STATIONARY SOLUTIONS AND CLOSED TIME-LIKE CURVES IN 2+1 DIMENSIONAL GRAVITY*

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Abstract

We give the general solution of the stationary problem of 2+1 dimensional gravity in presence of extended sources, also endowed with angular momentum. We solve explicitly the compact support property of the energy momentum tensor and we apply the results to the study of closed time-like curves. In the case of rotational symmetry we prove that the weak energy condition combined with the absence of closed time-like curves at space infinity prevents the existence of closed time-like curves everywhere in an open universe (conical space at infinity).

I. INTRODUCTION

A great deal of interest has been devoted to 2+1 dimensional gravity [1], more recently in connection with the problem of closed time-like curves (CTC) [2–7,9–14]. The aspect of exact solutions in 2+1 dimensional gravity has been addressed in detail, both in the case of point like sources [1] and in the case of extended sources [17,18]. Some inroads have also been done in the realm of time dependent solutions [15,17,18]. The radial gauge approach [16–18] has proven particularly fruitful in obtaining quadrature formulae for extended and non stationary sources. On the other hand most of the activity has been devoted to point like sources where the energy momentum tensor acquires a particularly simplified form. This has been done both for sources without and with spin; however point like sources with spin have been generally considered unphysical as they generate in their proximity metrics that support CTC. From this viewpoint considering extended sources is of great interest. In a previous paper [18] we gave in terms of quadratures the general resolvent formulae for the time dependent solutions, and we solved completely the support problem in the case
of rotational symmetry. Obviously finding solutions of Einstein’s equations whose energy momentum tensor has some prescribed support and symmetry properties is not enough as the energy momentum tensor should satisfy some energy condition [19]. This is not a trivial problem as it involves inequalities among the eigenvalues of the energy momentum tensor in the metric generated, through Einstein’s equations, by the energy momentum tensor itself. As we shall see the radial gauge, due to its physical nature, provides a powerful device for extracting useful information from the energy condition and in constructing sources that satisfy such energy conditions. In this paper we shall produce resolvent formulae, in term of quadratures for the general stationary problem with extended sources and give a complete treatment of the CTC that may appear in the case of rotational symmetry. One of the main results of the paper will be that the imposition of i) the weak energy condition (WEC) and ii) the absence of CTC at space infinity, prevents the occurrence of CTC anywhere in an open (conical) universe. To understand how far both conditions are also necessary to avoid CTC we provide solved examples of regular sources with non vanishing total angular momentum, which violate WEC, have no CTC at infinity but produce CTC for some finite radius. In addition the assumption that no CTC appears at space infinity is a necessary one, as we shall produce solved examples of regular sources that satisfy the WEC but produce CTC at infinity.

The paper is organized as follows: in sect.II we introduce the reduced radial gauge, which will be the main tool in the sequel of the paper, and write down Einstein’s equations in such a gauge for the most general stationary metric. In sect.III we give the resolvent formulae in terms of quadratures, that express the metric in terms of the energy momentum tensor, giving the explicit condition for the compactness of the support of the energy momentum tensor in the general case. We discuss also the simplifying features that occur in the case of rotational symmetry. In sect.IV we
address the problem of CTC for stationary solutions with rotational symmetry and prove the main result that the WEC combined with the absence of CTC at space infinity prevents the occurrence of CTC everywhere. In sect.V we summarize the main conclusion and outline possible possible developments. In appendix A we derive the resolvent formulae for the reduced radial gauge, in appendix B we give the general regularity conditions for the energy momentum tensor at the origin and in appendix C we give the derivation of the compact support condition for the energy momentum tensor in the general case.

II. REDUCED RADIAL GAUGE

In stationary problems the reduced radial gauge [17] is defined by

\[ \sum_i \xi^i \Gamma_{bi}^a = 0, \]  
\[ \sum_i \xi^i e_i^a = \sum_i \delta_i^a \xi^i, \]

where the sums run over the space indices. (In the following the indices \( i, j, k, l, m \) denote space indices).

It was shown in ref. [17] sect.4 that such a gauge is attainable and that the reference frame that realizes it is the (generalized) Fermi-Walker coordinate system for an observer that moves along an integral line of the time-like Killing vector field.

In any space-time dimension there are resolvent formulae that express the vierbeins and the connection in terms of the Riemann and torsion tensor in this gauge. They are (see appendix A)

\[ \Gamma_{bi}^a(\xi) = \xi^j \int_0^1 R_{bij}^a(\lambda \xi) d\lambda, \]

\[ \Gamma_{00}^a(\xi) = \Gamma_{00}^a(0) + \xi^i \int_0^1 R_{00i}^a(\lambda \xi) d\lambda, \]

where \( R_{ij}^a \) is the Riemann tensor and \( R_{00i}^a \) is the torsion tensor.
\[ e_i^a = \delta_i^a + \xi^j \xi^l \int_0^1 R^a_{jl}(\lambda \xi) \lambda(1-\lambda)d\lambda + \xi^j \int_0^1 S^a_{ji}(\lambda \xi)d\lambda, \tag{5} \]

\[ e_0^a = \delta_0^a + \xi^j \Gamma^a_{0j}(0) + \xi^i \xi^j \int_0^1 R^a_{ij}(\lambda \xi)(1-\lambda)d\lambda + \xi^j \int_0^1 S^a_{ji}(\lambda \xi)d\lambda. \tag{6} \]

In the following we shall consider theories with vanishing torsion. The peculiarity of 2+1 dimensions is the substantial identification of the Riemann with the Ricci tensor and through Einstein’s equation, with the energy momentum tensor. Explicitly

\[ \varepsilon_{abc} R^{ab} = -2\kappa T_c, \tag{7} \]

where \( \kappa = 8\pi G \), and thus

\[ R^{ab} = -\kappa \varepsilon^{abc} T_c = -\frac{\kappa}{2} \varepsilon^{abc} \varepsilon_{\rho\mu\nu} T_c^\rho d\xi^\mu \wedge d\xi^\nu, \tag{8} \]

where \( T_c \) is the energy momentum two form. Using such a relation one can express through a simple quadrature, the connections and the vierbeins in terms of the energy momentum tensor, which is the source of the gravitational field and thus one solves Einstein’s equation. On the other hand the energy momentum tensor is subject to the covariant conservation law and symmetry condition. Thus our problem is to construct the general form of conserved symmetric sources in the reduced radial gauge, which in addition should satisfy other physical requirement given by the support of the sources and the restrictions due to the energy condition [19].

The conservation and symmetry equations for the energy momentum tensor are

\[ \mathcal{D}T^a = 0, \tag{9} \]

\[ \varepsilon_{abc} T^b \wedge e^c = 0. \tag{10} \]

To solve these equations we use the technique developed in [18]. It is useful to introduce the cotangent vectors \( T_\mu = \frac{\partial \xi^0}{\partial \xi_\mu}, P_\mu = \frac{\partial \rho}{\partial \xi_\mu} \) and \( \Theta_\mu = \rho \frac{\partial \theta}{\partial \xi_\mu} \) where \( \rho \) and \( \theta \)
are the polar variables in the \((\xi^1, \xi^2)\) plane. The conservation equation is solved by
the general ansatz \[17\]

\[
\tau^\rho_c(\xi) = \frac{1}{\kappa} \left[ P^\mu \partial_\mu A^\rho_c(\xi) - \frac{1}{\rho} A^\rho_c(\xi) - \frac{1}{\rho} \Theta^\rho \Theta_\mu A^\mu_c(\xi) - P^\rho \left( \partial_\mu A^\mu_c(\xi) - \frac{1}{2} \varepsilon_{\alpha \beta \gamma} P^\alpha A^{\beta \gamma}(\xi) \right) \right],
\]

(11)

where \(A^\rho_c(\xi)\) is related to the connection \(\Gamma^a_{\beta \gamma}(\xi)\) in the reduced radial gauge in 2 + 1 dimensions by

\[
\Gamma^a_{\beta \gamma}(\xi) = \varepsilon^{abc} \varepsilon_{\mu \rho \sigma} P^\rho A^\sigma_c(\xi).
\]

(12)

The origin of (11) is the following: if we express through Einstein’s equations (7) the energy momentum tensor in terms of the connection written as in (12), which is the most general form for a radial connection in 2+1 dimensions, we obtain (11). The most general form of \(A^\rho_c(\xi)\), taking into account that the components of \(A^\rho_c\) along \(P^\rho\) are irrelevant to determine the geometry, is

\[
A^\rho_c(\xi) = T_c \left[ \Theta^\rho \beta_1 + T^\rho \frac{(\beta_2 - 1)}{\rho} \right] + \Theta_c \left[ \Theta^\rho \alpha_1 + T^\rho \frac{\alpha_2}{\rho} \right] + P_c \left[ \Theta^\rho \gamma_1 + T^\rho \frac{\gamma_2}{\rho} \right].
\]

(13)

We use for the coefficient of \(T_c T^\rho\) the form \(\beta_2 - 1\), as it simplifies the writing of subsequent formulae. Substituting (13) into (11) we obtain

\[
\tau^\rho_c = -\frac{1}{\kappa} \left\{ T_c \left( T^\rho \frac{\beta^\prime_2}{\rho} + \Theta^\rho \frac{\beta^\prime_1}{\rho} \right) + \Theta_c \left( T^\rho \frac{\alpha^\prime_2}{\rho} + \Theta^\rho \frac{\alpha^\prime_1}{\rho} \right) + P_c \left( T^\rho \frac{\gamma^\prime_2}{\rho} + \Theta^\rho \frac{\gamma^\prime_1}{\rho} \right) + \frac{1}{\rho} \left[ T_c \left( \alpha_1 \gamma_2 - \alpha_2 \gamma_1 - \frac{\partial \beta_1}{\partial \theta} \right) + \Theta_c \left( \beta_1 \gamma_2 - \beta_2 \gamma_1 - \frac{\partial \alpha_1}{\partial \theta} \right) + P_c \left( \alpha_1 \beta_2 - \alpha_2 \beta_1 - \frac{\partial \gamma_1}{\partial \theta} \right) \right] \right\}.
\]

(14)

Using eq. (7) and (14) into (5) and (6) we obtain

\[
e_{0}^a(\xi) = -T^a A_1 - \Theta^a B_1,
\]

(15)

\[
e_{c}^a(\xi) = -\frac{1}{\rho} \Theta^a \Theta_c B_2 - \frac{1}{\rho} T^a \Theta_c A_2 - P^a P_c,
\]

(16)
where $A_i$ and $B_i$ are defined by

$$
A_1(\xi) = \rho \int_0^1 \alpha_1(\lambda \xi) d\lambda - 1, \quad B_1(\xi) = \rho \int_0^1 \beta_1(\lambda \xi) d\lambda,
$$

$$
A_2(\xi) = \rho \int_0^1 \alpha_2(\lambda \xi) d\lambda \quad \text{and} \quad B_2(\xi) = \rho \int_0^1 \beta_2(\lambda \xi) d\lambda.
$$

Eqs. (15) and (16) can be summarized into

$$
e^a_{\mu}(\xi) = - T^a(T_\mu A_1 + \frac{1}{\rho} \Theta_\mu A_2) - \Theta^a(T_\mu B_1 + \frac{1}{\rho} \Theta_\mu B_2) - P^a P_\mu
$$

from which the metric is given by

$$
ds^2 = (A_1^2 - B_1^2) dt^2 + 2(A_1 A_2 - B_1 B_2) dtd\theta + (A_2^2 - B_2^2) d\theta^2 - d\rho^2.
$$

Contracting the energy momentum tensor (14) with the dreibein (18) we obtain the same quantity in the dreibein base

$$
\tau^{c\mu}(\xi) e^a_{\mu}(\xi) = T^{ca} = - \frac{1}{\kappa \rho} \left\{ T^a \left[ T^c \left( A_2 \beta'_1 - A_1 \beta'_2 \right) + \Theta^c \left( A_2 \alpha'_1 - A_1 \alpha'_2 \right) + P^c \left( A_2 \gamma'_1 - A_1 \gamma'_2 \right) \right] + \Theta^a \left[ T^c \left( B_2 \beta'_1 - B_1 \beta'_2 \right) + \Theta^c \left( B_2 \alpha'_1 - B_1 \alpha'_2 \right) + P^c \left( B_2 \gamma'_1 - B_1 \gamma'_2 \right) \right] + P^a \left[ T^c \left( \alpha_1 \gamma_2 - \alpha_2 \gamma_1 - \frac{\partial \beta_1}{\partial \theta} \right) + \Theta^c \left( \beta_1 \gamma_2 - \beta_2 \gamma_1 - \frac{\partial \alpha_1}{\partial \theta} \right) \right] \right\}.
$$

We shall consider solutions of Einstein’ s equations free of physical singularities, which implies that also the energy momentum tensor is regular. The regularity of $\Gamma^{ab}_{\mu}(\xi)$ and of $\tau^c_{\mu}(\xi)$ in a neighborhood of the origin impose conditions on the behavior of $\alpha_i$, $\beta_i$, $\gamma_i$ at the origin. We start from $\Gamma^{ab}_{\mu}(\xi)$. Its continuous behavior at the origin imposes the following behavior upon the basic functions $\alpha_i$, $\beta_i$, $\gamma_i$

$$
\alpha_1 \to \Gamma^{01}_{0}(0) \cos \theta + \Gamma^{02}_{0}(0) \sin \theta, \quad \alpha_2 \to o(\rho),
$$

$$
\beta_1 \to \Gamma^{12}_{0}(0), \quad \beta_2 \to 1 + o(\rho),
$$

$$
\gamma_1 \to -\Gamma^{02}_{0}(0) \cos \theta + \Gamma^{01}_{0}(0) \sin \theta, \quad \gamma_2 \to o(\rho).
$$
Eqs. (21) do not yet imply the regularity at the origin of the energy momentum tensor, which, containing derivatives of $\Gamma^{\mu}_{\alpha\beta}$, imposes stronger restrictions. Due to the singular nature of polar coordinates at the origin, it is more proper to discuss the regularity at the origin of the energy momentum tensor in cartesian coordinates and they are given in appendix B.

Coming back to the energy momentum tensor $T^{ab}$ the symmetry constraint (10) gives three differential relations

$$A_1\alpha_2' - A_2\alpha_1' + B_2\beta_1' - B_1\beta_2' = 0 \quad (22a)$$
$$\alpha_2\gamma_1 - \alpha_1\gamma_2 + A_2\gamma_1' - A_1\gamma_2' + \frac{\partial\beta_1}{\partial\theta} = 0 \quad (22b)$$
$$\beta_2\gamma_1 - \beta_1\gamma_2 + B_2\gamma_1' - B_1\gamma_2' + \frac{\partial\alpha_1}{\partial\theta} = 0. \quad (22c)$$

These relations can be integrated with respect to $\rho$ and taking into account the regularity conditions of $\alpha_i$, $\beta_i$ and $\gamma_i$ at the origin we reach

$$A_1\alpha_2 - A_2\alpha_1 + B_2\beta_1 - B_1\beta_2 = 0 \quad (23a)$$
$$A_2\gamma_1 - A_1\gamma_2 + \frac{\partial B_1}{\partial\theta} = 0 \quad (23b)$$
$$B_2\gamma_1 - B_1\gamma_2 + \frac{\partial A_1}{\partial\theta} = 0. \quad (23c)$$

In general, in absence of rotational symmetry, caustics may develop in the sense that geodesics emerging from the origin with different $\theta$ can intersect at some point for large enough $\rho$. This renders the map of $\rho, \theta$ into the physical points of space not one to one, but the geometry can be still regular in the sense that a proper change of coordinates removes the singularity. For an example of how this non single valuedness can show up and how it can be removed by changing coordinates, we refer to the appendix of ref. [17]. Such a problem does not arise in the case of rotational symmetry, that we discuss in sec. III B and apply in sect.IV.
III. SUPPORT PROPERTIES OF THE ENERGY MOMENTUM TENSOR

A. Generic case

First we give the solution of eqs. (23) in the generic case i.e. when \( \alpha_1 \) and \( \beta_1 \) depend on \( \theta \) (absence of rotational symmetry). In the system (23) we have 6 unknown functions \( \alpha_i, \beta_i, \gamma_i \) \((i = 1, 2)\) and 3 equations. We shall choose as independent functions \( \alpha_1, \beta_1, \gamma_1 \). This choice simplifies the discussion of the support property of the energy momentum tensor. The method of solution follows the one described in ref. [18] for the complete radial gauge. Multiplying (23b) by \( B_1 \) and (23c) by \( A_1 \) and subtracting we obtain

\[
A_2 B_1 - A_1 B_2 = \frac{1}{2\gamma_1} \frac{\partial}{\partial \theta} (A_1^2 - B_1^2) \equiv N(\rho, \theta) \quad (24)
\]
from which

\[
B_2 = \frac{A_2}{A_1} B_1 - \frac{N}{A_1} \quad (25)
\]
and

\[
\beta_2 = \frac{\partial}{\partial \rho} \left( \frac{A_2}{A_1} \right) B_1 + \frac{A_2}{A_1} \beta_1 - \frac{\partial}{\partial \rho} \left( \frac{N}{A_1} \right). \quad (26)
\]
Substituting into eq. (23a) we obtain

\[
\frac{\partial}{\partial \rho} \left( \frac{A_2}{A_1} \right) = \frac{B_1^2}{B_1^2 - A_1^2} \frac{\partial}{\partial \rho} \left( \frac{N}{A_1 B_1} \right) \quad (27)
\]
from which

\[
A_2 = \frac{N B_1}{B_1^2 - A_1^2} + 2A_1 I \quad (28)
\]
where

\[
I = \int_0^\rho d\rho' \frac{N(\alpha_1 \beta_1 - B_1 \alpha_1)}{(B_1^2 - A_1^2)^2} \quad (29)
\]
and
\[ \alpha_2 = \frac{B_1^2}{B_1^2 - A_1^2} \frac{\partial}{\partial \rho} \left( \frac{N}{B_1} \right) + 2\alpha_1 I. \] (30)

Substituting \( \alpha_2 \) and \( A_2 \) thus obtained into (26) we have
\[ \beta_2 = \frac{A_1^2}{B_1^2 - A_1^2} \frac{\partial}{\partial \rho} \left( \frac{N}{A_1} \right) + 2\beta_1 I, \] (31)

and finally using e.g. (23b) we have
\[ \gamma_2 = \frac{B_1^2}{B_1^2 - A_1^2} \frac{\partial}{\partial \theta} \left( \frac{A_1}{B_1} \right) + 2\gamma_1 I. \] (32)

Formulae (30), (31) and (32) give the solution of the problem in terms of the single
quadrature \( I \).

We must now discuss the support property of the source i.e. the necessary and
sufficient conditions that \( \alpha_1, \beta_1, \gamma_1 \) have to satisfy in order that the energy momentum
tensor vanishes outside a certain boundary \( \rho_0(\theta) \). From eq. (14) the vanishing of \( \tau_\rho^\rho \)
implies
\[ \frac{\partial \alpha_i}{\partial \rho} = \frac{\partial \beta_i}{\partial \rho} = \frac{\partial \gamma_i}{\partial \rho} = 0 \quad \text{for} \quad \rho > \rho_0(\theta) \] (33)

and
\[ \alpha_1 \beta_2 - \alpha_2 \beta_1 = 0 \] (34a)
\[ \alpha_2 \gamma_1 - \alpha_1 \gamma_2 = 0 \] (34b)
\[ \beta_2 \gamma_1 - \beta_1 \gamma_2 = 0 \] (34c)

for \( \rho > \rho_0(\theta) \). If \( \alpha_1, \beta_1, \gamma_1 \) satisfy eqs. (22b), (22c) and condition (33), also (34b)
and (34c) are satisfied. Thus we must simply impose (33) and (34a). In appendix C
we prove that necessary and sufficient condition for this to happen is that
\[ \alpha_1 B_1 - A_1 \beta_1 = \text{constant} \quad \text{for } \rho > \rho_0(\theta) \]  

\[ \alpha_1^2 - \beta_1^2 + \gamma_1^2 = \text{constant} \quad \text{for } \rho > \rho_0(\theta) \]

where the two constants do not depend both on \( \theta \) and on \( \rho \). Thus given \( \alpha_1 \) and \( \beta_1 \) that satisfy (33) and (35) one can easily construct a \( \gamma_1 \) that satisfy (36) and thus also \( \gamma_1' = 0 \) for \( \rho > \rho_0(\theta) \). There is no problem in choosing \( \alpha_1 \) and \( \beta_1 \) to satisfy (35). Then the explicit solution is given by eqs.(30),(31), (32). An alternative to the choice \( \alpha_1, \beta_1 \) and \( \gamma_1 \) for the fundamental functions, is \( \alpha_1, \beta_1 \) and \( N = A_2 B_1 - A_1 B_2 = \det(e) = \rho \det(e_\mu^a) \), where \( \det(e) \) and \( \det(e^a_\mu) \) are the determinants of the dreibeins in polar and cartesian coordinates. Such a choice avoids the necessity of dividing by \( \gamma_1 \) in (24). Thus (30),(31) and (32) hold also for \( \gamma_1 = 0 \) as it happens in the case of rotational symmetry. We notice furthermore that the energy momentum tensor can be written in algebraic form in terms of \( A_1, B_1 \) and \( N \) and their first and second derivatives.

**B. Case of rotational symmetry**

In this case as the function \( \alpha_i, \beta_i \) and \( \gamma_i \) (see appendix B of [18]) do not depend on \( \theta \), from the last two symmetry equations

\[ A_2 \gamma_1 - A_1 \gamma_2 = 0 \]
\[ B_2 \gamma_1 - B_1 \gamma_2 = 0 \]

(37)

we obtain \( \gamma_1 = \gamma_2 = 0 \), under the assumption that the determinant of the dreibein \( \det(e) = \frac{1}{\rho}(A_2 B_1 - A_1 B_2) \) never vanishes. Then the energy momentum tensor \( T^{\alpha \mu} \) becomes
\[
T^{ca} = -\frac{1}{\kappa \rho} \{ T^a [ T^e (A_2 \beta'_1 - A_1 \beta'_2) + \Theta^e (A_2 \alpha'_1 - A_1 \alpha'_2) ] +
\Theta^a [ T^e (B_2 \beta'_1 - B_1 \beta'_2) + \Theta^e (B_2 \alpha'_1 - B_1 \alpha'_2) ] +
P^a P^c (\alpha_1 \beta_2 - \alpha_2 \beta_1) \},
\] (38)

and the metric
\[
ds^2 = (A_1^2 - B_1^2) dt^2 + 2 (A_1 A_2 - B_1 B_2) dt d\theta + (A_2^2 - B_2^2) d\theta^2 - d\rho^2.
\] (39)

In the case of rotational symmetry the independence on \( \theta \) of the relations given in appendix B gives the following behavior for the functions \( \alpha_i, \beta_i \)

\[
\alpha_1 = O(\rho), \quad \alpha_2 = o(\rho^2), \quad \beta_1 = c + o(\rho), \quad \beta_2 = 1 + O(\rho^2).
\] (40)

It is easily checked that the behavior of eq.(40) makes the connection \( \Gamma^{ab}_{\mu} \) regular at the origin.

The vanishing of \( \tau^a_{\rho} \) outside a finite support, \( \rho > \rho_0 \), imposes using the same reasoning as above

\[
\alpha'_1 = \beta'_1 = \alpha'_2 = \beta'_2 = 0 \quad \text{for} \quad \rho > \rho_0
\] (41)

and

\[
\alpha_1 \beta_2 - \alpha_2 \beta_1 = 0 \quad \text{for} \quad \rho > \rho_0.
\] (42)

The only surviving symmetry equation is now

\[
A_1 \alpha_2 - A_2 \alpha_1 + B_2 \beta_1 - B_1 \beta_2 = 0.
\] (43)

We notice that eq.(43) is invariant under the three dimensional group of transformations

\[
\alpha_1 \rightarrow \alpha_1 + \omega_1 \alpha_2, \quad \beta_1 \rightarrow \beta_1 + \omega_2 \beta_2, \\
\alpha_2 \rightarrow c \alpha_2, \quad \beta_2 \rightarrow \beta_2
\] (44)
This is what is left of the larger $Sp(2) \times U(1)$ invariance of eq. (43) once one imposes that the symmetry transformation respects the regularity conditions of $\Gamma^b_{\gamma\xi}(\xi)$ at the origin. We observe that for $\omega_1 = \omega_2 = \omega$ and $c = 1$ the transformations (44) leave $T^{ca}$ invariant and as such also the support equations (41) and (42). It means that transformation (44) with $\omega_1 = \omega_2 = \omega$ and $c = 1$ corresponds to the same physical system described in a different frame. In fact the metric differs by a rigid rotation with angular velocity $\omega$. On the other hand the transformations with $\omega_1 \neq \omega_2$ and/or $c \neq 1$ correspond to different physical situations.

We can exploit the transformation (44) to generate all solutions for energy-momentum tensors with bounded support. In fact it is immediately seen that once three among the four functions $\alpha_i, \beta_i$ have zero derivatives for $\rho > \rho_0$, eq. (43) is solved by the remaining one also constant for $\rho > \rho_0$. The problem is to satisfy the bounded support condition (42). We shall show that all solutions with bounded support, through a rotation with $\omega_1 = \omega_2 = \omega, c = 1$, can be reduced to one of the following cases, where the bounded support condition is trivially satisfied.

1. $\alpha_2^0$ and/or $\beta_2^0 \neq 0$ and $\alpha_1^0 = \beta_1^0 = 0$.
2. $\alpha_2^0 = \beta_2^0 = 0$.

In fact if $\alpha_2^0 \neq 0$ choosing from (44) $\omega = -\alpha_1^0/\alpha_2^0$ we obtain $\alpha_1^0 \to 0$, but (42), which is invariant under such a transformation implies $\beta_1^0 \to 0$. Similarly one reasons for $\beta_2^0 \neq 0$.

**Case 1.** For $\alpha_2^{02} - \beta_2^{02} \neq 0$ we have that $g_{00}$ and $g_{0\theta}$ outside the source are constant while $g_{\theta\theta}$ behaves like $(\alpha_2^{02} - \beta_2^{02})\rho^2$. If $g_{00} > 0$ due to eq. (43) $\alpha_2^{02} - \beta_2^{02} < 0$ and we have the usual conical universe with angular deficit $\delta = 2\pi(1 - \sqrt{\beta_2^{02} - \alpha_2^{02}})$, related to the source mass by $\delta = 8\pi GM$ and angular momentum $g_{0\theta}/g_{00} = (A_1^0A_2^0 - B_1^0B_2^0)/(A_1^{02} - B_1^{02}) = 4GJ$. If $g_{00} = A_1^{02} - B_1^{02} < 0$ then we must have $\alpha_2^{02} - \beta_2^{02} > 0$.
because of eq.(43). Then by a proper rotation the metric can be reduced to

\[(\alpha_0^2 - \beta_0^2) \left( \rho - \rho_0 + \frac{\alpha_0^0 A_2^0 - \beta_0^0 B_2^0}{A_0^2 - B_0^2} \right)^2 \times \]

\[
\left( \frac{A_1^0 - B_1^0}{A_0^2 - B_0^2} \right) dt - d\theta \right)^2 + \left( \frac{A_1^0 A_2^0 - B_1^0 B_2^0}{A_0^2 - B_0^2} \right) d\theta^2 - d\rho^2
\]

which is the usual outer metric generated by a closed string with tension \([1]\). If \(\alpha_0^2 = \beta_0^2 \neq 0\) we have a “linear” universe in which \(g_{\theta\theta}\) behave linearly for \(\rho \to \infty\) (see [1]).

**Case 2.** \(g_{\theta\theta}\) and \(g_{\theta\theta}\) are constant for \(\rho > \rho_0\) and if \(\alpha_0^2 - \beta_1^2 \neq 0, g_{00}\) behaves like \(\rho^2\) for \(\rho \to \infty\). On the other hand \(g_{\theta\theta} \neq 0\) because otherwise \(A_{02}^0 = \pm B_{02}^0\) which implies through eq.(43) \(\alpha_1^0 = \pm \beta_1^0\) which contradicts \(\alpha_1^0 - \beta_1^0 \neq 0\). \((A_2^0 = B_2^0 = 0\) is excluded because otherwise for \(\rho > \rho_0\) the metric degenerates). Then by means of a rotation we can set \(g_{\theta\theta} = 0\) to reach, outside the source, the metric

\[ds^2 = k(c_1 + \rho)^2 dt^2 + g_{\theta\theta} d\theta^2 - d\rho^2.\]

which for positive \(k\) is the metric generated by a static string with tension \([1]\). For \(k < 0\) due to the signature we have \(g_{\theta\theta} = -\frac{(A_1 B_2 - A_2 B_1)^2}{g_{00}} > 0\) and outside the source the role of the angular variable is exchanged with that of time.

If \(\alpha_1^0 = \beta_1^0 \neq 0, g_{00}\) is linear at infinity, \(g_{\theta\theta}\) is constant and \(g_{\theta\theta}\) is zero. Finally if \(\alpha_i^0 = \beta_i^0 = 0\) we have a cylindric universe.

**IV. CLOSED TIME-LIKE CURVES AND THE WEAK ENERGY CONDITION.**

Recently considerable interest has been devoted to the problem of closed time-like curves (CTC) in 2+1 dimensional gravity. This is of interest also for the 3+1 dimensions as all solutions in 2+1 can be considered as solutions of 3+1 dimensional
gravity in presence of a space-like Killing vector field (cosmic strings). Gott [2] was able to produce examples of kinematical configurations of point particles without spin which produce CTC in 2+1 dimensions and this system came under close scrutiny [3–9]. Carroll, Fahri and Guth [5] proved that if the universe containing spinless point particles is open and has total time like momentum, no Gott pair can be created during its evolution thus supporting the idea that in an open time-like universe no CTC can form. ’t Hooft [9] for a system of spinless point particles, using a general construction of a complete series of time ordered Cauchy surfaces, concludes that, also in the case of a closed universe, if Gott pairs are produced as envisaged in [5], the universe collapses before a CTC can form. Tipler [10] and Hawking [11] assuming the weak energy condition proved in 3+1 dimensions that if CTC are formed in a compact region of space-time, then one must necessarily have the creation of singularities. On the other hand it is very simple to recognize [3] that a point like source with angular momentum produce sufficiently near the source CTC. In fact for a point like spinning particle the metric is given by

\[ ds^2 = (dt + 4J \, d\theta)^2 - \alpha^2 \rho^2 d\theta^2 - d\rho^2 \]  

where \( J \) is the angular momentum of the source in units \( c^3/G \) and \( \alpha = 1 - 4GM \). For \( 16J^2 - \alpha^2 \rho^2 > 0 \) i.e. for \( \rho < \left| \frac{4J}{\alpha} \right| \), we have the CTC \( \rho=\text{const} \, t=\text{const} \) and \( 0 \leq \theta < 2\pi \). Usually the appearance of such a CTC is ascribed to the unphysical nature of a point spinning particle as the energy momentum tensor is singular at the origin. A conjecture [3,11] states that if the metric is generated by a physical energy momentum tensor and proper boundary conditions are imposed, no CTC should develop. By physical sources one usually understands an energy momentum tensor that satisfies one or more among the weak, dominant and strong energy condition [19]. Using the results of the previous section, we shall show here that for a stationary source with
circular symmetry satisfying the WEC, if the universe is open, no CTC can appear provided there are no CTC at space infinity.

The proof follows from direct manipulations of Einstein’s equations in the form (43) combined with the WEC and it extends immediately to the case of cylindrical universes (Case 2 with $\alpha_1^0 = \beta_1^0 = 0$) and to linear universes (Case 1 with $(\alpha_2^0)^2 = (\beta_2^0)^2 \neq 0$).

It is immediately seen that a CTC implies the existence of a CTC with constant $\rho$ and $t$. In fact if we call $\sigma$ the parameter of the curve ($0 \leq \sigma < 1$), $t(\sigma)$ must satisfy $t(0) = t(1)$ and at the point $\sigma_0$ where $\frac{dt}{d\sigma} = 0$ we have that the tangent vector $(0, d\theta, d\rho)$ is time like i.e. also $(0, d\theta, 0)$ is time like, and thus the circle $t = 0$, $\rho = \rho(\sigma_0)$ is time-like. Thus to prove that CTC cannot exist it is sufficient to show that $g_{\theta\theta}$, which by assumption is negative at space infinity, cannot change sign.

To begin if the determinant of the dreibeins in the reduced radial gauge vanishes at a certain $\bar{\rho}$, it means that the manifold at $\rho = \bar{\rho}$ either closes or becomes singular. In fact let us consider the trace of the energy momentum tensor, which is an invariant. From eq. (38) we obtain

$$T^\mu_\mu = -\frac{1}{\kappa} \frac{(\det(e))''}{\det(e)} + \frac{\alpha_1 \beta_2 - \alpha_2 \beta_1}{\det(e)}$$

where $T^\mu_\mu$ is the trace of the energy momentum tensor, which is related to $T^{ab}$ by $T_{\mu\nu} = \rho \frac{T^{ab} e_{ab} e_{\mu\nu}}{\det(e)}$. $T^\mu_\mu$ is an invariant that we assume according to our general hypothesis to be regular.

We recall that $T^\mu_\mu = -\frac{1}{2\kappa} R$. We notice that the second term of the r.h.s. is the eigenvalue $\lambda_2$ of $T^{ab}$ and as such also an invariant. As a consequence $\frac{(\det(e))''}{\det(e)}$ is also an invariant and a regular function of $\rho$, according to our general assumption. Then if $\det(e)$ vanishes like $cr^\alpha$ with $r = \bar{\rho} - \rho$ we have $\alpha(\alpha - 1)r^{-2} = \text{regular}$ which implies either $\alpha = 0$ (then $\det(e)$ does not vanish) or $\alpha = 1$. In more detail, solving
\[
\frac{(\det(e))'}{\det(e)} = f(r) = \text{regular function of } r
\] (49)

one finds \(\det(e) = c r(1 + O(r^2))\). We distinguish two cases

1. \(A_2 \) and/or \(B_2 \neq 0 \) in \(\bar{\rho}\). Then there always exists a rotation which makes \(A_1 = B_1 = 0 \) in \(\bar{\rho}\). Thus without loss of generality we can consider the case \(A_2 \) and/or \(B_2 \neq 0 \) and \(A_1 = B_1 = 0 \). Then in a neighbourhood of \(\bar{\rho}\) the metric becomes

\[
ds^2 = r^2(\alpha_1^2 - \beta_1^2)dt^2 + 2r^2(\alpha_1\alpha_2 - \beta_1\beta_2)d\theta dt + (A_2^2 - B_2^2)d\theta^2 - dr^2.
\] (50)

If we impose, as required by eq. (49), that \(\det(g) = c r^2(1 + O(r^2))\) we must have in \(\bar{\rho}\), \((\alpha_1^2 - \beta_1^2)(A_2^2 - B_2^2) \neq 0\) and negative as required by eq. (43). If \(\alpha_1^2 - \beta_1^2 > 0\) and \(A_2^2 - B_2^2 < 0\) using the following transformation

\[
\tau = r \sinh \sqrt{\alpha_1^2 - \beta_1^2} t, \quad \zeta = r \cosh \sqrt{\alpha_1^2 - \beta_1^2} t
\] (51)

we reduce the metric to the regular form

\[
ds^2 = d\tau^2 - d\zeta^2 + (A_2^2 - B_2^2)d\theta^2 + 2\left(\frac{\alpha_1\alpha_2 - \beta_1\beta_2}{\sqrt{\alpha_1^2 - \beta_1^2}}\right)(\zeta d\tau - \tau d\zeta)d\theta.
\] (52)

But at the events \(\tau = 0\), \(\zeta = 0\), due to (51) we have only space-like displacements and thus the manifold is singular. If on the other hand \(\alpha_1^2 - \beta_1^2 < 0\) and \(A_2^2 - B_2^2 > 0\) using the following transformation

\[
x = r \cos \sqrt{\beta_1^2 - \alpha_1^2} t, \quad y = r \sin \sqrt{\beta_1^2 - \alpha_1^2} t
\] (53)

the metric takes the regular form

\[
ds^2 = -dx^2 - dy^2 + (A_2^2 - B_2^2)d\theta^2 + 2\left(\frac{\alpha_1\alpha_2 - \beta_1\beta_2}{\sqrt{\beta_1^2 - \alpha_1^2}}\right)(xdy - ydx)d\theta.
\] (54)

The only possibility to have a manifold in the neighbourhood of \(x = y = 0\) is that \(t\) is a periodic variable with period \(\frac{2\pi}{\sqrt{\beta_1^2 - \alpha_1^2}}\), i.e. the slice \(\rho = \text{const}\) would assume for
r small and positive the topology of $S_1 \times S_1$. But the topology near the origin $\rho = 0$ is $R \times S_1$ and as $\det(e) = g_{00}g_{\theta\theta} - g_{\theta\theta}^2$ does not vanish between 0 and $\bar{\rho}$ such a change of topology is not possible. Thus in $r = 0$ the manifold is singular.

2. $A_2 = B_2 = 0$ in $\bar{\rho}$. The metric for small $r$ becomes

$$ds^2 = (A_1^2 - B_1^2)dt^2 + 2r^2(\alpha_1\alpha_2 - \beta_1\beta_2)dtd\theta + r^2(\alpha_2^2 - \beta_2^2)d\theta^2 - dr^2$$

and for the same reason as in case 1 $(A_1^2 - B_1^2)(\alpha_2^2 - \beta_2^2) < 0$. Performing the rotation $\theta = \theta' - \frac{\alpha_1\alpha_2 - \beta_1\beta_2}{\alpha_2^2 - \beta_2^2}t$ we obtain

$$ds^2 = (A_1^2 - B_1^2 + O(r^2))dt^2 + r^2(\alpha_2^2 - \beta_2^2)d\theta'^2 - dr^2.$$  \hfill (56)

For $A_1^2 - B_1^2 > 0$ the universe closes like a cone in $\rho = \bar{\rho}$. Such a closure may be regular if $\alpha_2^2 - \beta_2^2 = 1$. For $A_1^2 - B_1^2 < 0$ by means of the transformation

$$x = r \cosh \sqrt{\alpha_2^2 - \beta_2^2} \theta \quad y = r \sinh \sqrt{\alpha_2^2 - \beta_2^2} \theta$$

identifying the points with $\theta = \pi$ with those with $\theta = -\pi$, we have the regular metric

$$ds^2 = c^2 dt^2 - dx^2 - dy^2.$$  \hfill (58)

But as $-\pi \leq \theta \leq \pi$, we have in $x = y = 0$ only space-like displacements and thus the manifold is singular.

The conclusion is that if $\det(e)(\bar{\rho}) = 0$ either the geometry is singular (divergence of the eigenvalues of the energy momentum tensor), or the manifold becomes singular, or the universe closes at $\rho = \bar{\rho}$.

We now show that the WEC combined with $\det(e) > 0$ and the absence of CTC at infinity, implies the absence of CTC for any $\rho$. In fact the validity of the WEC $\nu_a T^{ab}v_b \geq 0$ for the vectors $T^a + \Theta^a$ and $T^a - \Theta^a$ gives from eq.(38)

$$\frac{d}{d\rho}[(\alpha_2 \pm \beta_2)(B_1 \pm A_1) - (B_2 \pm A_2)(\alpha_1 \pm \beta_1)] \geq 0$$  \hfill (59)
that can be integrated to give

\[
E^{(\pm)}(\rho) \equiv (B_2 \pm A_2)(\alpha_1 \pm \beta_1) - (\alpha_2 \pm \beta_2)(B_1 \pm A_1) \geq
\]

\[
(B_2^0 \pm A_2^0)(\alpha_1^o \pm \beta_1^o) - (\alpha_2^0 \pm \beta_2^0)(B_1^0 \pm A_1^0)
\]

(60)

where \( E^{\pm}(\rho) \) does not depend on \( \rho \) for \( \rho > \rho_0 \). We study now the sign of \( E^{\pm}(\rho_0) \).

i) \( (\alpha_2^0)^2 - (\beta_2^0)^2 \neq 0 \) (conical universe).

Under the hypothesis that there are no CTC at infinity we have

\[
(\alpha_2^0)^2 - (\beta_2^0)^2 < 0,
\]

(61)

while the support equation for \( \tau^{\rho\rho} \) is

\[
(\alpha_1^0 + \beta_1^0)(\alpha_2^0 - \beta_2^0) - (\alpha_1^0 - \beta_1^0)(\alpha_2^0 + \beta_2^0) = 0.
\]

(62)

Using the symmetry equation (43) written in the form

\[
(A_2 + B_2)(\alpha_1 - \beta_1) + (A_2 - B_2)(\alpha_1 + \beta_1) -
\]

\[
(A_1 + B_1)(\alpha_2 - \beta_2) - (A_1 - B_1)(\alpha_2 + \beta_2) = 0
\]

(63)

and the determinant written as

\[
\det(e) = \frac{1}{2}[(A_2 - B_2)(A_1 + B_1) - (A_1 - B_1)(A_2 + B_2)].
\]

(64)

the r.h.s. of (60) becomes

\[
E^{(\pm)}(\rho_0) = -\frac{\alpha_2^0 \pm \beta_2^0}{\alpha_2^0 + \beta_2^0} \frac{d \det(e)}{d\rho} \big|_{\rho = \rho_0}
\]

(65)

that due to (61) and \( \frac{d \det(e)}{d\rho} \big|_{\rho = \rho_0} \geq 0 \) is non negative i.e. \( E^{(\pm)}(\rho_0) \geq 0 \).

ii) \( \alpha_2^0 = \beta_2^0 \neq 0 \) (linear universe).

Using the support equation we have \( \alpha_1^0 = \beta_1^0 \) which implies \( E^{(-)}(\rho_0) = 0 \). From the equation of motion (63) and \( \det(e) \neq 0 \) one obtains \( A_2^0 - B_2^0 \neq 0 \) and thus we have
\[ E^{(+)}(\rho_0) = -2 \frac{\alpha_0^0 + \beta_0^0}{A_2^0 - B_2^0} \det(e). \] (66)

But \( \det(e) > 0 \) while the sign of \( \frac{\alpha_0^0 + \beta_0^0}{A_2^0 - B_2^0} \) has to be negative if there are no CTC at infinity, because \( A_2^0 - B_2^0 \) in this case behaves linearly at infinity.

Similarly for \( \alpha_0^2 = -\beta_0^2 \) we obtain \( E^{(+)}(\rho_0) = 0 \) and \( E^{(-)}(\rho_0) > 0 \).

iii) If \( \alpha_1^0 = \alpha_2^0 = \beta_1^0 = \beta_2^0 = 0 \) (cylindrical universe) we have also \( E^{(\pm)}(\rho_0) = 0 \).

The only case that escapes our analysis is \( \alpha_2^0 = \beta_2^0 = 0 \) and \( \alpha_1^0 \) and/or \( \beta_1^0 \neq 0 \); this situation corresponds to a cylindrical universe generated by a string with tension and total angular momentum 0. Thus except for this case we have that the r.h.s. of equation (60) is \( E^{(\pm)}(\rho_0) \geq 0 \).

Let us now consider the following combination

\[ (A_2 - B_2)^2 E^{(+)}(\rho) + (A_2 + B_2)^2 E^{(-)}(\rho) \geq 0. \] (67)

A little algebra shows that the l.h.s. equals

\[ -2 \det(e)^2 \frac{d}{d\rho} \left( \frac{A_2^2(\rho) - B_2^2(\rho)}{\det(e)} \right). \] (68)

Thus we reached the conclusion that \( \frac{d}{d\rho} \left( \frac{A_2^2(\rho) - B_2^2(\rho)}{\det(e)} \right) \leq 0 \); it means that \( \left( \frac{A_2^2(\rho) - B_2^2(\rho)}{\det(e)} \right) \) is a non increasing function of \( \rho \). As \( \left( \frac{A_2^2(\rho) - B_2^2(\rho)}{\det(e)} \right) \) at the origin is zero we obtain that \( A_2^2(\rho) - B_2^2(\rho) \) is always negative and thus we cannot have CTC.

In the case of a regular open universe the hypothesis that there are no CTC at infinity and that at least an average form of the WEC is satisfied (see eq.(60)), are not only sufficient but also necessary for the absence of CTC. In fact we give now examples of regular sources that violate the WEC and produce CTC for finite radius but no CTC at infinity. In addition we shall construct examples of sources satisfying the WEC
and produce CTC for any radius larger than a certain $\rho_1$.

With regard to the first example we consider the following functions

$$\alpha_1 = 0, \quad \beta_2 = 1$$

and due to the eq. (43)

$$\alpha_2 = -B_1 + \rho \beta_1.$$  \hspace{1cm} (70)

The metric becomes

$$ds^2 = (1 - B_1^2)dt^2 - 2(A_2 + \rho B_1)dtd\theta + (A_2^2 - \rho^2)d\theta^2 - d\rho^2.$$  \hspace{1cm} (71)

In addition we choose $\beta_1(0) = 0$, $\beta_1(\rho) = 0$ for $\rho > \rho_0$ and $\int_0^{\rho_0} \beta_1(\rho) d\rho = 0$ i.e. $B_1(\rho) = 0$ for $\rho > \rho_0$. We have

$$\int_0^{\rho_0} \alpha_2(\rho) \ d\rho = A_2(\rho_0) = \int_0^{\rho_0} (-B_1 + \rho \beta_1) \ d\rho = \rho_0 B_1(\rho_0) - 2 \int_0^{\rho_0} B_1(\rho) \ d\rho = -2 \int_0^{\rho_0} B_1(\rho) \ d\rho$$

that we shall choose different from zero. Then the exterior metric ($\rho > \rho_0$) is

$$ds^2 = (dt - A_2(\rho_0)d\theta)^2 - \rho^2d\theta^2 - d\rho^2,$$  \hspace{1cm} (73)

and it obviously possesses CTC for $\rho < A_2(\rho_0)$. By properly taking the normalization of $\beta_1$ we can always have $|A_2(\rho_0)| > \rho_0$. The proved theorem tell us that our source must violate the WEC. This can be also directly seen from the fact that $T^{00} = -\frac{1}{\kappa \rho} A_2 \beta_1'$ changes sign at the point where $\beta_1'$ reaches the first zero starting from the origin because there $A_2(\rho)$ has a well defined sign due to the fact that $\alpha_2$ is monotonic up to the first zero of $\beta_1'$ (from (70) we have $\alpha_2' = \rho \beta_1'$ and $\beta_1'$ must possess a zero for $\rho$ belonging to $[0, \rho_0]$ because $\beta_1(0) = \beta_1(\rho_0) = 0$).

We come now to the second example i.e. of a source that satisfies the WEC and
generates a metric with CTC at infinity. To this purpose we take \( \alpha_1 = 0 \), \( \beta_1 = 1 \), \( \beta_2 = 1 + f'(\rho) \) with \( f'(\rho) = o(\rho) \) and \( f'(\rho) = -1 \) for \( \rho \geq \rho_0 = 1 \). Then

\[
\alpha_2 = f - \rho f' , \quad A_2 = 2F - \rho f \quad \text{and} \quad B_2 = \rho + f
\]

with \( f(\rho) = \int_0^\rho f'(\rho) \, d\rho \) and \( F(\rho) = \int_0^\rho f(\rho) \, d\rho \). The energy momentum tensor has the following form

\[
\begin{align*}
T^{00} &= -\frac{1}{\kappa \rho} f''(\rho) , & T^{0\theta} &= \frac{1}{\kappa \rho} \rho f''(\rho) \\
T^{\theta\theta} &= -\frac{1}{\kappa \rho} \rho^2 f''(\rho) , & T^{\rho\rho} &= \frac{1}{\kappa \rho} (f(\rho) - \rho f'(\rho)),
\end{align*}
\]

and the other elements equal to zero. The eigenvalues and relative eigenvectors are

\[
\begin{align*}
\lambda_0 &= -\frac{1}{\kappa \rho} f''(\rho)(1 - \rho^2) , & v^0 &= (1, -\rho, 0); \\
\lambda_1 &= 0 , & v^1 &= (\rho, -1, 0); \\
\lambda_2 &= \frac{1}{\kappa \rho} (\rho f'(\rho) - f(\rho)) , & v^2 &= (0, 0, 1).
\end{align*}
\]

For \( f''(\rho) \leq 0 \) \( \lambda_0 \geq 0 \) thus \( \lambda_0 \geq \lambda_1 \). The \( \lambda_2 \) is negative because \( \lambda'_2 = \rho f'' \leq 0 \) and \( \lambda_2(0) = 0 \). Thus \( \lambda_1 \geq \lambda_2 \) and we have satisfied the WEC [19]. Outside the source i.e. \( \rho \geq 1 \) \( g_{\theta \theta} \) becomes \( [A_2(1) + (\rho - 1)\alpha_2(1)]^2 - B_2^2(1) \) and thus as \( \alpha_2(1) \neq 0 \) (we recall that \( \alpha'_2 = -\rho f'' \) \( g_{\theta \theta} \) goes like \( \alpha_2(1) \rho^2 \) for \( \rho \to \infty \), thus giving CTC at infinity.

As mentioned above the only case of open universe with no CTC at infinity that escapes our analysis is the rather unphysical situation of a cylindrical universe with zero angular momentum of the type generated by a closed string with tension [3] for which on the other hand no CTC exists outside the source.

V. CONCLUSION

In this paper we developed a quadrature procedure for solving the general stationary problem in 2+1 dimensional gravity in presence of extended sources also endowed
with angular momentum. Such an approach reveals itself particularly useful in treating the problem of CTC in the case of rotational symmetry. The result is that an average form of the WEC (which is a consequence of the WEC itself) and the absence of CTC at space infinity exclude the existence of CTC everywhere for open universe, conical at infinity. Explicit counterexamples show that such conditions are not only sufficient but also necessary in the sense it is possible to find sources which do not satisfy the WEC, give rise to CTC for finite $\rho$ but generate no CTC at infinity, but also sources exist that satisfy the WEC and give rise to CTC at infinity. We mention that the radial gauge approach works also for the time dependent situation [18] and thus it appears a good candidate for treating the problem of evolution of continuous distribution of matter in 2+1 dimensional gravity.

APPENDIX A

It was shown in section 4 of [17] that the gauge

$$\sum_i \xi^i \Gamma^a_{bi} = 0 \quad (A1)$$

is attainable. We prove now formulae (3-6) of Sec.2 of the body of the paper which play a major role in the subsequent developments.

From (A1) and (A2) and the regularity of $\Gamma^a_{bi}$ and $e^a_\mu$ at the origin we obtain

$$\Gamma^a_{bi}(0) = 0 \quad \text{and} \quad e^a_i(0) = \delta^a_i. \quad (A3)$$

From the definition of the components $R^a_{b\mu}$ of the Riemann two form we obtain

$$\xi^i R^a_{bi\mu} = \xi^i \partial_i \Gamma^a_{b\mu} + \Gamma^a_{bi} \delta^i_\mu \quad (A4)$$
and thus
\[ \Gamma^a_{bi}(\xi) = \xi^i \int_0^1 R^a_{bij}(\lambda \xi) \lambda d\lambda. \] (A5)

For \( \Gamma^a_{b0}(\xi) \) we have
\[ \xi^i R^a_{b0} = \xi^i \partial_i \Gamma^a_{b0} \] (A6)

and thus
\[ \Gamma^a_{b0}(\xi) = \Gamma^a_{b0}(0) + \xi^i \int_0^1 R^a_{b0}(\lambda \xi) d\lambda \] (A7)

where \( \Gamma^a_{b0}(0) \) is an arbitrary constant of integration. For the vierbeins we have
\[ \xi^i \partial_i e^a_j - \xi^i \Gamma^a_{i0} = \xi^i S^a_{i0} \] (A8)

being \( S^a_{i0} \) the torsion, from which
\[ e^a_0(\xi) = \delta^a_0 + \xi^i \int_0^1 \Gamma^a_{i0}(\lambda \xi) d\lambda + \xi^j \int_0^1 S^a_{j0}(\lambda \xi) d\lambda. \] (A9)

Substituting (A5) into (A9) we obtain eq. (6) of sect.II. Similarly for \( e^a_i \) we obtain from
\[ \xi^i \partial_i e^a_j - \xi^i \partial_j e^a_i = \xi^i \Gamma^a_{ij} + \xi^i S^a_{ij} \] (A10)
\[ e^a_i = \delta^a_i + \xi^j \int_0^1 \Gamma^a_{ji}(\lambda \xi) \lambda d\lambda + \xi^j \int_0^1 S^a_{ji}(\lambda \xi) \lambda d\lambda \] (A11)

from which we derive eq. (5) of sect.II. It is worth mentioning that in all these equations as in eq. (5) and (6) of sect.II \( R^a_i \) and \( S^a \) are the Riemann and torsion two forms in the radial gauge and not in an arbitrary gauge and thus they cannot be chosen arbitrarily but they are subject to a reduced form of Bianchi identities analogous to those found in ref. [16].
We derive here the general conditions imposed upon the functions $\alpha_i$, $\beta_i$ and $\gamma_i$ by the regularity at the origin of the energy momentum tensor $\tau^\rho_\rho$. The simplest way to proceed is to express the unit vectors $\Theta$ and $P$ in terms of the cartesian versors $X$ and $Y$. Denoting with $T^{xx}, T^{xy}$ etc. the cartesian components we obtain the following behaviors

$$\alpha_1 = \Gamma^0_0(0) \cos \theta + \Gamma^0_{02}(0) \sin \theta - \kappa [T^{xx} \sin^2 \theta + T^{yy} \cos^2 \theta - (T^{xy} + T^{yx}) \sin \theta \cos \theta] \rho + o(\rho)$$

(B1)

$$\beta_1 = \Gamma^1_0(0) - \kappa (-T_{0x} \sin \theta + T_{0y} \cos \theta) \rho + o(\rho)$$

(B2)

$$\gamma_1 = \Gamma^0_0(0) \sin \theta - \Gamma^0_{02}(0) \cos \theta - \kappa [T^{xy} \cos^2 \theta - T^{yx} \sin^2 \theta + (T^{yy} - T^{xx}) \sin \theta \cos \theta] \rho + o(\rho)$$

(B3)

$$\alpha_2 = -\frac{\kappa}{2} (-T^{x0} \sin \theta + T^{y0} \cos \theta) \rho + o(\rho^2)$$

(B4)

$$\beta_2 = 1 - \frac{\kappa}{2} T^{00} \rho^2 + o(\rho^2)$$

(B5)

$$\gamma_2 = -\frac{\kappa}{2} (T^{x0} \cos \theta + T^{y0} \sin \theta) \rho^2 + o(\rho^2)$$

(B6)

Eqs. (33) and (34) of the text describe the compactness of the support of the energy momentum tensor $\tau^\rho_\rho$. We show here that necessary and sufficient condition for them to be satisfied is
\[ \alpha_1 B_1 - \beta_1 A_1 = \text{constant} \quad \text{for} \quad \rho > \rho_0(\theta) \quad (C1) \]

and

\[ \alpha_1^2 - \beta_1^2 + \gamma_1^2 = \text{constant} \quad \text{for} \quad \rho > \rho_0(\theta) \quad (C2) \]

where \( \alpha_1, \beta_1, \gamma_1 \) become all independent of \( \rho \) for \( \rho > \rho_0(\theta) \). The constants appearing in (C1) and (C2) do not depend both on \( \rho \) and \( \theta \).

First we prove (C1) and (C2) starting from (33) and (34). In fact multiplying (34b) by \(-\beta_1\), (34c) by \(\alpha_1\) and (34a) by \(-\gamma_1\) and then summing we obtain

\[ \frac{\partial}{\partial \theta}(\alpha_1^2 - \beta_1^2 + \gamma_1^2) = 0 \quad \text{for} \quad \rho > \rho_0(\theta). \quad (C3) \]

But as for \( \rho > \rho_0(\theta) \), \( \alpha_1, \beta_1, \gamma_1 \) do not depend on \( \rho \) we have equation (C2).

We come now to the proof of eq. (C1). For generic functions \( \alpha_i, \beta_i, \gamma_i \) that for \( \rho > \rho_0(\theta) \) do not depend on \( \rho \), the necessary and sufficient condition for having \( \alpha_2' = 0 \) for \( \rho > \rho_0(\theta) \), is the constancy in \( \rho \) of \( A_2 \alpha_1 - A_1 \alpha_2 \). In fact for \( \rho > \rho_0(\theta) \) \( A_i \) become

\[ A_i = \alpha_i^0(\rho - \rho_0) + A_i^0. \]

From eq. (27) this means that

\[ A_i^2 \frac{\partial}{\partial \rho} \left( \frac{A_2}{A_1} \right) = \frac{A_i^2 B_i^2}{B_i^2 - A_i^2 \frac{\partial}{\partial \rho} \left( \frac{N}{A_1 B_1} \right)} \equiv K(\rho, \theta) \quad (C4) \]

does not depend on \( \rho \) for \( \rho > \rho_0(\theta) \). Performing explicitly the derivative on the r.h.s. and substituting \( N \) given by (24) into (C4) and taking into account that \( \gamma_1 \) does not depend on \( \rho \) for \( \rho > \rho_0(\theta) \), we obtain always for \( \rho > \rho_0(\theta) \)

\[ A_i^2(B_1 \frac{\partial \alpha_1}{\partial \theta} - \beta_1 \frac{\partial A_1}{\partial \theta}) - B_i^2(A_1 \frac{\partial \beta_1}{\partial \theta} - \alpha_1 \frac{\partial B_1}{\partial \theta}) = \frac{H(\theta)}{2}(B_i^2 - A_i^2). \quad (C5) \]

For \( \rho > \rho_0(\theta) \) both members of (C5) become second degree polynomials in \( \rho \). Equating the coefficients we reach, always for \( \rho > \rho_0(\theta) \),

\[ B_1 \frac{\partial \alpha_1}{\partial \theta} - \beta_1 \frac{\partial A_1}{\partial \theta} = A_1 \frac{\partial \beta_1}{\partial \theta} - \alpha_1 \frac{\partial B_1}{\partial \theta} \equiv -\frac{H(\theta)}{2} \quad (C6) \]

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\[
\frac{\partial}{\partial \theta} (B_1 \alpha_1 - A_1 \beta_1) = 0. \tag{C7}
\]

Viceversa \( \alpha'_1 = \beta'_1 = \gamma'_1 = 0 \) and (C1) and (C2) imply the compactness conditions. In fact from eq. (C7) going back we have that \( A_2 \alpha_1 - A_1 \alpha_2 = f(\theta) \) i.e. \( \alpha'_2 = 0 \) for \( \rho > \rho_0(\theta) \). From eq. (22a) we have now \( \beta'_2 = 0 \). Multiplying (22b) by \( \beta_1 \) and (22c) by \( \alpha_1 \) and subtracting we obtain

\[
\gamma'_2 (\beta_1 A_1 - \alpha_1 B_1) = \gamma_1 (\alpha_2 \beta_1 - \alpha_1 \beta_2) + \frac{1}{2} \frac{\partial}{\partial \theta} (\beta_1^2 - \alpha_1^2). \tag{C8}
\]

But due to eq. (24), always for \( \rho > \rho_0(\theta) \), the r.h.s. of (C8) vanishes and thus \( \gamma'_2 = 0 \). The vanishing of \( \gamma'_i \) via eqs. (22b) and (22c) implies eqs. (34b) and (34c). From (C2) and the vanishing of the r.h.s. of (C8) we obtain

\[
\gamma_1 (\alpha_1 \beta_2 - \alpha_2 \beta_1) - \gamma_1 \frac{\partial \gamma_1}{\partial \theta} = 0 \tag{C9}
\]

i.e. (34a).
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