\delta B_\mu^\alpha(x)}. \tag{12.120} \]
Combine these two relations, we will get 
\[ \frac{\delta f_1[B]}{\delta \bar{\eta}_\alpha(x)} = \partial_\mu \frac{\delta f_1[B]}{\delta K_\mu^\alpha(x)}. \tag{12.121} \]
There $f_1[B]$ is a solution to eq.(12.106). Suppose that there is another functional $f_2$ that is given by, 
\[ f_2[K, C, \eta, L] = \int d^4x \ K_\mu^\alpha T_\mu^\alpha(C, \eta, L), \tag{12.122} \]
where $T_\mu^\alpha$ is a conserved current 
\[ \partial_\mu T_\mu^\alpha = 0. \tag{12.123} \]
It can be easily proved that $f_2[K, C, \eta, L]$ is also a solution of eq.(12.105). Because $\hat{\Gamma}_0$ satisfies eq.(12.106) (please see eq.(12.88)), operator $\hat{g}$ commutes with 
\[ \frac{\delta}{\delta \eta_\alpha(x)} - \partial_\mu \frac{\delta}{\delta K_\mu^\alpha(x)}. \]
It means that functional $f'[C, \eta, \bar{\eta}, K, L]$ in eq.(12.117) must satisfy eq.(12.106). According to these discussion, the solution of $f'[C, \eta, \bar{\eta}, K, L]$ has the following form, 
\[ f'[C, \eta, \bar{\eta}, K, L] = f_1[C, \eta, K_\mu^\alpha - \partial_\mu \bar{\eta}_\alpha, L] + \int d^4x \ K_\mu^\alpha T_\mu^\alpha(C, \eta, L). \tag{12.124} \]
In order to determine $f'[C, \eta, \bar{\eta}, K, L]$, we need to study dimensions of various fields and external sources. Set the dimensionality of mass to 1, i.e.

$$D[\hat{P}_\mu] = 1.$$  \hspace{1cm} (12.125)

Then we have

$$D[C^\alpha_\mu] = 1,$$ \hspace{1cm} (12.126)

$$D[d^4x] = -4,$$ \hspace{1cm} (12.127)

$$D[D_\mu] = 1,$$ \hspace{1cm} (12.128)

$$D[\eta] = D[\bar{\eta}] = 1,$$ \hspace{1cm} (12.129)

$$D[K] = D[L] = 2,$$ \hspace{1cm} (12.130)

$$D[g] = -1,$$ \hspace{1cm} (12.131)

$$D[\Gamma^{[N]}_{N+1,div}] = D[S] = 0.$$ \hspace{1cm} (12.132)

Using these relations, we can prove that

$$D[\hat{g}] = 1,$$ \hspace{1cm} (12.133)

$$D[f'] = -1.$$ \hspace{1cm} (12.134)

Define virtual particle number $N_g$ of ghost field $\eta$ is 1, and that of ghost field $\bar{\eta}$ is -1, i.e.

$$N_g[\eta] = 1,$$ \hspace{1cm} (12.135)

$$N_g[\bar{\eta}] = -1.$$ \hspace{1cm} (12.136)

The virtual particle number is a additive conserved quantity, so Lagrangian and action carry no virtual particle number,

$$N_g[S] = N_g[\mathcal{L}] = 0.$$ \hspace{1cm} (12.137)

The virtual particle number $N_g$ of other fields and external sources are

$$N_g[C] = N_g[D_\mu] = 0,$$ \hspace{1cm} (12.138)

$$N_g[g] = 0,$$ \hspace{1cm} (12.139)

$$N_g[\Gamma] = 0,$$ \hspace{1cm} (12.140)

$$N_g[K] = -1,$$ \hspace{1cm} (12.141)

$$N_g[L] = -2.$$ \hspace{1cm} (12.142)

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Using all these relations, we can determine the virtual particle number \( N_g \) of \( \hat{g} \) and \( f' \),

\[
N_g[\hat{g}] = 1, \quad (12.143)
\]

\[
N_g[f'] = -1. \quad (12.144)
\]

According to eq.(12.134) and eq.(12.144), we know that the dimensionality of \( f' \) is \(-1\) and its virtual particle number is also \(-1\). Besides, \( f' \) must be a Lorentz scalar.

Combine all these results, the only two possible solutions of \( f_1[C, \eta, K^\mu_{\alpha} - \partial^\mu \bar{\eta}_\alpha, L] \) in eq.(12.124) are

\[
(K^\mu_{\alpha} - \partial^\mu \bar{\eta}_\alpha)C^\alpha = (\tilde{\epsilon}^\alpha_{\bar{\eta}} L_{\alpha}), \quad (12.145)
\]

\[
\bar{\eta}^\alpha L_{\alpha}. \quad (12.146)
\]

The only possible solution of \( T^\alpha_{\mu} \) is \( C^\alpha_{\mu} \). But in general gauge conditions, \( C^\alpha_{\mu} \) does not satisfy the conserved equation eq.(12.123). Therefore, the solution to \( f'[C, \eta, \bar{\eta}, K, L] \) is linear combination of (12.145) and (12.146), i.e.

\[
f'[C, \eta, \bar{\eta}, K, L] = \int d^4x \left[ \beta_{N+1}(\varepsilon)(K^\mu_{\alpha} - \partial^\mu \bar{\eta}_\alpha)C^\alpha + \gamma_{N+1}(\varepsilon)\bar{\eta}^\alpha L_{\alpha} \right], \quad (12.147)
\]

where \( \varepsilon = (4 - D) \), \( \beta_{N+1}(\varepsilon) \) and \( \gamma_{N+1}(\varepsilon) \) are divergent parameters when \( \varepsilon \) approaches zero. Then using the definition of \( \hat{g} \), we can obtain the following result,

\[
\hat{g} f'[C, \eta, \bar{\eta}, K, L] = -\beta_{N+1}(\varepsilon) \int d^4x \left[ \frac{\delta f_1}{\delta C^\alpha_{\mu}(x)}C^\alpha_{\mu}(x) + \frac{\delta f_1}{\delta K^\mu_{\alpha}(x)}K^\mu_{\alpha}(x) - \bar{\eta}_\alpha \partial^\mu \bar{\eta}^\alpha \right] \\
- \gamma_{N+1}(\varepsilon) \int d^4x \left[ \frac{\delta f_1}{\delta L_{\alpha}(x)}L_{\alpha}(x) + \frac{\delta f_1}{\delta \eta^\alpha(x)}\eta^\alpha(x) \right]. \quad (12.148)
\]

The only possible solution to \( f[C] \) of eq.(12.117) which is constructed only from gravitational gauge fields is the action \( S[C] \) of gauge fields. Therefore, the most general solution of \( \bar{\Gamma}_{N+1, \text{div}} \) is

\[
\bar{\Gamma}_{N+1, \text{div}}^{[N]} = \alpha_{N+1}(\varepsilon)S[C] - \int d^4x \left[ \beta_{N+1}(\varepsilon) \frac{\delta f_1}{\delta C^\alpha_{\mu}(x)}C^\alpha_{\mu}(x) + \beta_{N+1}(\varepsilon) \frac{\delta f_1}{\delta K^\mu_{\alpha}(x)}K^\mu_{\alpha}(x) \\
+ \gamma_{N+1}(\varepsilon) \frac{\delta f_1}{\delta L_{\alpha}(x)}L_{\alpha}(x) + \gamma_{N+1}(\varepsilon) \frac{\delta f_1}{\delta \eta^\alpha(x)}\eta^\alpha(x) - \beta_{N+1}(\varepsilon)\bar{\eta}_\alpha \partial^\mu \bar{\eta}^\alpha \right]. \quad (12.149)
\]

In fact, the action \( S[C] \) is a functional of pure gravitational gauge field. It also contains gravitational coupling constant \( g \). So, we can denote it as \( S[C, g] \).

From eq.(5.31), eq.(5.36) and eq.(5.37), we can prove that the action \( S[C, g] \) has the following important properties,

\[
S[gC, 1] = g^2 S[C, g]. \quad (12.150)
\]
Differentiate both sides of eq.(12.150) with respect to coupling constant $g$, we can get

$$S[C, g] = \frac{1}{2} \int d^4 x . \frac{\delta S[C, g]}{\delta C_\mu^\alpha(x)} C_\mu^\alpha(x) - \frac{1}{2} \frac{\partial S[C, g]}{\partial g}. \tag{12.151}$$

It can be easily proved that

$$\int d^4 x C_\mu^\alpha(x) \frac{\delta}{\delta C_\mu^\alpha(x)} \left[ \int d^4 y \tilde{\eta}_\alpha(y) \partial^\mu D^\alpha_{\mu \beta} \eta^\beta(y) \right]$$

$$= \int d^4 x \left[ \partial^\mu \tilde{\eta}_\beta(x) \partial_\mu \eta^\beta(x) + \tilde{\eta}_\alpha(x) \partial^\mu D^\alpha_{\mu \beta} \eta^\beta(x) \right], \tag{12.152}$$

$$\int d^4 x C_\mu^\alpha(x) \frac{\delta}{\delta C_\mu^\alpha(x)} \left[ \int d^4 y K^\mu_\beta(y) D^\alpha_{\mu \beta} \eta^\beta(y) \right]$$

$$= -\int d^4 x \left[ K^\mu_\alpha(x) \partial_\mu \eta^\alpha(x) - K^\mu_\alpha(x) D^\alpha_{\mu \beta} \eta^\beta(x) \right], \tag{12.153}$$

$$g \frac{\partial}{\partial g} \left[ \int d^4 x \tilde{\eta}_\alpha(x) \partial^\mu D^\alpha_{\mu \beta} \eta^\beta(x) \right]$$

$$= \int d^4 x \left[ \partial^\mu \tilde{\eta}_\beta(x) \partial_\mu \eta^\beta(x) + \tilde{\eta}_\alpha(x) \partial^\mu D^\alpha_{\mu \beta} \eta^\beta(x) \right], \tag{12.154}$$

$$g \frac{\partial}{\partial g} \left[ \int d^4 x K^\mu_\alpha(x) D^\alpha_{\mu \beta} \eta^\beta(x) \right]$$

$$= -\int d^4 x \left[ K^\mu_\alpha(x) \partial_\mu \eta^\alpha(x) - K^\mu_\alpha(x) D^\alpha_{\mu \beta} \eta^\beta(x) \right], \tag{12.155}$$

Using eqs.(12.152 - 12.153), eq.(12.103) and eq.(12.45), we can prove that

$$\int d^4 x \frac{\delta S[C, g]}{\delta C_\mu^\alpha(x)} C_\mu^\alpha(x) = \int d^4 x \frac{\delta S[C, g]}{\delta C_\mu^\alpha(x)} + \int d^4 x \left[ -(\partial^\mu \tilde{\eta}_\alpha(x))(\partial_\mu \eta^\alpha(x)) \right]$$

$$-\tilde{\eta}_\alpha(x) \partial^\mu D^\alpha_{\mu \beta} \eta^\beta(x) + K^\mu_\alpha(x) \partial_\mu \eta^\alpha(x) - K^\mu_\alpha(x) D^\alpha_{\mu \beta} \eta^\beta(x) \right]. \tag{12.157}$$

Similarly, we can get,

$$g \frac{\partial S[C, g]}{\partial g} = g \frac{\partial S[C, g]}{\partial g} + \int d^4 x \left[ -(\partial^\mu \tilde{\eta}_\alpha(x))(\partial_\mu \eta^\alpha(x)) - \tilde{\eta}_\alpha(x) \partial^\mu D^\alpha_{\mu \beta} \eta^\beta(x) + K^\mu_\alpha(x) \partial_\mu \eta^\alpha(x) \right]$$

$$-K^\mu_\alpha(x) D^\alpha_{\mu \beta} \eta^\beta(x) - g L_\alpha(x) \eta^\beta(x)(\partial_\beta \eta^\alpha(x)) \right]. \tag{12.158}$$
Substitute eqs. (12.157 - 12.158) into eq. (12.151), we will get
\[ S[C, g] = \frac{1}{2} \int d^4 x \frac{\delta \Gamma_0}{\delta C^\alpha_\mu(x)} C^\alpha_\mu(x) - \frac{1}{2} g \frac{\partial \Gamma_0}{\partial g} + \int d^4 x \left\{ \frac{1}{2} g L_\alpha(x) \eta^\beta(x) (\partial_\beta \eta^\alpha(x)) \right\}. \] (12.159)

Substitute eq. (12.159) into eq. (12.149). we will get
\[ \Gamma_0^{[N]}_{N + 1, \text{div}} = \int d^4 x \left[ \frac{(\alpha_{N + 1}(\varepsilon) - \beta_{N + 1}(\varepsilon)) C^\alpha_\mu(x)}{2} \frac{\delta \Gamma_0}{\delta C^\mu_\alpha(x)} + \gamma_{N + 1}(\varepsilon) L_\alpha(x) \frac{\delta \Gamma_0}{\delta L_\alpha(x)} + \beta_{N + 1}(\varepsilon) \eta^\alpha(x) \frac{\delta \Gamma_0}{\delta \eta^\alpha(x)} \right] = 0. \] (12.160)

On the other hand, we can prove the following relations
\[ \int d^4 x \eta^\alpha(x) \frac{\delta \Gamma_0}{\delta \eta^\alpha(x)} = \int d^4 y \bar{\eta}_\beta(y) \partial^\mu D^\beta_\mu \eta^\alpha(y), \] (12.161)
\[ \int d^4 x \eta^\alpha(x) \frac{\delta \Gamma_0}{\delta \eta^\alpha(x)} = \int d^4 y K^\mu_\beta(y) D^\beta_\mu \eta^\alpha(y), \] (12.162)
\[ \int d^4 x \eta^\alpha(x) \frac{\delta \Gamma_0}{\delta \eta^\alpha(x)} = 2 \int d^4 x g L_\beta(x) \eta^\alpha(x) (\partial_\alpha \eta^\beta(x)), \] (12.163)
\[ = \int d^4 x \left\{ \bar{\eta}_\beta(x) \partial^\mu D^\beta_\mu \eta^\alpha(x) \right\}. \] (12.164)

Substitute eqs. (12.161 - 12.163) into eq. (12.164), we will get
\[ \int d^4 x \left\{ - \eta^\alpha \frac{\delta \Gamma_0}{\delta \eta^\alpha} + \bar{\eta}_\alpha \frac{\delta \Gamma_0}{\delta \bar{\eta}_\alpha} + K^\mu_\beta \frac{\delta \Gamma_0}{\delta K^\beta_\mu} + 2 L_\alpha \frac{\delta \Gamma_0}{\delta L_\alpha} \right\} = 0. \] (12.165)

Eq. (12.165) times \( \frac{\gamma_{N + 1} - \beta_{N + 1}}{2} \), then add up this results and eq. (12.160), we will get
\[ \Gamma^{[N]}_{N + 1, \text{div}} = \int d^4 x \left[ \frac{\alpha_{N + 1}(\varepsilon) - \beta_{N + 1}(\varepsilon)}{2} C^\alpha_\mu(x) \frac{\delta \Gamma_0}{\delta C^\mu_\alpha(x)} + L_\alpha(x) \frac{\delta \Gamma_0}{\delta L_\alpha(x)} \right] + \gamma_{N + 1}(\varepsilon) \eta^\alpha(x) \frac{\delta \Gamma_0}{\delta \eta^\alpha(x)} + \beta_{N + 1}(\varepsilon) \eta^\alpha(x) \frac{\delta \Gamma_0}{\delta \eta^\alpha(x)} \right] - \frac{\alpha_{N + 1}(\varepsilon) - \beta_{N + 1}(\varepsilon)}{2} g \frac{\delta \Gamma_0}{\delta g}. \] (12.166)
This is the most general form of $\hat{\Gamma}_{N+1,\text{div}}^{[N]}$ which satisfies Ward-Takahashi identities.

According to minimal subtraction, the counterterm that cancel the divergent part of $\hat{\Gamma}_{N+1}^{[N]}$ is just $-\hat{\Gamma}_{N+1,\text{div}}^{[N]}$, that is

$$\hat{S}^{[N+1]} = \hat{S}^{[N]} - \hat{\Gamma}_{N+1,\text{div}}^{[N]} + o(h^{N+2}), \quad (12.167)$$

where the term of $o(h^{N+2})$ has no contribution to the divergences of $(N+1)$-loop diagrams. Suppose that $\hat{\Gamma}_{N+1}^{[N]}$ is the generating functional of regular vertex which is calculated from $\hat{S}^{[N]}$. It can be easily proved that

$$\Gamma_{N+1}^{[N]} = \Gamma_{N+1}^{[N]} - \hat{\Gamma}_{N+1,\text{div}}^{[N]}, \quad (12.168)$$

Using eq.(12.100), we can get

$$\Gamma_{N+1}^{[N]} = \hat{\Gamma}_{N+1}^{[N]}, \quad (12.169)$$

$\Gamma_{N+1}^{[N]}$ contains no divergence which is just what we expected.

Now, let’s try to determine the form of $\hat{S}^{[N+1]}$. Denote the non-renormalized action of the system as $\hat{S}^{[0]} = [C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\mu^\alpha, L_\alpha, g, \alpha]$. (12.170)

As one of the inductive hypothesis, we suppose that the action of the system after $h^N$ order renormalization is

$$\hat{S}^{[N]} = [\sqrt{Z_1^{[N]}}, C_\mu^\alpha, \sqrt{Z_2^{[N]}}, \bar{\eta}_\alpha, \sqrt{Z_3^{[N]}}, \eta^\alpha, \sqrt{Z_4^{[N]}}, K_\mu^\alpha, \sqrt{Z_5^{[N]}}, L_\alpha, Z_g^{[N]} g, Z_\alpha^{[N]} \alpha], \quad (12.171)$$

Substitute eq.(12.166) and eq.(12.171) into eq.(12.167), we obtain

$$\hat{S}^{[N+1]} = \hat{S}^{[0]} \left[ \sqrt{Z_1^{[N]}}, C_\mu^\alpha, \sqrt{Z_2^{[N]}}, \bar{\eta}_\alpha, \sqrt{Z_3^{[N]}}, \eta^\alpha, \sqrt{Z_4^{[N]}}, K_\mu^\alpha, \sqrt{Z_5^{[N]}}, L_\alpha, Z_g^{[N]} g, Z_\alpha^{[N]} \alpha \right]$$

$$+ \int d^4x \left[ \left( \frac{\alpha_{N+1}(\varepsilon)}{2} - \beta_{N+1}(\varepsilon) \right) \left( C_\mu^\alpha(x) \frac{\delta \hat{F}_0}{\delta C_\mu^\alpha(x)} + L_\alpha(x) \frac{\delta \hat{F}_0}{\delta L_\alpha(x)} \right) \right. $$

$$+ \beta_{N+1}(\varepsilon) + \gamma_{N+1}(\varepsilon) \left( \eta^\alpha(x) \frac{\delta \hat{F}_0}{\delta \eta^\alpha(x)} + \bar{\eta}_\alpha(x) \frac{\delta \hat{F}_0}{\delta \bar{\eta}_\alpha(x)} \right. + \left. K_\mu^\alpha(x) \frac{\delta \hat{F}_0}{\delta K_\mu^\alpha(x)} \right) \right]$$

$$+ \frac{\alpha_{N+1}(\varepsilon)}{2} g \frac{\delta \hat{S}_0}{\delta g} + o(h^{N+2}).$$

(12.172)
Using eq.(12.103), we can prove that

\[ \int d^4 x C^\alpha_\mu(x) \frac{\delta \tilde{\Gamma}_0}{\delta C^\alpha_\mu(x)} = \int d^4 x C^\alpha_\mu(x) \frac{\delta S^{[0]}}{\delta C^\alpha_\mu(x)} + 2 \alpha \frac{\partial S^{[0]}}{\partial \alpha}, \]  
\[ (12.173) \]

\[ \int d^4 x L_\alpha(x) \frac{\delta \tilde{\Gamma}_0}{\delta L_\alpha(x)} = \int d^4 x L_\alpha(x) \frac{\delta S^{[0]}}{\delta L_\alpha(x)}, \]  
\[ (12.174) \]

\[ \int d^4 x \bar{\eta}_\alpha(x) \frac{\delta \tilde{\Gamma}_0}{\delta \bar{\eta}_\alpha(x)} = \int d^4 x \bar{\eta}_\alpha(x) \frac{\delta S^{[0]}}{\delta \bar{\eta}_\alpha(x)}, \]  
\[ (12.175) \]

\[ \int d^4 x \eta^\alpha(x) \frac{\delta \tilde{\Gamma}_0}{\delta \eta^\alpha(x)} = \int d^4 x \eta^\alpha(x) \frac{\delta S^{[0]}}{\delta \eta^\alpha(x)}, \]  
\[ (12.176) \]

\[ \int d^4 x K^\mu_\alpha(x) \frac{\delta \tilde{\Gamma}_0}{\delta K^\mu_\alpha(x)} = \int d^4 x K^\mu_\alpha(x) \frac{\delta S^{[0]}}{\delta K^\mu_\alpha(x)}, \]  
\[ (12.177) \]

\[ g \frac{\partial \tilde{\Gamma}_0}{\partial g} = g \frac{\partial S^{[0]}}{\partial g}. \]  
\[ (12.178) \]

Using these relations, eq.(12.172) is changed into,

\[ \tilde{S}^{[N+1]} = \tilde{S}^{[0]} \left[ \sqrt{Z_1^{[N]} C^\alpha_\mu, \sqrt{Z_2^{[N]} \bar{\eta}_\alpha}, \sqrt{Z_3^{[N]} \eta^\alpha}, \sqrt{Z_4^{[N]} K^\mu_\alpha}, \sqrt{Z_5^{[N]} L_\alpha, Z_g^{[N]} g, Z_\alpha^{[N]} \alpha}} \right. \]

\[ - \int d^4 x \left[ \left( \frac{\alpha_{N+1}(\varepsilon)}{2} - \beta_{N+1}(\varepsilon) \right) \left( C^\alpha_\mu(x) \frac{\delta \tilde{S}^{[0]}}{\delta C^\alpha_\mu(x)} + L_\alpha(x) \frac{\delta \tilde{S}^{[0]}}{\delta L_\alpha(x)} \right) \right. \]

\[ + \left. \beta_{N+1}(\varepsilon) - \gamma_{N+1}(\varepsilon) \right] \left( \eta^\alpha(x) \frac{\delta \tilde{S}^{[0]}}{\delta \eta^\alpha(x)} + \bar{\eta}_\alpha(x) \frac{\delta \tilde{S}^{[0]}}{\delta \bar{\eta}_\alpha(x)} + K^\mu_\alpha(x) \frac{\delta \tilde{S}^{[0]}}{\delta K^\mu_\alpha(x)} \right) \]

\[ + \frac{\alpha_{N+1}(\varepsilon)}{2} g \frac{\delta \tilde{S}^{[0]}}{\partial g} - 2 \left( \frac{\alpha_{N+1}(\varepsilon)}{2} - \beta_{N+1}(\varepsilon) \right) \alpha \frac{\delta \tilde{S}^{[0]}}{\delta \alpha} + o(h^{N+2}) \]  
\[ (12.179) \]

We can see that this relation has just the form of first order functional expansion.

Using this relation, we can determine the form of \( \tilde{S}^{[N+1]} \). It is

\[ \tilde{S}^{[N+1]} [C^\alpha_\mu, \bar{\eta}_\alpha, \eta^\alpha, K^\mu_\alpha, L_\alpha, g, \alpha] \]

\[ = \tilde{S}^{[0]} \left[ \sqrt{Z_1^{[N+1]} C^\alpha_\mu, \sqrt{Z_2^{[N+1]} \bar{\eta}_\alpha}, \sqrt{Z_3^{[N+1]} \eta^\alpha}, \sqrt{Z_4^{[N+1]} K^\mu_\alpha}, \sqrt{Z_5^{[N+1]} L_\alpha, Z_g^{[N+1]} g, Z_\alpha^{[N+1]} \alpha}} \right] \]
\[ (12.180) \]
where
\[
\sqrt{Z_1^{[N+1]}} = \sqrt{Z_6^{[N+1]}} = \sqrt{Z_7^{[N+1]}} = 1 - \sum_{L=1}^{N+1} \left( \frac{\alpha_L(\varepsilon)}{2} - \beta_L(\varepsilon) \right),
\]
(12.181)

\[
\sqrt{Z_2^{[N+1]}} = \sqrt{Z_3^{[N+1]}} = \sqrt{Z_4^{[N+1]}} = 1 - \sum_{L=1}^{N+1} \frac{\beta_L(\varepsilon) + \gamma_L(\varepsilon)}{2},
\]
(12.182)

\[
Z_g^{[N+1]} = 1 + \sum_{L=1}^{N+1} \frac{\alpha_L(\varepsilon)}{2}.
\]
(12.183)

Denote
\[
\sqrt{Z^{[N+1]}} \triangleq \sqrt{Z_1^{[N+1]}} = \sqrt{Z_6^{[N+1]}} = \sqrt{Z_7^{[N+1]}},
\]
(12.184)

\[
\sqrt{Z^{[N+1]}} \triangleq \sqrt{Z_2^{[N+1]}} = \sqrt{Z_3^{[N+1]}} = \sqrt{Z_4^{[N+1]}},
\]
(12.185)

The eq.(12.180) is changed into
\[
\sim^{[N+1]} \left[ \begin{array}{cccc} C^\alpha_{\mu}, \bar{\eta}_\alpha, \eta^\alpha, K^\mu_\alpha, L_\alpha, g, \alpha \end{array} \right] 
= S^{[0]} \left[ \sqrt{Z^{[N+1]}} C^\alpha_{\mu}, \sqrt{\sim^{[N+1]}} \bar{\eta}_\alpha, \sqrt{\sim^{[N+1]}} \eta^\alpha, \sqrt{Z^{[N+1]}} K^\mu_\alpha, \sqrt{Z^{[N+1]}} L_\alpha, Z_g^{[N+1]} g, Z^{[N+1]} \alpha \right].
\]
(12.186)

Using eq.(12.186), we can easily prove that
\[
\bar{\Gamma}^{[N+1]} \left[ \begin{array}{cccc} C^\alpha_{\mu}, \bar{\eta}_\alpha, \eta^\alpha, K^\mu_\alpha, L_\alpha, g, \alpha \end{array} \right] 
= \bar{\Gamma}^{[0]} \left[ \sqrt{Z^{[N+1]}} C^\alpha_{\mu}, \sqrt{\sim^{[N+1]}} \bar{\eta}_\alpha, \sqrt{\sim^{[N+1]}} \eta^\alpha, \sqrt{Z^{[N+1]}} K^\mu_\alpha, \sqrt{Z^{[N+1]}} L_\alpha, Z_g^{[N+1]} g, Z^{[N+1]} \alpha \right].
\]
(12.187)

Now, we need to prove that all inductive hypotheses hold at \( L = N + 1 \). The main inductive hypotheses which is used in the above proof are the following three:

1. the lowest divergence of \( \bar{\Gamma}^{[N]} \) appears in the \((N+1)\)-loop diagram;
2. \( \bar{\Gamma}^{[N]} \) satisfies Ward-Takahashi identities eqs.(12.93-12.94);
3. after \( h^{th} \) order renormalization, the action of the system has the form of eq.(12.171).
First, let’s see the first hypothesis. According to eq.(12.169), after introducing \((N+1)th\) order counterterm, the \((N+1)\)-loop diagram contribution of \(\bar{\Gamma}^{[N+1]}\) is finite. It means that the lowest order divergence of \(\bar{\Gamma}^{[N+1]}\) appears in the \((N+2)\)-loop diagram. So, the first inductive hypothesis hold when \(L = N+1\).

It is known that the non-renormalized generating functional of regular vertex

\[
\bar{\Gamma}^{[0]} = \bar{\Gamma}^{[0]}[C, \bar{\eta}, \eta, K, L, g, \alpha] \tag{12.188}
\]

satisfies Ward-Takahashi identities eqs.(12.88-12.89). If we define

\[
\bar{\Gamma}' = \bar{\Gamma}^{[0]}[C', \bar{\eta}', \eta', K', L', g', \alpha'], \tag{12.189}
\]

then, it must satisfy the following Ward-Takahashi identities

\[
\partial_{\mu} \frac{\delta \bar{\Gamma}'}{\delta K'_{\alpha}^{\mu}(x)} = \frac{\delta \bar{\Gamma}'}{\delta \bar{\eta}'_{\alpha}(x)}, \tag{12.190}
\]

\[
\int d^4 x \left\{ \frac{\delta \bar{\Gamma}'}{\delta K'_{\alpha}^{\mu}(x)} \frac{\delta \bar{\Gamma}'}{\delta C'_{\alpha}^{\mu}(x)} + \frac{\delta \bar{\Gamma}'}{\delta L'_{\alpha}(x)} \frac{\delta \bar{\Gamma}'}{\delta \eta'_{\alpha}(x)} \right\} = 0. \tag{12.191}
\]

Set,

\[
C'^{\alpha}_{\mu} = \sqrt{Z^{[N+1]}} C^{\alpha}_{\mu}, \tag{12.192}
\]

\[
K'^{\alpha}_{\mu} = \sqrt{Z^{[N+1]}} K^{\alpha}_{\mu}, \tag{12.193}
\]

\[
L'_{\alpha} = \sqrt{Z^{[N+1]}} L_{\alpha}, \tag{12.194}
\]

\[
\eta'^{\alpha} = \sqrt{Z^{[N+1]}} \eta^{\alpha}, \tag{12.195}
\]

\[
\bar{\eta}'_{\alpha} = \sqrt{Z^{[N+1]}} \bar{\eta}_{\alpha}, \tag{12.196}
\]

\[
g' = Z^{[N+1]} g, \tag{12.197}
\]

\[
\alpha' = Z^{[N+1]} \alpha. \tag{12.198}
\]

In this case, we have

\[
\bar{\Gamma}' = \bar{\Gamma}^{[0]} \left[ \sqrt{Z^{[N+1]}} C^{\alpha}_{\mu}, \sqrt{Z^{[N+1]}} \bar{\eta}_{\alpha}, \sqrt{Z^{[N+1]}} \eta^{\alpha}, \right.
\]

\[
\left. \sqrt{Z^{[N+1]}} K^{\alpha}_{\mu}, \sqrt{Z^{[N+1]}} L_{\alpha}, Z^{[N+1]} g, Z^{[N+1]} \alpha, , \eta_1, \eta_2 \right] \tag{12.199}
\]

\[
= \bar{\Gamma}^{[N+1]}.
\]
Then eq.(12.190) is changed into
\[ \frac{1}{\sqrt{Z^{[N+1]}}} \frac{\partial^\mu}{\partial K_\mu^\alpha(x)} = \frac{1}{\sqrt{Z^{[N+1]}}} \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \eta_\alpha(x)}. \] (12.200)

Because \( \frac{1}{\sqrt{Z^{[N+1]}}} \) does not vanish, the above equation gives out
\[ \partial^\mu \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta K_\mu^\alpha(x)} = \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \eta_\alpha(x)}. \] (12.201)

Eq.(12.191) gives out
\[ \int d^4x \left\{ \frac{1}{\sqrt{Z^{[N+1]}}} \left[ \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta K_\mu^\alpha(x)} \right] \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta C_\mu^\alpha(x)} + \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \eta_\alpha(x)} \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \eta_\alpha(x)} \right\} = 0. \] (12.202)

Because \( \frac{1}{\sqrt{Z^{[N+1]}}} \) does not vanish, we can obtain
\[ \int d^4x \left\{ \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta K_\mu^\alpha(x)} \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta C_\mu^\alpha(x)} + \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \eta_\alpha(x)} \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \eta_\alpha(x)} \right\} = 0. \] (12.203)

Eq.(12.201) and eq.(12.203) are just the Ward-Takahashi identities for \( L = N + 1 \). Therefore, the second inductive hypothesis holds when \( L = N + 1 \).

The third inductive hypothesis has already been proved which is shown in eq.(12.186). Therefore, all three inductive hypothesis hold when \( L = N + 1 \). According to inductive principle, they will hold when \( L \) is an arbitrary non-negative number, especially they hold when \( L \) approaches infinity.

In above discussions, we have proved that, if we suppose that when \( L = N \) eq.(12.171) holds, then it also holds when \( L = N + 1 \). According to inductive principle, we know that eq.(12.186 - 12.187) hold for any positive integer \( N \), that is
\[ \tilde{S} \left[ C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\mu^\alpha, \eta \right] \\
= \tilde{S}^{[0]} \left[ \sqrt{Z} C_\mu^\alpha, \sqrt{Z} \bar{\eta}_\alpha, \sqrt{Z} \eta^\alpha, \sqrt{Z} K_\mu^\alpha, \sqrt{Z} L_\alpha, Z g, Z \right], \] (12.204)

\[ \bar{\Gamma} \left[ C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\mu^\alpha, \eta \right] \\
= \bar{\Gamma}^{[0]} \left[ \sqrt{Z} C_\mu^\alpha, \sqrt{Z} \bar{\eta}_\alpha, \sqrt{Z} \eta^\alpha, \sqrt{Z} K_\mu^\alpha, \sqrt{Z} L_\alpha, Z g, Z \right], \] (12.205)
where
\[
\sqrt{Z} \triangleq \lim_{N \to \infty} \sqrt{Z^N} = 1 - \sum_{L=1}^{\infty} \left( \frac{\alpha_L(\varepsilon)}{2} - \beta_L(\varepsilon) \right),
\] (12.206)
\[
\sqrt{Z} \triangleq \lim_{N \to \infty} \sqrt{Z^N} = 1 - \sum_{L=1}^{\infty} \left( \frac{\beta_L(\varepsilon) + \gamma_L(\varepsilon)}{2} \right),
\] (12.207)
\[
Z_g \triangleq \lim_{N \to \infty} Z^N_g = 1 + \sum_{L=1}^{\infty} \frac{\alpha_L(\varepsilon)}{2}.
\] (12.208)

\(\tilde{S}[C_{\mu}, \bar{\eta}_\alpha, \eta^\alpha, K_{\alpha}^\mu, L_\alpha, g, \alpha]\) and \(\tilde{\Gamma}[C^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K^\mu_{\alpha}, L_\alpha, g, \alpha]\) are renormalized action and generating functional of regular vertex. The generating functional of regular vertex \(\tilde{\Gamma}\) contains no divergence. All kinds of vertex that generated from \(\tilde{\Gamma}\) are finite. From eq.(12.204) and eq.(12.205), we can see that we only introduce three known parameters which are \(\sqrt{Z}\), \(\sqrt{Z}\) and \(Z_g\). Therefore, gravitational gauge theory is a renormalizable theory.

From eq.(12.205) and eqs.(12.88 - 12.89), we can deduce that the renormalized generating functional of regular vertex satisfies Ward-Takahashi identities,
\[
\partial^\mu \frac{\delta \tilde{\Gamma}}{\delta K^\mu_{\alpha}(x)} = \frac{\delta \tilde{\Gamma}}{\delta \bar{\eta}_\alpha(x)},
\] (12.209)
\[
\int d^4x \left\{ \frac{\delta \tilde{\Gamma}}{\delta K^\mu_{\alpha}(x)} \frac{\delta \tilde{\Gamma}}{\delta C_{\mu}^\alpha(x)} + \frac{\delta \tilde{\Gamma}}{\delta L_\alpha(x)} \frac{\delta \tilde{\Gamma}}{\delta \eta^\alpha(x)} \right\} = 0.
\] (12.210)

It means that the renormalized theory has the structure of gauge symmetry. If we define
\[
C_{0\mu}^\alpha = \sqrt{Z} C_{\mu}^\alpha,
\] (12.211)
\[
\eta_0^\alpha = \sqrt{Z} \eta^\alpha,
\] (12.212)
\[
\bar{\eta}_{0\alpha} = \sqrt{Z} \bar{\eta}_\alpha,
\] (12.213)
\[
K_{0\alpha}^\mu = \sqrt{Z} K_{\alpha}^\mu,
\] (12.214)
\[
L_{0\alpha} = \sqrt{Z} L_\alpha,
\] (12.215)
\[
g_0 = Z_g g,
\] (12.216)
\[
\alpha_0 = Z \alpha.
\] (12.217)

Therefore, eqs.(12.204 - 12.205) are changed into
\[
\tilde{S}[C_{\mu}^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_{\alpha}^\mu, L_\alpha, g, \alpha] = \sim[0] S[0][C_{0\mu}^\alpha, \bar{\eta}_{0\alpha}, \eta_0^\alpha, K_{0\alpha}^\mu, L_{0\alpha}, g_0, \alpha_0],
\] (12.218)
\[
\Gamma[C^\alpha_\mu, \bar{\eta}_\alpha, \eta^\alpha, K^\mu_\alpha, L_\alpha, g, \alpha] = \bar{\Gamma}^{[0]}[C^\alpha_{0\mu}, \bar{\eta}_{0\alpha}, \eta^\alpha_0, K^\mu_{0\alpha}, L_{0\alpha}, g_0, \alpha_0].
\] (12.219)

\(C^\alpha_{0\mu}, \bar{\eta}_{0\alpha}\) and \(\eta^\alpha_0\) are renormalized wave function, \(K^\mu_{0\alpha}\) and \(L_{0\alpha}\) are renormalized external sources, and \(g_0\) is the renormalized gravitational coupling constant.

The action \(\tilde{\Sigma}\) which is given by eq.(12.218) is invariant under the following generalized BRST transformations,

\[
\delta C^\alpha_{0\mu} = -D^\alpha_{0\beta} \eta^\beta_0 \delta \lambda,
\] (12.220)

\[
\delta \eta^\alpha_0 = g_0 \eta^\sigma_0 (\partial_\sigma \eta^\alpha_0) \delta \lambda,
\] (12.221)

\[
\delta \bar{\eta}_{0\alpha} = \frac{1}{\alpha_0} \eta_{\alpha\beta} f^\beta_0 \delta \lambda,
\] (12.222)

\[
\delta \eta^{\mu\nu} = 0,
\] (12.223)

where,

\[
D^\alpha_{0\mu\beta} = \delta^\alpha_\beta \partial_\mu - g_0 \delta^\alpha_\beta C^\sigma_{0\mu} \partial_\sigma + g_0 (\partial_\beta C^\alpha_{0\mu}),
\] (12.224)

\[
f^\alpha_0 = \partial^\mu C^\alpha_{0\mu}.
\] (12.225)

Therefore, the normalized action has generalized BRST symmetry, which means that the normalized theory has the structure of gauge theory.

13 Einstein-like Field Equation with Cosmological Trem

In the above chapters, the quantum gravity is formulated in the traditional framework of quantum field theory, i.e., the physical space-time is always flat and the space-time metric is always selected to be the Minkowski metric. In this picture, gravity is treated as physical interactions in flat physical space-time. Our gravitational gauge transformation does not act on physical space-time coordinates, but act on physical fields, so gravitational gauge transformation does not affect the structure of physical space-time. This is one picture of gravity, or call it one representation of gravity theory. For the sake of simplicity, we call it physical representation of gravity.

There is another representation of gravity which is widely used in Einstein’s general relativity. This representation is essentially a geometrical representation of gravity. For gravitational gauge theory, if we treat \(G^\alpha_{\mu}\) (or \(G^{-1\mu}_{\alpha}\)) as a fundamental physical input of the theory, we can also set up the geometrical representation of gravity[30]. For gravitational gauge theory, the geometrical nature of gravity
essentially originates from the geometrical nature of the gravitational gauge transformation. In the geometrical picture of gravity, gravity is not treated as a kind of physical interactions, but is treated as the geometry of space-time. So, there is no physical gravitational interactions in space-time and space-time is curved if gravitational field does not vanish. In this case, the space-time metric is not Minkowski metric, but \( g^{\alpha\beta} \) (or \( g_{\alpha\beta} \)). In other words, Minkowski metric is the space-time metric if we discuss gravity in physical representation while metric tensor \( g^{\alpha\beta} \) (or \( g_{\alpha\beta} \)) is space-time metric if we discuss gravity in geometrical representation. So, if we use Minkowski metric in our discussion, it means that we are in physical representation of gravity; if we use metric tensor \( g^{\alpha\beta} \) (or \( g_{\alpha\beta} \)) in our discussion, it means that we are in geometrical representation.

In one representation, gravity is treated as physical interactions, while in another representation, gravity is treated as geometry of space-time. But we know that there is only one physics for gravity, so two representations of gravity must be equivalent. This equivalence means that the nature of gravity is physics-geometry duality, i.e., gravity is a kind of physical interactions which has the characteristics of geometry; it is also a geometry of space-time which is essentially a kind of physical interactions. Now, let’s go into the geometrical representation of gravity and use \( g^{\alpha\beta} \) and \( g_{\alpha\beta} \) as space-time metric tensors. In this picture, we can obtain an Einstein-like field equation with cosmological term. In this chapter, we will first calculate out the affine connection, curvature tensor, Ricci tensor and curvature scalar from gravitational gauge field. Then we will deduce the Einstein-like field equation with cosmological constant.

In the previous discussions, we have given out the relations between space-time metric and gravitational gauge fields, which is shown in eq.(5.19 - 5.20). Using these relations, we can change the lagrangian for scalar field into the following form

\[
\mathcal{L} = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \frac{m^2}{2} \phi^2. \tag{13.1}
\]

In this lagrangian, we can not see any gravitational gauge field in it directly. It can be selected as the lagrangian of scalar field in the geometrical representation of gravity. In this case, gravity is treated as geometry of space-time. Generally speaking, in gravitational gauge theory, we can change all lagrangians into the form which is expressed in terms of \( G_\mu^\alpha \). In other words, if we did not discuss quantum behavior of gravity, from mathematical point of view, we can use \( G_\mu^\alpha \) as a fundamental quantity to represent gravitational field. Please note that, in gauge theory of gravity, \( G_\mu^\alpha \) is the quantum counterpart of Cartan tetrad in Cartan geometry. If we use \( G_\mu^\alpha \) as a fundamental quantity of gravity theory, we can set up the geometrical representation of gauge theory of gravity, which is quite similar to the general relativity in Cartan.
Eq. (13.1) gives out the lagrangian in curved space-time. But we can make a local transformation to transform it into flat space-time. Making the following coordinate transformation

$$dx^\mu \rightarrow dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu, \quad (13.2)$$

where \(\frac{\partial x'^\mu}{\partial x^\nu}\) is given by,

$$\frac{\partial x'^\mu}{\partial x^\nu} = (G^{-1})^\mu_\nu. \quad (13.3)$$

It can be proved that

$$g^{\alpha\beta} \rightarrow g'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} g^{\mu\nu} = \eta^{\alpha\beta}. \quad (13.4)$$

Therefore, under this coordinates transformation, the space-time metric becomes flat, in other words, we go into an inertial reference system. In this inertial reference system, the Lagrangian eq. (13.1) becomes

$$\mathcal{L} = -\frac{1}{2} \eta^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \frac{m^2}{2} \phi^2. \quad (13.5)$$

Eq. (13.5) is just the Lagrangian for real scalar fields in flat Minkowski space-time.

Using space-time metric tensors \(g_{\alpha\beta}\) and \(g'^{\alpha\beta}\), we can calculate the affine connection and curvature tensor. The affine connection \(\Gamma^\lambda_\mu^\nu\) is defined by

$$\Gamma^\lambda_\mu^\nu = \frac{1}{2} g^{\lambda\sigma} \left( \frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right). \quad (13.6)$$

Using the following relation,

$$gF^\lambda_\rho_\sigma = G^\nu_\rho G^\mu_\sigma [(G^{-1}\partial_\mu G)_\nu^\lambda - (G^{-1}\partial_\nu G)_\mu^\lambda], \quad (13.7)$$

where \(F^\lambda_\rho_\sigma\) is the component field strength of gravitational gauge field, we get

$$\Gamma^\lambda_\mu^\nu = -\frac{1}{2} [(G^{-1}\partial_\mu G)_\nu^\lambda + (G^{-1}\partial_\nu G)_\mu^\lambda] + \frac{1}{2} \eta^{\alpha_1\beta_1} \eta^{\alpha_3\beta_3} F^\rho_\mu_\beta_3 G^\lambda_\rho_\alpha_1 G^{-1}\beta_3 G^{-1}\mu_1 G^{-1}\beta_1 + G^{-1}\beta_1 G^{-1}\alpha_1 G^{-1}\mu_1 \mu_1. \quad (13.8)$$

From this expression, we can see that, if there is no gravity in space-time, that is

$$C^\alpha_\mu = 0, \quad F^\lambda_\mu_\nu = 0, \quad (13.9)$$

then the affine connection \(\Gamma^\lambda_\mu^\nu\) will vanish, which is what we expect in general relativity.
The curvature tensor $R^\lambda_{\mu\nu\kappa}$ is defined by

$$R^\lambda_{\mu\nu\kappa} \triangleq \partial_\kappa \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\eta_{\mu\nu} \Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa} \Gamma^\lambda_{\nu\eta},$$

(13.10)

the Ricci tensor $R_{\mu\kappa}$ is defined by

$$R_{\mu\kappa} \triangleq R^\lambda_{\mu\lambda\kappa},$$

(13.11)

and the curvature scalar $R$ is defined by

$$R \triangleq g^{\mu\kappa} R_{\mu\kappa}.$$

(13.12)
The explicit expression for Ricci tensor $R_{\mu\kappa}$ is

$$
R_{\mu\kappa} = -\left(\partial_\nu \partial_\mu G \cdot G^{-1}\right)_\alpha^\alpha + 2\left(\partial_\kappa G \cdot G^{-1} \cdot \partial_\mu G \cdot G^{-1}\right)^\lambda_\alpha
+ \eta^{\sigma\rho} \eta_{\alpha\beta} \left(\partial_\mu G \cdot G^{-1}\right)_\rho^\rho \left(\partial_\kappa G \cdot G^{-1}\right)_\sigma^\sigma
+ \frac{1}{2} g^{\nu\rho} \eta_{\alpha\beta} \left(G^{-1} \cdot \partial_\nu \partial_\mu G \cdot G^{-1} \cdot \partial_\kappa G \cdot G^{-1}\right)^\alpha_\beta
+ \frac{1}{2} g^{\lambda\nu} \eta_{\alpha\beta} \left(G^{-1} \cdot \partial_\nu \partial_\lambda G \cdot G^{-1}\right)_\beta^\alpha
+ \frac{1}{2} g^{\lambda\nu} \eta_{\alpha\beta} \left(G^{-1} \cdot \partial_\nu \partial_\lambda G \cdot G^{-1}\right)_\alpha^\beta
+ \frac{1}{2} g^{\nu\rho} \eta_{\alpha\beta} G^{-1}_{\mu\kappa} \left(G^{-1} \cdot \partial_\nu \partial_\kappa G \cdot G^{-1}\right)^\beta_\mu
+ \frac{1}{2} g^{\nu\rho} \eta_{\alpha\beta} G^{-1}_{\mu\kappa} \left(G^{-1} \cdot \partial_\nu \partial_\kappa G \cdot G^{-1}\right)^\beta_\kappa
+ g^{\nu\rho} \eta_{\alpha\beta} \left(G^{-1} \cdot \partial_\nu \partial_\kappa G \cdot G^{-1}\right)^\alpha_\beta
+ \frac{1}{2} \left(G^{-1} \cdot \partial_\kappa G \cdot G^{-1}\right)^\beta_\nu
- \frac{1}{2} \left(G^{-1} \cdot \partial_\kappa G \cdot G^{-1} \cdot \partial_\nu G\right)^\lambda_\nu
+ \frac{1}{2} \left(G^{-1} \cdot \partial_\kappa G \cdot G^{-1} \cdot \partial_\nu G\right)^\lambda_\nu
+ \frac{1}{2} \left(G^{-1} \cdot \partial_\nu G \cdot G^{-1}\right)^\beta_\kappa
+ \frac{1}{2} \left(G^{-1} \cdot \partial_\nu G \cdot G^{-1} \cdot \partial_\kappa G\right)^\beta_\kappa
+ \frac{1}{2} \left(G^{-1} \cdot \partial_\nu G \cdot G^{-1}\right)^\beta_\nu
- \frac{1}{2} \left(G^{-1} \cdot \partial_\nu G \cdot G^{-1} \cdot \partial_\kappa G\right)^\beta_\nu
- \frac{1}{2} \left(G^{-1} \cdot \partial_\nu G \cdot G^{-1} \cdot \partial_\kappa G\right)^\beta_\nu
+ \frac{1}{2} \left(G^{-1} \cdot \partial_\nu G \cdot G^{-1}\right)^\beta_\nu
- \frac{1}{2} \left(G^{-1} \cdot \partial_\nu G \cdot G^{-1} \cdot \partial_\kappa G\right)^\beta_\nu
- \frac{1}{2} \left(G^{-1} \cdot \partial_\kappa G \cdot G^{-1}\right)^\beta_\nu.
$$
The explicit expression for scalar curvature $R$ is

$$
R = 4g^{\mu\kappa}(\partial_\mu G \cdot G^{-1} \cdot \partial_\kappa G \cdot G^{-1})^\alpha_\alpha - 2g^{\mu\kappa}(\partial_\mu \partial_\kappa G \cdot G^{-1})^\alpha_\alpha + \frac{3}{2} \eta^{\alpha\beta}\eta_{\alpha\beta}g^{\mu\kappa}(\partial_\mu G \cdot G^{-1})^\alpha_\mu (\partial_\kappa G \cdot G^{-1})^\beta_\kappa - 2\eta^{\alpha\beta}G^\mu_\gamma (\partial_\mu G \cdot G^{-1})^\gamma_\alpha - 2\eta^{\alpha\beta}G^\kappa_\gamma (\partial_\kappa G \cdot G^{-1})^\gamma_\alpha
$$

$$
-2\eta^{\alpha\beta}G^\gamma_\alpha (\partial_\gamma G \cdot G^{-1})^\alpha_\beta + 2\eta^{\alpha\beta}G^\alpha_\beta (\partial_\alpha \partial_\beta G)^\gamma_\kappa + \frac{5}{2} \eta^{\alpha\beta}\eta^{\gamma\mu}\eta^{\rho\sigma}G^\gamma_\mu G^\rho_\sigma (\partial_\gamma G \cdot G^{-1})^\alpha_\mu (\partial_\rho G \cdot G^{-1})^\beta_\sigma + \frac{1}{2} \eta^{\alpha\beta}\eta^{\gamma\mu}\eta^{\rho\sigma}G^\gamma_\mu G^\rho_\sigma (\partial_\gamma G \cdot G^{-1})^\alpha_\mu (\partial_\rho G \cdot G^{-1})^\beta_\sigma
$$

$$
-2g\eta^{\alpha\beta}F^\rho_{\alpha\beta}G^\rho_\mu (\partial_\mu G \cdot G^{-1})^\alpha_\beta + g\eta^{\alpha\beta}F^\rho_{\alpha\beta}G^\rho_\mu (\partial_\mu G \cdot G^{-1})^\beta_\alpha + g\eta^{\alpha\beta}F^\mu_\beta F^\rho_{\alpha\beta} (G^{-1} \cdot \partial_\nu G \cdot G^{-1})^\alpha_\nu + g^2 \eta^{\alpha\beta}F^\rho_{\alpha\beta}F^\mu_{\alpha\beta} G^\mu_\rho (G^{-1} \cdot G^{-1})^\alpha_\mu - \frac{2}{3} \eta^{\alpha\beta}\eta^{\gamma\mu}\eta^{\rho\sigma}F^\rho_{\alpha\beta}F^\mu_{\alpha\beta} (G^{-1} \cdot G^{-1})^\gamma_\rho (G^{-1} \cdot G^{-1})^\gamma_\mu
$$

$$
- \frac{2}{3} \eta^{\alpha\beta}\eta^{\gamma\mu}\eta^{\rho\sigma}F^\rho_{\alpha\beta}F^\mu_{\alpha\beta} (G^{-1} \cdot G^{-1})^\gamma_\rho (G^{-1} \cdot G^{-1})^\gamma_\mu
$$

(13.14)

From these expressions, we can see that, if there is no gravity, that is $C^\alpha_\mu$ vanishes, then $R_{\mu\nu}$ and $R$ all vanish. It means that, if there is gravity, the space-time is flat, which is what we expected in general relativity.

Because scalar curvature $R$ is invariant under general coordinates transformation, it transforms covariantly under gravitational gauge transformation. Its behavior under gravitational gauge transformation is the same as that of the lagrangian $L_0$. Just from the requirement of gravitational gauge symmetry, the most general lagrangian which gives an action with gravitational gauge symmetry is

$$
L_0 = \frac{c_1}{4} \eta^{\rho\sigma}g_{\alpha\beta}F^\alpha_\mu F^\beta_\rho + c_2 R,
$$

(13.15)

where $c_1$ and $c_2$ are two constant parameters. But, if we include $R$ in our lagrangian, our field equation for gravitational gauge field will become extremely complicated, for the expression of scalar curvature $R$ in terms of gravitational gauge field $C^\alpha_\mu$ is very complicated. So, we did not consider scalar curvature $R$ in this model. Another reason that we did not consider $R$ in this model is that the lagrangian given by eq.(5.31) is the lagrangian expected by gauge field theory and it is enough to give reasonable results and predictions on all possible problems, such as classical limit, classical tests, cosmological constant, cosmological model, etc. In a meaning, the model discuss in this paper is the minimum model for quantum gauge theory of gravity.

Now, let’s discuss some transformation properties of these tensors under general coordinates transformation. Make a special kind of local coordinates translation,

$$
x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu (x')
$$

(13.16)
Under this transformation, the covariant derivative and gravitational gauge fields transform as

\[ D_\mu(x) \rightarrow D'_\mu(x') = \hat{U}_\epsilon(x') D_\mu(x') \hat{U}_\epsilon^{-1}(x'), \] (13.17)

\[ C_\mu(x) \rightarrow C'_\mu(x') = \hat{U}_\epsilon(x') C_\mu(x') \hat{U}_\epsilon^{-1}(x') + \frac{i}{g} \hat{U}_\epsilon(x') \left( \frac{\partial}{\partial x'^\mu} \hat{U}_\epsilon^{-1}(x') \right). \] (13.18)

It can be proved that

\[ G_\mu^\alpha(x) \rightarrow G'^\alpha_\mu(x') = \Lambda_\alpha^\beta G^\beta_\mu(x), \] (13.19)

where

\[ \Lambda_\alpha^\beta = \frac{\partial x'^\alpha}{\partial x^\beta}. \] (13.20)

We can see that Lorentz index \( \mu \) does not take part in transformation. In fact, all Lorentz indexes do not take part in this kind of transformation. Therefore,

\[ \eta^{\mu\nu} \rightarrow \eta'^{\mu\nu} = \eta^{\mu\nu}. \] (13.21)

Using all these relations, we can prove that

\[ g^{\alpha\beta}(x) \rightarrow g'^{\alpha\beta}(x') = \Lambda_\alpha^{\alpha_1} \Lambda_\beta^{\beta_1} g^{\alpha_1\beta_1}(x), \] (13.22)

\[ R_{\alpha\beta\gamma\delta}(x) \rightarrow R'_{\alpha\beta\gamma\delta}(x') = \Lambda_\alpha^{\alpha_1} \Lambda_\beta^{\beta_1} \Lambda_\gamma^{\gamma_1} \Lambda_\delta^{\delta_1} R_{\alpha_1\beta_1\gamma_1\delta_1}(x), \] (13.23)

\[ R_{\alpha\beta}(x) \rightarrow R'_{\alpha\beta}(x') = \Lambda_\alpha^{\alpha_1} \Lambda_\beta^{\beta_1} R_{\alpha_1\beta_1}(x), \] (13.24)

\[ R(x) \rightarrow R'(x') = R(x). \] (13.25)

These transformation properties are just what we expected in general relativity. All these quantities have the same transformation properties as those in general relativity.

Equivalence principle is one of the most important fundamental principles of general relativity. But, as we have studied in previous chapters, the inertial energy-momentum is not equivalent to the gravitational energy-momentum in gravitational gauge theory. This result is an inevitable result of gauge principle. But all these differences are caused by gravitational gauge field. In leading term approximation, the inertial energy-momentum tensor \( T^\mu_{\alpha a} \) is the same as the gravitational energy-momentum tensor \( T^\mu_{\beta a} \). Because gravitational coupling constant \( g \) is extremely small and the strength of gravitational field is also weak, it is hard to detect the difference between inertial mass and gravitational mass. Using gravitational gauge field theory, we can calculate the difference of inertial mass and gravitational mass for different kinds of matter and help us to test the validity of equivalence principle. This is a fundamental problem which will help us to understand the nature of gravitational interactions. In general relativity, equivalence principle is the foundation of geometrical nature of gravity. But now, the foundation of geometrical nature
of gravity is no longer equivalence principle, but geometrical nature of translation transformation.

Define
\[
\Lambda \triangleq \frac{1}{2} (R + 4g^2 \mathcal{L}_0).
\] (13.26)

Then action given by eq.(3.22) will be changed into
\[
S = S_g + S_M,
\] (13.27)

where
\[
S_g = -\frac{1}{16\pi G} \int d^4 x \sqrt{-g} (R - 2\Lambda),
\] (13.28)
\[
S_M = \int d^4 x \mathcal{L}_M,
\] (13.29)

where \(G\) is the Newtonian gravitational constant, which is given by
\[
G = \frac{g^2}{4\pi},
\] (13.30)

\(R\) is the scalar curvature, \(\Lambda\) is the cosmological term, \(\mathcal{L}_M\) is the lagrangian density for matter fields. Scalar curvature \(R\) can be expressed by gravitational gauge field \(C^\alpha_{\mu}\). We have added the action for matter fields into eq.(13.27) and denoted the action for pure gravitational gauge field as \(S_g\). Using the following relations
\[
\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu},
\] (13.31)
\[
\sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \partial_\lambda W^\lambda,
\] (13.32)
\[
T_{\mu\nu}^m = \frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}(x)},
\] (13.33)

where \(T_{\mu\nu}^m\) is the energy-momentum tensor of matter fields and \(W^\lambda\) is a contravariant vector, we can obtain the Einstein’s field equation with cosmological term \(\Lambda\),
\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu},
\] (13.34)

where \(T_{\mu\nu}\) is the revised energy-momentum tensor, whose definition is
\[
T_{\mu\nu} \triangleq T_{m\mu\nu} - \frac{1}{4\pi G} \frac{\delta \Lambda}{\delta g^{\mu\nu}}.
\] (13.35)
In eq.(13.35), the definition of $\frac{\delta \Lambda}{\delta g^{\mu \nu}}$ is not clear, because $\Lambda$ is a function of $G^\alpha_\mu$, not a function of $g^{\mu \nu}$. So, we need to give out an explicite definition of $\frac{\delta \Lambda}{\delta g^{\mu \nu}}$. According to eq.(5.19), we have

$$\frac{\partial g^{\mu \nu}}{\partial G^\lambda_\alpha} = \delta^\mu_\lambda \eta^{\alpha \beta} G^\nu_\beta + \delta^\nu_\lambda \eta^{\alpha \beta} G^\mu_\beta.$$  \hspace{1cm} (13.36)

Therefore, we have

$$\frac{\delta \Lambda}{\delta G^\alpha_\lambda} = 2 \eta^{\alpha \beta} G^\nu_\beta \frac{\delta \Lambda}{\delta g^{\lambda \nu}}.$$  \hspace{1cm} (13.37)

It gives out

$$\frac{\delta \Lambda}{\delta g^{\mu \nu}} = \frac{1}{4} \eta_{\alpha \beta} \left( G^{-1 \beta}_\nu \frac{\delta \Lambda}{\delta g^{\mu \nu}} + G^{-1 \beta}_\mu \frac{\delta \Lambda}{\delta g^{\nu \mu}} \right),$$  \hspace{1cm} (13.38)

which gives out the explicite expression for $\frac{\delta \Lambda}{\delta g^{\mu \nu}}$. Eq.(13.34) is quite like the Einstein’s field equation with cosmological constant in form, but it is not the traditional Einstein field equation, so we call it the Einstein-like field equation with cosmological term.

### 14 Discussions

In this paper, a new kind of gauge gravity is formulated in the framework of traditional quantum field theory, where gravity is treated as a kind of physical interactions and space-time is always kept flat. This treatment satisfies the fundamental spirit of traditional quantum field theory, and go along this way, four different kinds of fundamental interactions can be unified on the same fundamental baseline[43]. The most advantage of this approach is that the renormalizability of the quantum gravity is easy to be proved. Its transcendental foundation is gauge principle. Gravitational gauge interactions is completely determined by gauge symmetry. In other words, the Lagrangian of the system is completely determined by gauge symmetry. Using the language of Cartan tetrad, we set up the geometrical formulation of this new quantum gauge theory of gravity and to study its geometrical foundation. So, gravity theory has both physical picture and geometrical picture, which is the reflection of the physics-geometry duality of gravity. In this chapter, we give some simple discussions on some interested problem related to gauge theory of gravity. The content of this chapter is not in the main topic of this paper, so our discussion on them are quite simple, and detailed discussion can be found in some related literature.

1. [Schwarzchild-like Solution and Classical Tests]

It is known that General Relativity is tested by three main classical tests, which are perihelion procession, deflection of light and gravitational red-shift.
All these three tests are related to geodesic curve equation and schwarzchild solution in general relativity. If we know geodesic curve equation and space-time metric, we can calculate perihelion procession, deflection of light and gravitational red-shift. In this chapter, we discuss this problem from the point of view of gauge theory of gravity.

In order to discuss classical tests of gravity, for the sake of convinence, we use the geometrical representation of gravity. As we have stated above, in the geometrical representation of gravity, gravity is not treated as physical interactions in space-time. In the geometrical representation of gravity, we use the same manner which is widely used in general relativity to discuss classical tests and to explain the predictions with observations. In the geometrical representation of gravity, if there is no other physical interactions in space time, any mass point can not feel any physical forces when it moves in space-time. So, it must move along the curve which has the least length. Suppose that a particle is moving from point A to point B along a curve. Define

\[ T_{BA} = \int_A^B \sqrt{-g_{\mu\nu} \frac{dx^\mu}{dp} \frac{dx^\nu}{dp}} dp, \]  

(14.1)

where \( p \) is a parameter that describe the orbit. The real curve that the particle moving along corresponds to the minimum of \( T_{AB} \). The minimum of \( T_{AB} \) gives out the following geodesic curve equation

\[ \frac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{dp} \frac{dx^\lambda}{dp} = 0, \]  

(14.2)

where \( \Gamma^\mu_{\nu\lambda} \) is the affine connection

\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left( \frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right). \]  

(14.3)

Eq. (14.2) gives out the curve that a free particle moves along in curved space-time if we discuss physics in the geometrical representation of gravity.

Now, we need to calculate a schwarzchild-like solution in gauge theory of gravity. In chapter 4, we have obtained a solution of \( C_\alpha^\mu \) for static spherically symmetric gravitational fields in linear approximation of \( gC_\alpha^\mu \). But experimental tests, especially perihelion procession, are sensitive to second order of \( gC_\alpha^\mu \). The best way to do this is to solve the equation of motion of gravitational gauge field in the second order approximation of \( gC_\alpha^\mu \). But this equation of motion is a non-linear second order partial differential equations. It is rather
difficult to solve them. So, we had to find some other method to do this. The perturbation method is used to do this. After considering corrections from gravitational energy of the sun in vacuum space and gravitational interactive energy between the sun and the Mercury, the equivalent gravitational gauge field in the second order approximation is

\[ gC^0_0 = -\frac{GM}{r} - \frac{3G^2M^2}{2r^2} + O \left( \frac{G^3M^3}{r^3} \right). \] (14.4)

Then using eq.(5.20), we can obtain the following solution

\[ d\tau^2 = \left[ 1 - \frac{2GM}{r} + O \left( \frac{G^2M^2}{r^2} \right) \right] dt^2 - \left[ 1 + \frac{2GM}{r} + O \left( \frac{G^2M^2}{r^2} \right) \right] dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2, \] (14.5)

where we have use the following gauge for gravitational gauge field,

\[ C^\mu_\mu = 0. \] (14.6)

This solution is quite similar to schwarzschild solution, but it is not schwarzschild solution, so we call it schwarzschild-like solution. If we use Eddington-Robertson expansion, we will find that for the present schwarzschild-like solution[?],

\[ \alpha = \beta = \gamma = 1. \] (14.7)

They have the same values as schwarzschild solution in general relativity and all three tests are only sensitive to these three parameters, so gauge theory of gravity gives out the same theoretical predictions as those of general relativity[44]. More detailed discussions on classical tests can be found in literature [44]. (This result hold for those models which have other choice of \( \eta_2 \) and \( J(C) \) which is duscussed in [27, 28].)

2. [Gravitational Wave] In gravitational gauge theory, the gravitational gauge field is represented by \( C_\mu \). From the point of view of quantum field theory, gravitational gauge field \( C_\mu \) is a vector field and it obeys dynamics of vector field. In other words, gravitational wave is vector wave. Suppose that the gravitational gauge field is very weak in vacuum, then in leading order approximation, the equation of motion of gravitational wave is

\[ \partial^\mu F^\alpha_0 = 0, \] (14.8)

where \( F^\alpha_0 \) is given by eq.(6.9). If we set \( gC^\alpha_\mu \) equals zero, we can obtain eq.(14.8) from eq.(5.50). Eq.(14.8) is very similar to the famous Maxwell equation in vacuum. Define

\[ F^i_j = -\varepsilon_{ijk}B^\alpha_k, \quad F^\alpha_0 = E^\alpha_i, \] (14.9)
then eq.(14.8) is changed into
\[ \nabla \cdot \vec{E} = 0, \tag{14.10} \]
\[ \frac{\partial}{\partial t} \vec{E} - \nabla \times \vec{B} = 0. \tag{14.11} \]
From definitions eq.(14.9), we can prove that
\[ \nabla \cdot \vec{B} = 0, \tag{14.12} \]
\[ \frac{\partial}{\partial t} \vec{B} + \nabla \times \vec{E} = 0. \tag{14.13} \]
If there were no superscript \( \alpha \), eqs.(14.10-14.13) would be the ordinary Maxwell equations. In ordinary case, the strength of gravitational field in vacuum is extremely weak, so the gravitational wave in vacuum is composed of four independent vector waves.

Though gravitational gauge field is a vector field, its component fields \( C^\alpha_\mu \) have one Lorentz index \( \mu \) and one group index \( \alpha \). Both indexes have the same behavior under Lorentz transformation. According to the behavior of Lorentz transformation, gravitational field likes a tensor field. We call it pseudo-tensor field. The spin of a field is determined according to its behavior under Lorentz transformation, so the spin of gravitational field is 2. In conventional quantum field theory, spin-1 field is a vector field, and vector field is a spin-1 field. In gravitational gauge field theory, this correspondence is violated. The reason is that, in gravitational gauge field theory, the group index contributes to the spin of a field, while in ordinary gauge field theory, the group index do not contribute to the spin of a field. In a word, gravitational field is a spin-2 vector field.

3. [Gravitational Magnetic Field] [From eq.(14.10-14.13), we can see that the equations of motion of gravitational wave in vacuum are quite similar to those of electromagnetic wave. The phenomenological behavior of gravitational wave must also be similar to that of electromagnetic wave. In gravitational gauge theory, \( \vec{B}^\alpha \) is called the gravitational magnetic field. It will transmit gravitational magnetic interactions between two rotating objects. In first order approximation, the equation of motion of gravitational gauge field is
\[ \partial^\mu F^{\alpha \mu}_{\nu} = -g \eta^{\alpha \beta} T^\tau_{\eta \beta}. \tag{14.14} \]
Using eq.(14.9) and eq.(14.14), we can get the following equations
\[ \nabla \cdot \vec{E} = -g \eta^{\alpha \beta} T^\tau_{\eta \beta}. \tag{14.15} \]
\[
\frac{\partial}{\partial t} \overrightarrow{E} - \nabla \times \overrightarrow{B} = +g\eta^\alpha_2 T^{g\beta},
\]

where \( T^{g\beta} \) is a simplified notation whose explicit definition is given by the following relation

\[
(T^{g\beta})^i = T^{i}_{g\beta}.
\]

On the other hand, it is easy to prove that (omit self interactions of graviton)
\[
\partial_\mu F^\alpha_\nu_\lambda + \partial_\nu F^\alpha_\lambda_\mu + \partial_\lambda F^\alpha_\mu_\nu = 0.
\]

From eq.(14.18), we can get
\[
\nabla \cdot \overrightarrow{B} = 0,
\]
\[
\frac{\partial}{\partial t} \overrightarrow{B} + \nabla \times \overrightarrow{E} = 0.
\]

Eq.(14.15) means that energy-momentum density of the system is the source of gravitational electric fields, eq.(14.16) means that time-varying gravitational electric fields give rise to gravitational magnetic fields, and eq.(14.20) means that time-varying gravitational magnetic fields give rise to gravitational electric fields. Suppose that the angular momentum of an rotating object is \( J_i \), then there will be a coupling between angular momentum and gravitational magnetic fields. The interaction Hamiltonian of this coupling is proportional to \( \overrightarrow{P}_\alpha \overrightarrow{J} \cdot \overrightarrow{B}^\alpha \). The existence of gravitational magnetic fields is important for cosmology. It is known that almost all galaxies in the universe rotate. The global rotation of galaxy will give rise to gravitational magnetic fields in space-time. The existence of gravitational magnetic fields will affect the moving of stars in (or near) the galaxy. This influence contributes to the formation of the galaxy and can explain why almost all galaxies have global large scale structures. In other words, the gravitational magnetic fields contribute great to the large scale structure of galaxy and universe.

4. [Lorentz Force] There is a force when a particle is moving in a gravitational magnetic field. In electromagnetic field theory, this force is usually called Lorentz force. As an example, we discuss gravitational interactions between gravitational field and Dirac field. Suppose that the gravitational field is static. According to eqs.(7.2-7.3), the interaction Lagrangian is
\[
\mathcal{L}_I = gJ(C)\overrightarrow{\gamma}^\mu \partial_\alpha \psi^C_\alpha \mu.
\]

For Dirac field, the gravitational energy-momentum of Dirac field is
\[
T^{\mu}_g = \overrightarrow{\psi} \gamma^\mu \partial_\alpha \psi.
\]
Substitute eq.(14.22) into eq.(14.21), we get
\[ \mathcal{L}_I = gJ(C)T_{\alpha\mu}^\alpha C_{\mu}. \] (14.23)

The interaction Hamiltonian density \( \mathcal{H}_I \) is
\[ \mathcal{H}_I = -\mathcal{L}_I = -gJ(C)T_{\alpha\mu}(y, \bar{x})C_{\mu}(y). \] (14.24)

Suppose that the moving particle is a mass point at point \( \bar{x} \), in this case
\[ T_{\alpha\mu}(y, \bar{x}) = T_{\alpha\mu} \delta(\bar{y} - \bar{x}), \] (14.25)
where \( T_{\alpha\mu} \) is independent of space coordinates. Then, the interaction Hamiltonian \( H_I \) is
\[ H_I = \int d^3 \bar{y} \mathcal{H}_I(y) = -g \int d^3 \bar{y} J(C)T_{\alpha\mu}(y, \bar{x})C_{\mu}(y). \] (14.26)

The gravitational force that acts on the mass point is
\[ f_i = g \int d^3 y J(C)T_{\alpha\mu}(y, \bar{x})F_{i\alpha} + g \int d^3 y J(C)T_{\alpha\mu}(y, \bar{x})\frac{\partial}{\partial y_{\mu}}C_{\alpha}. \] (14.27)

For quasi-static system, if we omit higher order contributions, the second term in the above relation vanish. For mass point, using the technique of Lorentz covariance analysis, we can proved that
\[ P_{\alpha\mu} = \gamma T_{\alpha\mu}, \] (14.28)
where \( U^\mu \) is velocity, \( \gamma \) is the rapidity, and \( P_{\alpha\mu} \) is the gravitational energy-momentum. According eq.(14.25), \( P_{\alpha\mu} \) is given by
\[ P_{\alpha\mu} = \int d^3 \bar{y} T_{\alpha\mu}(y) = T_{0\alpha}. \] (14.29)

Using all these relations and eq.(14.9), we get
\[ \bar{f} = -gJ(C)P_{\alpha\mu} \bar{E} - gJ(C)P_{\alpha\mu} \bar{v} \times \bar{B}. \] (14.30)

For quasi-static system, the dominant contribution of the above relation is
\[ \bar{f} = gJ(C)M \bar{E} + gJ(C)M \bar{v} \times \bar{B}, \] (14.31)
where \( \bar{v} = \bar{U} / \gamma \) is the velocity of the mass point. The first term of eq.(14.31) is the classical Newton’s gravitational interactions. The second term of eq.(14.31)
is the Lorentz force. The direction of this force is perpendicular to the direction of the motion of the mass point. When the mass point is at rest or is moving along the direction of the gravitational magnetic field, this force vanishes. Lorentz force is important for cosmology, because the rotation of galaxy will generate gravitational magnetic field and this gravitational magnetic field will affect the motion of stars and affect the large scale structure of galaxy.

5. **Negative Energy** First, let’s discuss inertial energy of pure gravitational wave. Suppose that the gravitational wave is not so strong, so the higher order contribution is very small. We only consider leading order contribution here. For pure gravitational field, we have

\[
\frac{\partial L_0}{\partial \partial_\mu C_\nu} = -\eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\beta\gamma} F_{\rho\sigma}^\gamma + g \eta^{\lambda\rho} \eta^{\nu\sigma} \eta_{2\beta\gamma} C_\chi^\mu F_{\rho\sigma}^\gamma. \tag{14.32}
\]

From eq.(5.43), we can get the inertial energy-momentum tensor of gravitational field in the leading order approximation, that is

\[
T_{\alpha\lambda} = J(C)[\eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\beta\gamma} F_{\rho\sigma}^\gamma \partial_\alpha C_\beta + \delta_{\alpha}^\mu \mathcal{L}_0]. \tag{14.33}
\]

Using eq.(14.9), Lagrangian given by eq.(5.31) can be changed into

\[
\mathcal{L}_0 = \frac{1}{2} \eta_{2\alpha\beta}(\overrightarrow{E} \cdot \overrightarrow{E} + \overrightarrow{B} \cdot \overrightarrow{B}). \tag{14.34}
\]

Space integral of time component of inertial energy-momentum tensor gives out inertial energy \(H_i\) and inertial momentum \(\overrightarrow{P}_i\). They are

\[
H_i = \int d^3 \overrightarrow{x} J(C) \left[ \frac{1}{2} \eta_{2\alpha\beta}(\overrightarrow{E} \cdot \overrightarrow{E} + \overrightarrow{B} \cdot \overrightarrow{B}) \right], \tag{14.35}
\]

\[
\overrightarrow{P}_i = \int d^3 \overrightarrow{x} J(C) \eta_{2\alpha\beta} \overrightarrow{E} \times \overrightarrow{B}. \tag{14.36}
\]

In order to obtain eq.(14.35), eq.(14.10) is used. Let consider the inertial energy-momentum of gravitational field \(C_0^\mu\). Because,

\[
\eta_{200} = -1, \tag{14.37}
\]

eq.(14.35) gives out

\[
H_i(C^0) = -\frac{1}{2} \int d^3 \overrightarrow{x} J(C)(\overrightarrow{E} \cdot \overrightarrow{E} + \overrightarrow{B} \cdot \overrightarrow{B}). \tag{14.38}
\]

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\( H_1(C^0) \) is a negative quantity. It means that the inertial energy of gravitational field \( C^0_\mu \) is negative. The gravitational energy-momentum of pure gravitational gauge field is given by eq.(5.51). In leading order approximation, it is
\[
T^\mu_{\alpha 0} = \eta^{\mu \rho} \eta^{\sigma \tau} \eta_{2\beta \gamma} F^\alpha_{\rho \sigma} (\partial_\alpha C^\beta_\tau) + \eta^\mu_{\alpha} E_0
\]
(14.39)

After omitting surface terms, the gravitational energy of the system is
\[
H_g = \int d^3 \vec{x} \left[ \frac{1}{2} \eta_{2\alpha \beta} (\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}) - \eta^{ij} \partial_0 (C^0_j E^0_i) \right].
\]
(14.40)

The gravitational energy of gravitational field \( C^0_\mu \) is,
\[
H_g(C^0) = - \frac{1}{2} \int d^3 \vec{x} (E \cdot E + B \cdot B + 2\eta^{ij} \partial_0 (C^0_j E^0_i)).
\]
(14.41)

6. [Repulsive Force] The classical gravitational interactions are attractive interactions. But in gravitational gauge theory, there are repulsive interactions as well as attractive interactions. The gravitational force is given by eq.(14.31). The first term corresponds to classical gravitational force. It is
\[
f_i = gJ(C)T^0_{\alpha 0} (\partial_\alpha C^0_\tau).
\]
(14.42)

For quasi-static gravitational field, it is changed into
\[
f_i = -gJ(C) P_{\alpha 0} E^\alpha_i
\]
(14.43)
\[
= gJ(C) (M_1 E^0_i - P_{\alpha 0} E^1_i),
\]
where \( M_1 \) is the gravitational mass of the mass point which is moving in gravitational field. Suppose that the gravitational field is generated by another mass point whose gravitational energy-momentum is \( Q^\alpha_g \) and gravitational mass is \( M_2 \). For quasi-static gravitational field, we can get
\[
E^\alpha_i = - \frac{g}{4\pi r^3} Q^\alpha_g r_i
\]
(14.44)
Substitute eq.(14.44) into eq.(14.43), we get

\[ \bar{f} = J(C) \frac{g^2}{4\pi r^3} \bar{r} \left( -E_{1g}E_{2g} + \bar{P}_g \cdot \bar{Q}_g \right), \]  

(14.45)

where \( E_{1g} \) and \( E_{2g} \) are gravitational energy of two mass point. From eq.(14.45), we can see that, if \( \bar{P}_g \cdot \bar{Q}_g \) is positive, the corresponding gravitational force between two momentum is repulsive. This repulsive force is important for the stability of some celestial object. For relativistic system, all mass point moving at a high speed which is near the speed of light. Then the term \( \bar{P}_g \cdot \bar{Q}_g \) has approximately the same order of magnitude as that of \( E_{1g}E_{2g} \), therefore, for relativistic systems, the gravitational attractive force is not so strong as the force when all mass points are at rest.

7. **[Equivalence Principle]** Equivalence principle is one of the most important foundations of general relativity, but it is not a logic starting point of gravitational gauge field theory. The logic starting point of gravitational gauge field theory is gauge principle. However, one important inevitable result of gauge principle is that gravitational mass is not equivalent to inertial mass. The origin of violation of equivalence principle is gravitational field. If there were no gravitational field, equivalence principle would strictly hold. But if gravitational field is strong, equivalence principle will be strongly violated. For some celestial objects which have strong gravitational field, such as quasar and black hole, their gravitational mass will be higher than their inertial mass. But on earth, the gravitational field is very weak, so the equivalence principle almost exactly holds. We need to test the validity of equivalence principle in astrophysics experiments.

In this paper, we have discussed a completely new quantum gauge theory of gravity. Finally, we give a simple summary to the whole theory.

1. In leading order approximation, the gravitational gauge field theory gives out classical Newton’s theory of gravity.

2. In first order approximation and for vacuum, the gravitational gauge field theory gives out Einstein’s general theory of relativity.

3. Gravitational gauge field theory is a renormalizable quantum theory.

4. Combining cosmological principle with the field equation of gravitational gauge field, we can also set up a cosmological model with is consistent with recent observations[45].
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