Closed string field theory, strong homotopy Lie algebras

and the operad actions of moduli spaces

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I’d like to thank Bob Penner for organizing such a well-coordinated conference. ‘A funny thing happened on the way’ to this conference. Between the time that Bob set it up initially and last Monday (November 2, 1992), there has been a flood of new results. Being on leave at the University of Pennsylvania, I was able to learn first hand of Y.-Z. Huang’s success in following my suggestion that something like an ‘operad’ was present in his approach to VOA’s (vertex operator algebras). Gregg Zuckerman invited me to Yale, intuiting that if he and I and Bong Lian got together, something synergistic would take place. Meanwhile, thanks to e-mail, I had received notes of a talk Ezra Getzler had given in Sydney which in turn led me to learn more of his work in progress with J.D.S. Jones and that of Ginzburg-Kapranov. When I told Peter May of this renaissance in operad theory in these several applications, he responded with a preprint [KM] where the general theory is advanced, precisely by the consideration of partial algebras over operads, analogous to the partial operads which are essential to Huang’s point of view.

Today I will try to do what physicists call a ‘review’, a survey talk trying to bring all these strands together.

{Post conference: This talk was a preliminary effort, based on rough drafts which had arrived only in the previous week. Things have not let up since I gave the talk; I will do my best to make this report up to date as of the day submitted. For historical purposes, remarks unknown to me in November 1992 are inserted as is this remark. A major influence during the development of the talk and this paper has been provided by seminars at the University of Pennsylvania and especially by my co-conspirators: Y.-Z. Huang, T. Kimura and A. Voronov. }

Let me begin, however, with a topic where some time ago I was able to give a mathematical interpretation of the corresponding physics.

1. Closed string field theory

When particles are conceived of as points, much of particle physics can be described in terms of Lie algebras and their representations. Closed string field theory, on the other hand, leads to a gen-

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eralization of Lie algebra which arose naturally within mathematics in the study of deformations of algebraic structures [SS]. It also appeared in work on higher spin particles [BBvD]. Representation theoretic analogs arose in the mathematical analysis of the Batalin-Fradkin-Vilkovisky approach to constrained Hamiltonians [S6].

String field theory is multi-layered, often presented as involving topology, geometry, algebra and analysis, especially analysis in the sense of Riemann surfaces. The bottom layer is the topology of string configurations.

The obvious picture of a closed string is that of a closed curve in a (Riemannian) manifold \( M \). The first subtlety one encounters with this picture is that the physics is often described in terms of a parameterization of such a curve, e.g. a map of the standard circle \( S^1 \) (parameterized from 0 to \( 2\pi \)) into \( M \), but the physics should not depend on the parameterization, although expressed in terms of it. Think of arc length of a parameterized curve. Thus the space of closed strings \( \mathcal{C} \) can be described as the space of equivalence classes (under reparameterization) of maps of the standard (parameterized) circle into \( M \):

\[
\text{Map}(S^1, M)/\text{Diff}^+ S^1.
\]

We begin by reviewing the joining of two strings to form a third. The picture is the familiar ‘pair of pants’:

Figure 1

The details in terms of parameterization extend a method due to Lashof [L] in the case of based loops and to Witten [W1] for strings. The idea is that two closed strings \( Y \) and \( Z \) join to form a third \( Y \ast Z \) if a semi-circle of one agrees with a reverse oriented semi-circle of the other. (Notice this avoids Witten’s marking of the circle.) The join \( Y \ast Z \) is formed from the complementary semi-circles of each. To be more precise, consider the configuration of three arcs \( A_i, i = 0, 1, 2 \) with the three initial points identified and the three terminal points identified, as in a circle together with a diameter.

Figure 2

(One is tempted to call this a (theta) \( \Theta \) curve, but string field theory is likely to involve theta
functions in the sense of number theory; there's enough confusion of terminology already!) To emphasize the symmetry and to fix parameterizations, consider the arcs to be great semi-circles on the unit 2-sphere in $\mathbb{R}^3$ parameterized by arc length from north pole to south pole. Denote the union of the three arcs by $\Theta$. Denote by $\bar{A}_i$ the arc parameterized in the reverse direction. Let $C_i$ denote any isometry $C_i : S^1 \leftarrow \Theta$ which agrees with $A_j$ on one semi-circle and with $\bar{A}_k$ on the other for a cyclic permutation $(i,j,k)$ of $(0,1,2)$. (Up to rotation, $C_i$ is $A_j$ followed by $\bar{A}_k$.)

Given any map $X : \Theta \rightarrow M$, let $X_i = X \circ C_i$ and think of $X_0$ as the fusion of $X_1$ and $X_2$.

When physicists speak of closed string field theory, I would like to think they are referring to ‘fields’ which are some sort of functions, or more generally some sort of sections of some bundle (unspecified) on the space $C$ of closed strings, $\text{Map}(S^1, M)$. Actually Zwiebach (cf. [SZ], [KKS], [K], [Wies], [WZ], [Z]) stipulates that the string fields $\phi_1, \phi_2, \cdots$ are elements of $\mathcal{H}$, a Hilbert space of a combined conformal field theory of matter and ‘ghosts’. The presence of ghosts indicates the presence of a ‘BRST operator’, which corresponds to the exterior derivative along the leaves of the reparameterization orbits.

A convolution product for $\phi, \psi$ function(al)s on $C$, the space of closed strings, is defined in terms of such maps (although the details are unimportant for this paper):

Define the convolution product $\phi \ast \psi$ as follows:

$$(\phi \ast \psi)(X_0) = \int \phi(X_1)\psi(X_2)dX$$

where the integrals are over all isometries $C_i$ and over all maps $X : \Theta \rightarrow M$ such that $X_0 = X \circ C_0$. Thus $\phi \ast \psi$ depends on all ways of decomposing $X_0$ into two loops $X_1$ and $X_2$. (The range $M$ is ‘space’ and not ‘space-time’ so there is no problem with a Lorentz metric. In the case of the standard metric on $M = \mathbb{R}^d$, notice that this integral is over paths from $X_0(\theta)$ to $X_0(\theta + \pi)$ and hence is well defined as a Brownian bridge.)

Given a product, we ask immediately for its algebraic properties. With an appropriate grading, the convolution product is graded commutative but is nothing like associative; rather it comes close to satisfying a graded Jacobi identity. For this reason, we change notation and define

$$[\phi, \psi] = \phi \ast \psi.$$ 

For three fields $\phi_i$, $i = 1, 2, 3$, there exists a trilinear $[\phi_1, \phi_2, \phi_3]$ such that

$$(1) \quad [\phi_1, [\phi_2, \phi_3]] \pm [[\phi_1, \phi_2], \phi_3] \pm [\phi_2, [\phi_1, \phi_3]] = d_1[\phi_1, \phi_2, \phi_3] \pm [d_1 \phi_1, \phi_2, \phi_3] \pm [\phi_1, d_1 \phi_2, \phi_3] \pm [\phi_1, \phi_2, d_1 \phi_3].$$
where \( d \) is the vertical differential along the reparameterization orbits. The form of the (super) Jacobi identity used here is equivalent to other standard forms (e.g. the cyclic form) since the bracket is super anti-commutative. In more general algebras, it will not be equivalent to other standard forms; this derivation form has the easiest set of signs to remember and is the appropriate one when considering operators.

From this equation, we see that the Jacobi identity holds not strictly but rather modulo the right hand side. In physical language, the Jacobi identity holds modulo a BRST exact term. In the language of homological algebra, \( d_3 \) is a chain homotopy, so we say that \((V,\lbrack,\rbrack)\) satisfies the Jacobi identity up to homotopy or \((V,d_1,\lbrack,\rbrack,\lbrack,\rbrack,\lbrack,\rbrack)\) is a homotopy Lie algebra.

The genesis of this trilinear is an interpretation which sees the theta curve as imbedded in a world sheet, which led several physicists, starting with Kaku [K], to consider a tetrahedral configuration in which the perimeter of each face is regarded as isometrically a circle (to be mapped via a closed string). This requires that that pairs of opposite edges have equal lengths, say \((a_1,a_2,a_3)\), with \(0 \leq a_i \leq \pi\) and \(\Sigma a_i = 2\pi\). Denote such a tetrahedron by \(\Delta^3(a_1,a_2,a_3)\). Now consider an orientation preserving \(C_i\) taking \(S^1\) isometrically to the boundary of the \(i\)-th face of \(\Delta^3(a_1,a_2,a_3)\).

Figure 4

My interpretation of the trilinear is then:

\[
[\phi_1,\phi_2,\phi_3](X_0) = \frac{1}{6} \sum \int \int \phi_{i_1}(X_1)\phi_{i_2}(X_2)\phi_{i_3}(X_3)dX
\]

where the sum is over all permutations \((i_1,i_2,i_3)\) and the integration with respect to \(dX\) is over all maps \(X: \Delta^3(a_1,a_2,a_3) \to M\) such that \(X \circ C_0 = X_0\), for all \(\Delta^3(a_1,a_2,a_3)\) and then the integration is over all isometries \(C_i, i = 1,2,3\) and the space of all \(\Delta^3(a_1,a_2,a_3)\). The integral with respect to \(dX\) is less standard than a Brownian bridge, but has been addressed carefully by Wiesbrock [Wies]. The homotopy algebra we are concerned with is carried by the more ordinary integration over the moduli space with isometries. That the trilinear does indeed satisfy (1) follows from an application of Stokes’ Theorem, the right hand side arising as the boundary terms when one of the \(a_i\) equals \(\pi\) (cf. Figure 5).

2. Restricted Polyhedra

To proceed further, it proved necessary to consider polyhedra of more than four faces. Extension to polyhedra with 5 faces was worked out by Saadi and Zwiebach [SZ] and their lead was carried through to general polyhedra by Kugo, Kunitomo and Suehiro [KKS]. Here ‘polyhedron’ refers to a cell decomposition of the oriented 2-sphere in which each face (=2-cell) has boundary
(perimeter) consisting of a finite number of edges (1-cells). Each face and hence its perimeter carries the orientation induced from $S^2$. The polyhedra are restricted geometrically in that each edge is assigned a length such that:

(2.a) Saadi and Zwiebach: the perimeter of each face has length $2\pi$ (which implies each edge has length $\leq \pi$), and

(2.b) Kugo, Kunitomo and Suehiro: any simple closed edge path has length $\geq 2\pi$.

(At the conference, Bonahon called my attention to the remarkable fact that exactly these restrictions are the hypotheses for Rivin’s Theorem which has just recently appeared in Bull AMS, though based on his earlier work [R].)

**Theorem:** (Characterization of ideal convex polyhedra) The dual polyhedron $P^*$ of a convex ideal polyhedron $P$ in $H^3$ satisfies the following conditions (in terms of ‘weights’ $w(e^*)$ for edges of $P^*$):

**Condition 1.** $0 < w(e^*) < \pi$ for all edges $e^*$ of $P^*$.

**Condition 2.** If the edges $e_1^*, e_2^*, \ldots, e_k^*$ form the boundary of a face of $P^*$, then

$$w(e_1^*) + w(e_2^*) + \cdots + w(e_k^*) = 2\pi.$$  

**Condition 3.** If $e_1^*, e_2^*, \ldots, e_k^*$ form a simple circuit which does not bound a face of $P^*$, then

$$w(e_1^*) + w(e_2^*) + \cdots + w(e_k^*) > 2\pi.$$  

Conversely, any abstract polyhedron $P^*$ with weighted edges satisfying the conditions 1-3 is the Poincaré dual of a convex ideal polyhedron $P$ with the exterior dihedral angles equal to the weights.

We return to restricted polyhedra and consider the “moduli” spaces thereof in some detail. There is only one restricted trihedron, Θ.

For tetrahedra, the restrictions are precisely that opposite edges have equal lengths, say $(a_1, a_2, a_3)$, with $0 \leq a_i \leq \pi$ and $\Sigma a_i = 2\pi$ (Figure 4). In other words, the “moduli” space of restricted tetrahedra is given by the union of two 2-simplices (one for each orientation of the tetrahedron) with vertices in common.

![Figure 5](image-url)

More generally, let $\mathcal{V}_N$ be the “moduli” space of all restricted $(N+1)$-hedra $P$ with an arbitrary ordering of the faces from 0 to N. (We have tried to keep our notation close to Zwiebach’s; in particular, his $n = N + 1$. As a space, $\mathcal{V}_N$ is given the topology of the local coordinates which are the edge lengths - in fact, this is a cell decomposition. This moduli space $\mathcal{V}_N$ of all restricted
(N + 1)-hedra as defined is manifestly a union of cells indexed by the topological type of the polyhedron (and the ordering). For a given topological type, the restrictions 1) and 2) with edge lengths > 0 describe an open convex subspace (polytope) $\mathcal{V}_N$ of $R^E$ where $E$ is the number of edges of the (N + 1)-hedron. Strictly speaking, when an edge length goes to zero, the topological type of the (N + 1)-hedron changes and thus two cells are glued together along a cell of one lower dimension.

Thus $\mathcal{V}_N$ is described as a finite cell complex in which each cell has boundary composed of a finite number of cells of lower dimension. The cells of maximal dimension correspond to 3-valent polyhedra and the dimension of these cells is $2N - 4 = 2(N + 1) - 6$, with faces corresponding to (N + 1)-hedra with one 4-valent vertex, etc. when an edge length goes to zero.

I have emphasized the topology of $\mathcal{V}_N$ as a cell complex. Zwiebach instead regards $\mathcal{V}_N$ as a subspace of the traditional moduli space $\mathcal{M}_{0,N+1}$ of Riemann spheres with $N + 1$ punctures by filling in each face of the polyhedron with a punctured disk realized in terms of a metric of minimal area $|Z|$. Thus $\mathcal{V}_N$ can be regarded as a subspace of $\mathcal{M}_{0,N+1}$ “cut-off” from the degenerations of Riemann spheres having double points appearing in various compactifications of $\mathcal{M}_{0,N+1}$ [Kn] [D] [FM].

Decorations

We are going to “decorate” such polyhedra with specific parameterizations $C_i : S^1 \hookrightarrow P$ which are isometries with the perimeter of the i-th face - analogous to a local coordinate at one of (N + 1) punctures on a Riemann surface of genus 0.

Let $\mathcal{P}_N$ denote the space of ordered restricted (N + 1)-hedra with such specified isometries $C_i : S^1 \hookrightarrow P$. It is naturally a bundle over $\mathcal{V}_N$ with fibres $(S^1)^{N+1}$ (homeomorphic to $\mathcal{V}_N \times (S^1)^{N+1}$). We will use the notation $\bar{P} = (P, C_0, \ldots, C_N) \in \mathcal{P}_N$. Note that Zwiebach’s $\mathcal{P}_{0,n}$ is more fully decorated with arbitrary complex local coordinates at the punctures. Though natural in conformal field theory, these more general coordinates are not needed for the structures we consider.

3. The N-ary operations

We are now ready to define N-ary operations, multi-linear ‘brackets’, which are key to the closed string field theory of KKS and Zwiebach, in that they satisfy the crucial relation

$$d[\phi_1, \ldots, \phi_N] + \Sigma_i^N \pm [\phi_1, \ldots, d\phi_i, \ldots, \phi_N] = \Sigma_{Q=2}^{N-1} \pm [[\phi_{i_1}, \ldots, \phi_{i_Q}], \phi_{i_{Q+1}}, \ldots, \phi_{i_N}]$$

where the signs are the usual one for interchanging graded objects.
Closed SFT and operads

{ Post conference: Here is a revisionist version of my description of a portion of closed string field theory, which hopefully has benefitted from discussions with Alexander Voronov (since my talk). The data consists of a Hilbert space $\mathcal{H}$, graded by ‘ghost number’, an integer; in fact, $\mathcal{H}$ is a differential graded Hilbert space; the differential is of degree $+1$, is denoted $Q$ and is called a BRST operator. For our purposes, the essential aspect of a conformal field theory is that it provides a chain map from $\mathcal{H}^\otimes N+1$ to the ordinary differential forms on $\mathcal{P}_N$:

$$ \mathcal{H}^\otimes N+1 \to \Omega^\bullet(\mathcal{P}_N). $$

(As so often in physics, $\mathcal{H}$ is identified with its linear dual $\mathcal{H}^*$ and similarly for iterated tensor products.)

If the result is then pulled down to $|\text{cal}V_N$ via a section and integrated over $\mathcal{V}_N$, we obtain

$$ <\psi_0\ldots\psi_N> \quad \text{for} \quad \psi_i \in \mathcal{H} $$

which, if the ghost numbers of the $\psi_i$ have the correct total, is a number, called the correlator or $N+1$-point function of the $\psi_i$. The $N$-fold brackets are then determined by the correlators or vice versa via:

$$ <\psi_0\ldots\psi_N> = <\psi_0[\psi_1,\ldots,\psi_N]> $$

or

$$ [\psi_1,\ldots,\psi_N] := \Sigma \psi^\alpha <\psi_\alpha\psi_1\ldots\psi_N>. $$

4. The bracket relations

The cell complex $\mathcal{P}_N$ does have a “boundary” corresponding to saturation of the inequalities (2.b). KKS refer to such polyhedra as critical, i.e. if there is an edge path enclosing at least two faces and of length precisely $2\pi$. Separating the polyhedron $P$ along this edge path produces two restricted polyhedra $Q$ and $R$.

Figure 7

The separation can be regarded as giving a partition of the set $\{0,\ldots,N\}$ or, if the faces of $P$ are ordered, as an unshuffle giving induced orderings on $Q$ and $R$. (An unshuffle of $\{1,\ldots,n\}$ is a permutation that keep $i_1,\ldots,i_j$ and $i_{j+1},\ldots,i_n$ in the same relative order. A shuffle of two ordered sets (decks of cards) is a permutation of the ordered union which preseves the order of each of the given subsets; an unshuffle reverses the process, cf. Maxwell’s demon.)
Conversely, if we have two decorated restricted polyhedra $Q$ and $R$, we can form a connected sum $Q \# R$ by deleting two faces $F_Q \in Q$ and $F_R \in R$ and identifying their perimeters by an orientation reversing isometry. (Since we regard $S^1$ as parameterized once and for all from 0 to $2\pi$, the orientation reversing isometry is uniquely specified by requiring that the images of 0 coincide.) Generically, the result will be a restricted polyhedron $P$ (although occasionally the identification may produce vertices of valence greater than 3). If the faces of $Q$ and $R$ are ordered, we can define such a connected sum for each shuffle.

Thus we can describe boundary “facets” of $P_N$ by inclusions

$$\mathcal{P}_Q \times S^1 \mathcal{P}_R \hookrightarrow \mathcal{P}_N$$

where $Q + R = N + 1$, the inclusions being indexed by $(Q, R)$-shuffles and the quotient by $S61$ corresponding to the fact that one decorated polyhedron $P$ can result from different decorations before identification as long the orientation reversing isometry is the same.

Similarly, we can describe boundary “facets” of $V_N$ by inclusions

$$\mathcal{V}_Q \times \mathcal{V}_R \times S^1 \hookrightarrow \mathcal{V}_N$$

where $Q + R = N + 1$, the inclusions being indexed by $(Q, R)$-shuffles and the factor of $S^1$ specifying the possible rotations possible in identifying isometrically the unparameterized perimeters.

The proof of the defining relation $(J_N)$ is again essentially an application of Stokes' Theorem. This time the boundary term can be written as

$$\int_{\mathcal{V}_R} \int_{\mathcal{V}_Q}$$

giving rise to the terms

$$[[\phi_{i_1}, \ldots, \phi_{i_Q}], \phi_{i_{Q+1}}, \ldots, \phi_{i_N}].$$

5. Strongly homotopy Lie algebras

To make sense out of the formula $(J_N)$, we re-examine the concept of Lie algebra, which can be expressed in several different ways. The most familiar are: in terms of generators and relations and in terms of a bilinear “bracket” on a vector space $V$ satisfying the Jacobi identity. A much more subtle description appears in the homological study of Lie algebras and is implicit in the somewhat more familiar dual formulation of the Chevalley-Eilenberg cochain complex for Lie algebra cohomology [CE]. We can deal directly with the vectors rather than with the multi-linear ‘forms’ at the expense of introducing a new point of view and consideration of skew-symmetric tensors ($poly$-vectors).

A Lie algebra is equivalent to the following data:

A vector space $V$ (assumed finite dimensional for simplicity of exposition).

The skew (or alternating) tensor products of $V$, denoted

$$\bigwedge V = \{\bigwedge V\},$$
with $\bigwedge^0 V = k$, the field of scalars, typically the reals, $\mathbb{R}$ or the complex numbers, $\mathbb{C}$.

A linear map

$$d : \bigwedge V \to \bigwedge V$$

which lowers $n$ by one and is a co-derivation determined by $d|\bigwedge^2 V$ such that $d^2 = 0$.

(That $d$ is a co-derivation determined by $d|\bigwedge^2 V$ means just that $d(v_1) = 0$ and

$$d(v_1 \wedge \cdots \wedge v_n) = \sum_{i<j} (-1)^{i+j} d(v_i \wedge v_j) \wedge v_1 \wedge \cdots \hat{v}_i \cdots \hat{v}_j \cdots \wedge v_n$$

where $\hat{v}_i$ denotes the deletion of $v_i$.)

It may not be immediately obvious, but $d$ restricted to $\bigwedge^2$ is to be interpreted as a bracket: $d(v_1 \wedge v_2) = [v_1, v_2]$ and $d^2 = 0$ is equivalent to the Jacobi identity.

The generalization we need ‘up to higher homotopy of all orders’ is comparatively straightforward from this point of view of the skew tensor powers of $V$; an sh Lie algebra (strongly homotopy Lie algebra) [LS] is equivalent to a straightforward generalization in which $d$ is replaced by a coderivation

$$D = d_1 + d_2 + d_3 + \ldots$$

where $d_i$ lowers $n$ by $i - 1$, in particular, $d_n(v_1 \wedge \cdots \wedge v_n) \in V$. The only subtlety is in the signs. The vector space $V$ is a graded vector space and the ‘skew symmetry’ is better expressed as graded symmetry where the grading has been shifted by 1. We denote this by

$$\bigwedge^s V.$$

(A further generalization in which there is a term $d_0$ interpreted as a fixed “background” field occurs in Zwiebach’s investigation of background dependence, but for this there is no mathematical precursor, to our knowledge.)

We say that $D$ is a coderivation to summarize the several conditions:

$$d_j(v_1 \wedge \cdots \wedge v_n) = \sum \pm d_j(v_{i_1} \wedge \cdots \wedge v_{i_j}) \wedge v_{i_{j+1}} \wedge \cdots \wedge v_{i_n},$$

where the sum is over all unshuffles of $\{1, \ldots, n\}$.

Notice that the old $d$ corresponds to $d_2$ since $d(v_1 \wedge v_2) = [v_1, v_2] \in V$. On the other hand, for the new $D$, the component $d_2$ no longer is of square zero by itself and hence corresponds to a bracket which does NOT necessarily satisfy the Jacobi identity. Let us look in detail at what can happen instead:

Expand $D^2 = 0$ in its homogeneous components and set them separately equal to zero. We have then:

0) $d_1^2 = 0$

so $(V, d_1)$ is a complex or differential (graded) module. (Typically $d_1$ raises (or lowers) degree by 1.)

1) $d_1 d_2 + d_2 d_1 = 0$

so, with appropriate sign conventions, $d_2$ gives a bracket $[v_1, v_2] \in V$ for which $d_1$ is a derivation.

2) $d_1 d_3 + d_2 d_2 + d_3 d_1 = 0$
or equivalently
\[ d_2 d_2 = -(d_1 d_3 + d_3 d_1). \]

If we further adopt the notation:
\[ d_3(v_1 \land v_2 \land v_3) = [v_1, v_2, v_3], \]
then we have
\[ [[v_1, v_2], v_3] \pm [[v_1, v_3], v_2] \pm [[v_2, v_3], v_2] = -d_1[v_1, v_2, v_3] \pm [d_1 v_1, v_2, v_3] \pm [v_1, d_1 v_2, v_3] \pm [v_1, v_2, d_1 v_3]. \]

In the language of homological algebra, we say that \((V, d_1, d_2, d_3)\) is a homotopy Lie algebra. The adverb “strongly” is added to refer to the other \(d_i\), the higher homotopies.

If we adopt the notation that \(d_1 = Q\) and in general:
\[ d_n(v_1 \land \cdots \land v_n) = [v_1, v_2, \ldots, v_n], \]
then the appropriate homogeneous piece of \(D^2 = 0\) is (up to sign conventions and up to some constants related to conventions on the definition of \(\land V\)) precisely the equation \((J_N)\).

In the higher spin particle algebra of \([BBvD]\), variations \(\delta _\epsilon \) do not respect a strict bracket \([\epsilon _1, \epsilon _2]\) but rather an sh Lie structure on the space of \(\epsilon \)‘s. In the Batalin-Fradkin-Vilkovisky operator for constraints forming an ‘open’ algebra with structure functions, one sees a similar structure \([S6]\).

\{ Post conference: Schechtman has informed me that Drinfel’d described sh Lie algebras in essentially these terms in a letter to Schechtman in 1988.\}

6. The operad structure

We return to the geometry underlying this algebra in closed string field theory. From now on, the face labelled 0 of a restricted polyhedron will play a distinguished role. Think of inputs going in through the other \(n\) faces with face 0 reserved for output. Then decorated restricted polyhedra (elements of \(P\)) can be combined exactly as rooted trees can by “grafting” roots to branch tips. The sequence of spaces \(P_k\) provides an example of May’s notion of an operad \([M1]\): this is a sequence of spaces \(O(k)\) satisfying certain conditions modelled on those satisfied by \(Map(X^k, X)\) with the obvious action of the symmetric group \(\Sigma _k\).

A space \(X\) is acted on by an operad \(O\) means there is a sequence of \(\Sigma _k\)-equivariant maps \(\theta _k : O(k) \times X^k \to X\) satisfying certain compatibility conditions. More generally, operads can be defined in any symmetric monoidal category (so that \(X^k\) makes sense)[Ke], for example, the category of topological spaces, the category of graded vector spaces (with tensor product), or the category of chain complexes (once more with tensor product).

Definition. An operad \(O\) in such a category is a sequence \(O(k), k \geq 1\), of objects with \(\Sigma _k\)-action and maps
\[ \gamma : O(k) \times O(j_1) \times \cdots \times O(j_k) \to O(j_1 + \cdots + j_k), \]
satisfying conditions of ‘operad-associativity’ for iterating \(\gamma\) and “operad-symmetry” (respecting the \(\Sigma _k\) action) - which are obvious in the following example built from \(Map(X^k, X)\).

THE basic example, in any such category, is the endomorphism operad
\[ \mathcal{E}_X = \{ \mathcal{E}_X(k) = Map(X^k, X)\} \]
with $\gamma$ given by composing a map with $k$ ordered inputs with the outputs of $k$ (other) maps in the usual way.

Two very important operads related to ordinary algebra are

$$\Sigma = \{ \Sigma(k) = \Sigma_k \}$$

with the structure map $\gamma$ given by a generalization of wreath product, and

$$\mathcal{N} = \{ k \}$$

interpreted as singletons.

Notice that the distinct ways of multiplying elements $(x_1, \ldots, x_k)$ of an associative algebra can be indexed by the elements $\pi \in \Sigma_k$, namely the iterated products $x_{\pi(1)} \cdots x_{\pi(k)}$. Thus an action of $\Sigma$ on a set $X$ makes $X$ an associative monoid. Similarly, an action of $\mathcal{N}$ makes $X$ a commutative associative monoid since all of the $x_{\pi(1)} \cdots x_{\pi(k)}$ are to be equal.

Operads were originally invented [M1] for the study of iterated (based) loop spaces: for two excellent overviews of this theory, see Adams [Ad] and May [M2]. Before that invention (and hence without the name), I created an operad [S4, S3] that made explicit the higher homotopies required of the multiplication on an $H$-space for it to be homotopy equivalent to a loop space. I introduced a sequence of convex polyhedra $K_k$ (which have come to be known as associahedra), of dimension $k - 2$, with the property that a connected space $X$ has the homotopy type of a loop space if and only if there is a sequence of maps

$$\theta_k : K_k \times X^k \to X$$

satisfying certain compatibility conditions. Such a space is called an $A_\infty$-space. The associahedron $K_2$ is a point, so that $\theta_2$ gives $X$ the structure of an $H$-space. Furthermore, this product is homotopy associative, in the sense that $K_3$ is an interval, and the two products $(ab)c$ and $a(bc)$ correspond to the two endpoints of $K_2$, thus $\theta_3$ gives a canonical homotopy between $(ab)c$ and $a(bc)$. The associahedron $K_4$ is the now familiar pentagon; $K_5$ is pictured in Figure 8 in a visualization I owe to John Harer.

**Figure 8**

The decorated restricted polyhedra (elements of $\mathcal{P}$) form an operad in the obvious way; just as the isometries $C_i$ were used to ‘glue’ two restricted polyhedra to form a third, so the isometries allow us to glue $k$ polyhedra in $\mathcal{P}_{j_i}$ to a $k+1$-polyhedron in $\mathcal{P}_k$ to form a $(j_1 + \cdots + j_k + 1)$-polyhedron.
For n-fold iterated loop spaces, Boardman and Vogt [BoVo] introduced a useful sequence of operads, the little n-cubes operad $\mathcal{C}_n(k)$, which have the homotopy type of the configuration spaces $F(\mathbb{R}^n, k)$ of k-tuples of distinct points in $\mathbb{R}^n$; a space with an action of the operad $\mathcal{C}_n$ is called an $E_n$-space and has the homotopy type of an n-th loop space, at least if it is connected. Certain cases are of particular interest: $n = 1$ recovers the theory of $A_\infty$-spaces, $n = \infty$ leads to infinite loop spaces and $n = 2$ is intimately related to the braid groups $B_k$, since $F(\mathbb{R}^2, k)/\Sigma_k$ has the homotopy type of $K(B_k, 1)$.

Henceforth we will be concerned only with $n = 2$ for which we have the following notation and formal definition:

**Definition:** The little squares operad $\mathcal{C}_2$: Let $\mathcal{C}_2(k)$ be the space of all maps from $\coprod_{i=1}^k I^2$ to $I^2$ which are affine on each coordinate of each square and such that the images of the $k$ squares are disjoint. The symmetric group $\Sigma_k$ acts by permuting the cubes in the domain of the map. The operad structure is given by the maps

$$\gamma : \mathcal{C}_2(k) \times \mathcal{C}_2(j_1) \times \cdots \times \mathcal{C}_2(j_k) \to \mathcal{C}_2(j_1 + \cdots + j_k)$$

defined by

$$\gamma(c, d_1, \ldots, d_k)(x_1, \ldots, x_j) = c(d_1(x_1, \ldots, x_{j_1}), \ldots, d_k(x_{j_1+\cdots+j_{k-1}+1}, \ldots, x_j)).$$

Closely related is the geometric little disks operad $\mathcal{D} = \{\mathcal{D}(k)\}[M1]$: Let $D$ be the unit disc in $\mathcal{C}$. Let $\mathcal{D}(k)$ be the space of all maps $(z_1, \ldots, z_k)$ from $\coprod_{i=1}^k D$ to $D$ which are obtained by dilation and translation such that the images of the $k$ discs are disjoint. The operad structure is again defined by composition.

We can think of the little disks as being cut out of the standard disk and hence think of certain Riemann surfaces of genus 0 with $k + 1$ parameterized boundary components.

The decorated restricted polyhedral operad $\mathcal{P}$ is very similar except that, as Albert Schwarz remarked, they are ‘Riemann surfaces with boundary only’ and the boundary components have various (isometric) parameterizations. We can similarly ‘decorate’ the elements of $\mathcal{D}$ by allowing rotations in the maps $\coprod_{i=1}^k D$ to $D$; we denote the resulting operad by $\tilde{\mathcal{D}}$.

Huang [Hu] adopts a related point of view, that of Riemann surfaces with punctures and local coordinates, more appropriate to sewing rather than glueing, with the crucial difference that he constructs a partial operad, $\mathcal{H}$, i.e. the structure map $\gamma$ is not always defined but when it is, the compatibility does hold. (Further details are in Section 9.)

The undecorated operads $\mathcal{C}$ and $\mathcal{D}$ have the homotopy type of the non-operad of configuration spaces $F(\mathbb{R}^2, k)$. The decorated operads $\mathcal{P}$ and $\tilde{\mathcal{D}}$ have the homotopy type of torus bundles over the non-operad of configuration spaces $F(\mathbb{R}^2, k)$.

### 7. Operads via homology or chains

Operads in one category give rise to operads in another by applying a suitable functor. In particular, for an operad in a category of topological spaces, homology with field coefficients is such a functor. (Getzler [Get] starts with smooth or complex spaces and Huang [Hu] with complex manifolds.) Until quite recently, an operad to characterize Lie algebras had not been given in as formal a fashion as for associative or commutative associative algebras, but, as pointed out by Getzler and Jones [GJ], there was one implicit in the work of Fred Cohen on the homology of configuration space. Arnol’d [Ar] and later but independently Fred Cohen [C1] determined the
homology of \( F(\mathbb{R}^2, k) \). Cohen [C2] also provided the following related result of direct relevance for us. For any graded vector space \( V \), let \( sV \) denote the graded vector space such that \((sV)_{n+1} = V_n\); the operator \( s \) shifts the grading by 1.

**Theorem:** For homology with coefficients in a field and any graded vector space \( V \) over that field,
\[
\bigoplus_{k} sH_{k-1}(F(\mathbb{R}^2, k)) \otimes_{\Sigma_k} V^\otimes k
\]
is isomorphic to the free graded Lie algebra generated by \( sV \).

Thus, in the category of graded vector spaces, we can regard \( L = \{L(k) = H_{k-1}(F(\mathbb{R}^2, k))\} \) as an operad \( L \) such that \( L \) acting on a graded vector space gives \( sV \) the structure of a graded Lie algebra or, alternatively, it is sometimes said that \( V \) has a Lie bracket of degree \(-1\) or in physspeak, an ‘anti-bracket’.

An alternate approach to sh Lie algebras (inspired by work of Beilinson and Ginzburg [BG1]) has been developed by Hinich and Schechtman [HS2]. Rather than use \( L \) as described above, they describe it in terms of certain sub-modules of the free Lie algebras on \( n \) variables and call it the ‘trivial Lie operad’ \( L \). ‘Weak versions’, equivalent to sh Lie algebras, are described in terms of operads quasi-isomorphic to \( L \) and a universal one is constructed. This universal one bears a nice relation to appropriate compactifications of \( M_{0,N+1} \), as explained further in Section 13. In particular, this relates to a good chain complex model for \( F(\mathbb{R}^2, k) \).

Similarly, Cohen’s results show the full homology \( G = \{G(k) = H_*(F(\mathbb{R}^2, k))\} \) also forms an operad, which characterizes a **Gerstenhaber algebra** (a.k.a. braid algebra in the terminology of Getzler and Jones). The formal definition is:

**Definition.** A **Gerstenhaber or braid algebra** is a graded vector space \( V \) with a product \( V^p \times V^q \to V^{p+q} \) and a bracket \( \{ , \} : V^p \times V^q \to V^{p+q-1} \) such that

- (a) \( uv = (-1)^{|u||v|}vu \)
- (b) \( (uv)w = u(vw) \)
- (c) \( \{u, v\} = (-1)^{|u||v|}\{v, u\} \)
- (d) \( \{u, \{v, w\}\} = \{\{u, v\}, w\} + (-1)^{|u||v|}\{v, \{u, w\}\} \)
- (e) \( \{u, vw\} = \{u, v\}w + (-1)^{|u||v|}v\{u, w\} \).

The homology of a 2-fold loop space \( H_*(\Omega^2X) \) is a Gerstenhaber algebra, although Cohen [C1] does not use that name; the product is the Pontryagin product coming from the H-space structure of \( \Omega^2X \), while the Lie bracket is called the Browder operation. In fact, if \( X \) is the 2-fold suspension of a space \( Y \), then \( H_*(\Omega^2X) \) is the free Gerstenhaber algebra generated by the homology of \( Y \).

Gerstenhaber’s creation of the first example of this structure occurred in a purely algebraic context. The Hochschild cohomology \( H^\bullet(A, A) \) of an associative algebra is a Gerstenhaber or braid algebra; the product is the cup product, while the bracket is Gerstenhaber’s [Ger]. In the special case that \( A \) is the algebra of differentiable functions \( C^\infty(M) \) on a manifold \( M \), the Hochschild cohomology was shown by Hochschild-Kostant-Rosenberg [HKR] to be naturally isomorphic to the space of multivectors \( \Gamma(M, \bigwedge^\bullet TM) \). With this identification, the cup product may be identified
with the wedge product on $\Gamma(M, \wedge^* TM)$, while the Gerstenhaber bracket may be identified with the Schouten-Nijenhuis-Richardson bracket.

But what of the decorated operads? What structure are they characterizing? Before answering that in the terms of Getzler and Jones, let me first comment on Zuckerman’s talk.

8. The G-algebra of BRST string theory

In the first talk of this conference, Zuckerman reported on his recent work with Lian [LZ] in which they develop the structure of a Gerstenhaber algebra on the BRST cohomology of a chiral algebra. To be precise, he stated:

**Theorem LZ 1:** The BRST cohomology of a chiral algebra (a.k.a. chiral cohomology) $H^*$ admits the structure of a Gerstenhaber algebra and the action of $b_0$ is a derivation of the Gerstenhaber bracket.

In more detail, they construct a product $u \cdot v$ and a bracket $\{u, v\}$ such that the BRST operator $Q$ is a derivation of both the product and bracket. Moreover:

**Theorem LZ 2.2:** On the chiral cohomology $H^*$, we have $u \cdot v : H^p \times H^q \to H^{p+q}$ and $\{., .\} : H^p \times H^q \to H^{p+q-1}$ such that

(a) $u \cdot v = (-1)^{|u||v|} v \cdot u$

(b) $(u \cdot v) \cdot w = u \cdot (v \cdot w)$

(c) $\{u, v\} = (-1)^{|su||sv|}\{v, u\}$

(d) $(-1)^{|su||sw|}\{u, \{v, w\}\} + (-1)^{|sv||su|}\{w, \{u, v\}\} + (-1)^{|sv||su|}\{v, \{w, u\}\} = 0$

(e) $\{u, v \cdot w\} = \{u, v\} \cdot w + (-1)^{|su||v|}\{u, \{w, v\}\}$

(f) $b_0\{u, v\} = \{b_0u, v\} + (-1)^{|su|}\{u, b_0v\}$.

On the BRST complex itself (‘off-shell’ in physspeak), all the identities of a Gerstenhaber algebra hold up to homotopy. Of course, it is the structure of these homotopies that excites my interest. Many of these identities up-to-homotopy are the same as they are in the Hochschild complex, as given by Gerstenhaber [Ger]. A striking difference is that the Hochschild product of cochains is strictly associative, while that of Lian and Zuckerman is only associative up to homotopy:

$$(b') (u \cdot v) \cdot w - u \cdot (v \cdot w) = Qn(u, v, w) + n(Qu, v, w) + (-1)^{|u||v|}n(u, Qv, w) + (-1)^{|u|+|v|}n(u, v, Qw)$$

where $n$ is a trilinear operation defined using $b_1$.

The Lian-Zuckerman homotopy versions of (a) and (c)-(e) above are:

$$(a') u \cdot v = (-1)^{|u||v|} v \cdot u = Qm(u, v) + m(Qu, v) + (-1)^{|u|}m(u, Qv)$$

where $m$ is a bilinear operation again defined using $b_1$,

$$(c') \{u, v\} + (-1)^{|su||sv|}\{v, u\} = (-1)^{|su|}(Qm'(u, v) - m'(Qu, v) - (-1)^{|u|}m'(u, Qv))$$

where $m'$ is built using the homotopy $m$ and $b_0$

$$(d') \{u, \{v, w\}\} = \{\{u, v\}, w\} + (-1)^{|sv||su|}\{v, \{u, w\}\}$$

if the antighost field is defined with care,

$$(e') \{u, v \cdot w\} = \{u, v\} \cdot w + (-1)^{|su||v|}v \cdot \{u, w\}$$
but
\[\{u \cdot v, w\} - u \cdot \{v, w\} = (-1)^{|x|(|v|+|w|)}\{u, w\} \cdot v = \]
\[(-1)^{|u|+|v|} (Qn''(u, v, w) - n''(Qu, v, w) - (-1)^{|u|} n''(u, Qv, w) - (-1)^{|u|+|v|} n''(u, v, Qw))\]
where \(n''\) is a trilinear operation defined using \(m'\) and the product.

Notice that the two versions of \((e')\) are different, but exactly as they are in the Hochschild complex, or earlier in Steenrod’s relation between \(\sim\) and \(\sim_1\). By simply skew-symmetrizing the bracket, \((c')\) can be replaced by \((c)\); the price to be paid is that \((d')\) then holds only up to homotopy. This however is preferable if higher homotopies are to be studied as in string field theory.

Finally, the bracket and product are related via \(b_0\):
\[(f') = (f) : (-1)^{|u|} \{u, v\} = b_0(u \cdot v) - (b_0u) \cdot v - (-1)^{|u|} u \cdot (b_0v).\]

A remark about this identity: it clearly measures the failure of \(b_0\) to be a derivation of the dot product. The same idea appears in the Batalin-Vilkovisky “anti-bracket” formalism \([\text{BaVi}]\), but in a seemingly different context. In \([\text{W2}]\), Witten showed that the Batalin-Vilkovisky master equation can be formulated using a certain fundamental differential operator \(\Delta\) in field space, together with an anti-bracket which measures the failure of \(\Delta\) to be a derivation of an operator product. The \(b_0\) operator here plays the role of \(\Delta\). This extra operator on a Gerstenhaber algebra is the signature of what Penkava and Schwarz and Getzler and Jones have named a Batalin-Vilkovisky algebra, which we discuss below.

These homotopies suggest that there is an fact a strong homotopy Gerstenhaber algebra structure present - in a sense yet to be defined, but see below. First, here is an alternate approach due to Y.-Z. Huang.

### 9. Partial operads and partial algebras

Huang \([\text{Hu, HL}]\) considers a partial operad \(\mathcal{K}\) (that is, the structure maps are defined only on suitable subsets of the usual range, cf. \([\text{Ste}]\)), but one with analytic structure, cf. \([\text{Get}]\). His partial operad is the moduli space of Riemann spheres with \(n + 1\) ordered punctures and a local coordinate vanishing at each puncture, further restricted in that the 0-th puncture is negatively oriented and the other punctures are positively oriented.

The operad structure map \(\gamma\) is defined by sewing Riemann spheres with punctures and local coordinates with the orientations specified above, the sewing being of the 0-th puncture of one to one of the positively oriented punctures of the other. The sewing of two such spheres with punctures and local coordinates is defined by cutting disks at the specified punctures using the local coordinates and then identifying the boundaries of the disks using the map \(z \rightarrow 1/z\). The sewing is defined only when the local coordinate neighborhoods can be extended such that the disks we want to cut are inside the extended neighborhoods, and there are no other punctures inside these disks. Therefore \(\gamma\) is only partially defined.

Huang considers vertex operator algebras (which correspond to chiral algebras without ghosts or anti-ghosts) and establishes that a VOA-structure on a vector space \(V\) implies a projective action of \(\mathcal{K}\) on \(V\), except that a double dual of \(V\) is involved. Conversely such an action on \(V\) gives \(V\) a VOA-structure. Since the introduction of ghosts and anti-ghosts leads to a super chiral algebra with the additional subtlety only of some signs, Huang’s techniques also give a projective \(\mathcal{K}\) action on the underlying vector space \(V\) of an appropriate BRST complex.
In the preprint of Kriz and May [KM], the general theory is advanced, not by the consideration of partial operads but rather by the consideration of partial actions of an operad, i.e. a partial algebra over an operad $\mathcal{O}$ has similarly coherent maps

$$\mathcal{O}(k) \otimes A_k \to A$$

where $A_k \subset A^k$.

Kriz and May explain: “The original motivation of this paper was to show how to construct May algebras from partial algebras”, May algebras (the terminology is due to Hinich and Schechtman in earlier work) being algebras over a special class of “$E_\infty$-operads”. “This work was inspired by letters from Deligne to Bloch and May. Deligne suggested that Bloch’s Chow complex, which is a partial algebra, should give rise to a quasi-isomorphic (graded) May algebra...”, with an eye to its derived category of modules providing a “a suitable site in which to define (integral) mixed Tate motives...”.

May also pointed out that his “little convex bodies” were described as a partial operad on page 172 of [M3], but later was replaced by a strict operad by Steiner [Ste] in a way presaging Huang and Lepowsky’s rescaling.

10. Batalin-Vilkovisky algebras

Since punctured Riemann spheres or restricted polyhedra (or, up to homotopy equivalence, configurations of points in the plane) govern the structure of Gerstenhaber algebras, what then is the import of the ‘decorations’ which correspond to torus bundles over the component undecorated moduli spaces of the operads? At the homology level, Getzler and Jones see the answer as the structure of a Batalin-Vilkovisky algebra.

**Definition:** A Batalin-Vilkovisky algebra is a graded commutative algebra with an operator $\Delta : A_* \to A_{*+1}$ such that $\Delta^2 = 0$ and

$$\Delta(abc) = \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{|a|-1}|b|b\Delta(ac)
- (\Delta a)bc - (-1)^{|a|}a(\Delta b)c - (-1)^{|a|+|b|}ab(\Delta c).$$

The defining condition describes $\Delta$ as a ‘second order derivation’, that is, the deviation of $\Delta$ from being a derivation is in turn a derivation [PS]. In fact, Penkava and Schwarz point out that in the super-algebra context a Batalin-Vilkovisky algebra is the same thing as a super-commutative associative algebra with an odd second order derivation $\Delta : A_* \to A_{*+1}$ such that $\Delta^2 = 0$. They prefer the alternate:

**Definition:** A Batalin-Vilkovisky algebra is a Gerstenhaber algebra with operator $\Delta : A_* \to A_{*+1}$, such that $\Delta^2 = 0$, and with product, bracket and $\Delta$ related by the formula

$$[a, b] = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}(\Delta a)b - a(\Delta b).$$

Furthermore, in a Batalin-Vilkovisky algebra, $\Delta$ satisfies the formula

$$\Delta [a, b] = [\Delta a, b] + (-1)^{|a|-1}[a, \Delta b].$$

11. The Batalin-Vilkovisky formalism
Why were physicists (Batalin and Vilkovisky) [BaVi] interested in such structures? The answer is that they were interested in the Lagrangian approach to field theories. (For a thorough treatment which is very accessible mathematically but written by physicists, see the book by Henneaux and Teitelboim [HT].) This means that they were concerned with a variational problem, finding the critical points of a functional $S$ defined on some space of fields, most frequently functions on the jet bundle over some mapping space. If a classical Lagrangian ‘action’ $S$ is invariant with respect to ‘gauge symmetries’ (e.g. diffeomorphisms of the space of maps in question), there results a perturbative expansion
\[ S_t = \Sigma t^i S_i \]
where $S_0$ is the original $S$. Batalin and Vilkovisky enlarge the original space of field functionals to a differential graded algebra with an ‘anti-bracket’ $( , )$ such that the desired $S_t$ is a solution of the equation $(S_t, S_t) = 0$, in complete analogy with the usual integrability equation in algebraic deformation theory. Indeed, after the fact, we can recognize the classical Batalin-Vilkovisky algebra as a Gerstenhaber algebra. However, just as in the Batalin-Fradkin-Vilkovisky approach to constrained Hamiltonian problems [BFV], the motivation was not from the classical situation, but rather from the desire to quantize it. To do so, they introduce a new operator $\Delta$ and the B-V ‘master equation’ becomes
\[ (S_t, S_t) = -i\hbar \Delta S_t. \]
It is this structure that Getzler [Get], Getzler and Jones [GJ] and Penkava and Schwarz [PS] formalize in their definition of a B-V algebra.

Although they derive this structure from an action of (the homology of) any of the operads homotopy equivalent to $P$, the master equation solutions of Zwiebach’s closed string field theory involve the moduli spaces of Riemann surfaces of genus $g$ with $k + 1$ punctures. Moreover, the notion of an operad needs to be extended to allow sewing of two local coordinates on the same Riemann surface (of genus $g$, producing a surface of genus $g + 1$); I am unaware of any known mathematical generalization of operad which handles this structure.

12. Closed string field theory: Reprise with inner product

Now that we are back to closed string field theory, we should notice that the relevant action $S_t$ involves terms of the form
\[ < \phi_0 \phi_1 \ldots \phi_N > = < \phi_0 | [\phi_1 \ldots \phi_N] > \]
which can be regarded as fundamental or as determined by the $N$-ary brackets via the inner product $< | >$. The fundamental inner product $< \phi_0 | \phi_1 >$ involves integration over $S^1$ as well as a non-trivial inner product on the Hilbert space $\mathcal{H}$ which is the space of fields.

Thus we have an example of an sh Lie algebra with inner product as in Kontsevich’s graph cohomology [Ko]. A corresponding sh associative algebra with inner product appears in the open string field theory of the Kyoto group [HIKKO], although there
\[ < \phi_0 \phi_1 \phi_2 \phi_3 \phi_4 > = 0 \]
for reasons explained in [S1].

One of the subtleties of graph cohomology is getting the signs right! For the associahedra, the geometry is obvious but the combinatorics of the signs were originally a bear! Kontsevich gets
around this very cleverly by having an orientation of the vector space $\mathbb{R}^E \oplus H^1$ to handle the signs (where $E$ is the number of edges in the graph and $H^1$ is that cohomology of the graph).

13. ‘Standard’ Constructions and moduli space compactifications

We have seen higher homotopy analogs of strict (differential graded) algebras, both Lie and associative. Just as sh Lie algebras can be defined in terms of a coderivation on $\bigwedge sV$, so an sh associative algebra can be defined in terms of a coderivation on the tensor coalgebra on the suspension of the vector space. But what are the analogs for other structures, for example, Gerstenhaber algebras?

\{ Post conference: At the time of the conference, I speculated that the Lian-Zuckerman homo-
topies were part of an action of an operad for a strong homotopy analog of $\mathcal{G}$, i.e. that there exist approprate higher homotopies of all orders. Since the above descriptions of $\mathcal{L}$ and $\mathcal{G}$ differ in the use of part versus all of the homology of configuration space, it is to be hoped that the same will be true for their sh analogs in using part versus all of the cells of $\mathcal{P}_N$. (This is work in progress by the Penn-Princeton string quartet: Huang, Kimura, Stasheff and Voronov.)\}

One way to define strong homotopy analogs of algebraic structure controlled by an operad is to define actions of the operad up to strong homotopy which in turn means that the sequence of $\Sigma_k$-equivariant maps $\theta_k : O(k) \times X^k \to X$ satisfy the appropriate compatibility conditions up to strong homotopy. This is a reasonably effective procedure since the higher homotopy conditions are essentially cubical in their combinatorics. This approach follows that of Sugawara [Su] who defined strong homotopy maps of associative spaces. Lada [Lad] and a later work of Hinich and Schechtman [HS1] provide details.

I did not resort to this definition for sh Lie algebras, believing that use of the ‘standard’ construction is conceptually and computationally more useful because of its (comparatively) small size. What then of a ‘standard construction’ approach to BV algebras?

Answers of one sort are provided by Getzler and Jones [GJ] and in an alternate form by Ginzburg and Kapranov [GK]. Ginzburg and Kapranov are concerned primarily with a special class of quadratically related operads, called Koszul, which includes all of the examples of interest in this talk. They introduce the notion of the cobar construction on an operad, which strictly speaking involves a linear dual, whereas Getzler and Jones deal somewhat more naturally with the bar construction, at the expense of introducing the dual notion of a co-operad. \{ Post conference: Both ‘bar’ and ‘cobar’ are slightly misleading as the constructions make crucial use of the symmetric group actions and so are really closer to the standard constructions used on Lie and commutative algebras.\}

A co-operad in a symmetric monoidal category $\mathcal{C}$ is an operad in the opposite symmetric monoidal category $\mathcal{C}^{op}$. For example, if $O$ is an operad in the category of chain complexes, then the linear dual $O^*$ is a co-operad in the category of cochain complexes.

If $O$ is an operad in the category of chain complexes such that $O(1) = \mathcal{C}$, then, “inspired by constructions of Boardman and Vogt [BoVo] and Ginzburg and Kapranov [GK]”, Getzler and Jones define the bar co-operad $BO$ of $O$. As graded vector space, it is essentially

$$BO(k) = \sum_{T \in T(k)} \bigotimes_{v \in T} sO(\text{Inp}(v))$$

where $T(k)$ is the set of (isomorphism classes) of trees with $k$ numbered inputs and $v$ denotes an
internal vertex of \( T \) while \( \text{Inp}(v) \) denotes the set of incoming edges to \( v \), trees being directed so there is only one outgoing edge at any vertex. In order to describe the action of the permutation group \( \Sigma_k \) on \( B\Omega \), they (like Ginzburg-Kapranov) have changed the indexing from integers to finite sets. The differential is induced from the operation on trees which contracts an internal edge to a point, thus identifying its two vertices. The tricky question of signs is handled in [GK] by consistent use of the top exterior power of the vector space spanned by the internal edges of a tree.

Similarly, there is a **cobar operad** construction applicable to any co-operad and bar and cobar stand in the usual adjoint relation up to homotopy. Ginzburg and Kapranov are concerned with such adjoint relations or ‘duality’, in particular with that between Lie algebras and commutative associative algebras. For any operad \( \mathcal{O} \), there is a notion of a strong homotopy object over the ‘dual’ operad \( \mathcal{O}^* \) as precisely a strict object over the graded linear dual of \( B\mathcal{O} \) (assuming some appropriate finite dimensionality). The Hinich-Schechtman operad quasi-isomorphic to \( \mathcal{L} \) can be described as the graded linear dual of \( BN \). Since \( N(k) \) is just a singleton, \( BN \) is described entirely in terms of trees.

It is this description in terms of trees that provides the link with the compactifications of moduli spaces or configuration spaces in the work of Beilinson-Ginzburg [BG1,2]. In particular, for Riemann surfaces of genus 0 with \( N + 1 \) labelled punctures, the appropriate compactification is stratified with strata indexed by isomorphism classes of rooted trees with \( N \) labelled inputs. { *Post conference:* These compactifications provide operads as do Zwiebach’s compact “cut-off” subspaces. }

### 14. When there is no internal differential

Although the cohomology associated to a topological chiral algebra inherits a strict B-V algebra structure for which higher homotopies are not needed, there may still exist a sequence of higher order operations on the cohomology. These may derive from higher homotopies in terms of the fields (before passing to cohomology); this is what occurs in Zwiebach[Z]. Alternatively, these may occur even if the fields form a strict differential graded algebra with no terms of higher order. The latter is familiar in algebraic topology in the context of Massey products or their H-space analogs (in the associative case). For example, the linking of two circles may be detected by the cohomology product of the complement of the link, but the non-trivial linking of three circles, no two of which are linked, in the configuration known as the Borromean rings can be detected by a tri-linear Massey product.

How can we make sense out of an sh Lie algebra on the (co)homology level, where there is no apparent differential \( d_1 \)? We can consider it to be 0 and then interpret the relations \( (J_N) \) accordingly. We still have left a sequence of compatibility conditions for the brackets of all orders, beginning with the Jacobi identity for the bi-linear bracket. It is this version that occurs in the ‘homotopy Lie algebra’ of Witten-Zwiebach [WZ], although it is expressed in the form

\[
0 = \{V, V\}
\]

where \( V \) is a ‘vector field’ acting as a derivation \( \{V, \cdot \} \) corresponding precisely to our \( D \) with \( d_1 = 0 \). Moreover, Witten and Zwiebach write their vector field in terms of a basis of ghosts as:

\[
c_{abc}^a \eta^b \eta^c \partial_a + c_{bca}^a \eta^b \eta^c \partial_a + \cdots
\]

where \( \partial_a = \partial/\partial \eta^a \). The term \( c_{bca}^a \eta^b \eta^c \partial_a \) is a basis dependent way of writing the usual Lie algebra co-boundary \( d_2 \) and the terms of higher order correspond precisely to the further \( d_i \).
One explanation of such higher order structure is given by the characteristic technique of Homological Perturbation Theory (HPT) [HPT]: When a chain complex (a differential graded module) $C$ has the structure of a strict algebra, higher order operations in homology can arise as follows. Assume we are over a ground field or that the modules of concern are free over a ground ring. Suppose that a chain complex (a differential graded module) $C$ has the structure of a strict algebra. Choose a splitting $C = H \oplus X$ where $X$ is a contractible chain complex; indeed, choose a contracting homotopy. HPT then provides an algorithm for constructing suitably compatible higher order operations on the (co)homology $H$.

In the “harmonic” case, there is a particularly good Hodge decomposition, a choice of splitting $H \hookrightarrow C$ which is a strict map of the structures. In physical language, Zwiebach says this as: “the product of physical states can not give an unphysical state”, meaning, at the chain level, that $H$ as a subspace of $C$ is closed under all the $N$-ary operations.

Finally let me call attention to a potential problem with terminology, namely, a conflict with the terminology in algebraic topology if ‘homotopy Lie algebra’ is used to indicate not only the first order homotopy for the Jacobi identity, but also the higher homotopies which I indicate by ‘strong’ or ‘strongly’.
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