Large-N Quenching in the Kazakov-Migdal Model.

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Abstract

To study the behavior of the Kazakov-Migdal at large N the quenched momen-
tum prescription with constraints for treating the large N limit of gauge theories is
used. It is noted that it leads to a quartic dependence of an action on unitary matrix
instead of a quadratic dependence discussed in previous considerations. Therefore
the model is not exactly solvable in the weak coupling limit. An approximation pro-
cedure for investigation of the model is outlined. In this approximation an indication
to a phase transition for $d < 4,8$ with $\beta_{cr} = \frac{1}{d-4,8}$ is obtained.

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1 Introduction

Recently Kazakov and Migdal [1] have made an important progress in attacking the old problem of an evaluation of the large N limit for gauge models [2]. They proposed a lattice gauge model which is solvable in the large N limit under an assumption of translation invariance of a master field and has a nontrivial critical behaviour. Different aspects of the Kazakov and Migdal (KM) model were considered [3] - [10]. A relation of this model to QCD and, in particular, the property of asymptotic freedom still should be clarified.

In this note a standard quenched momentum prescription [11] - [19] for treating the large N limit is used for the KM model. It has been established [12, 14] that in the case of gauge theories the quenching of the momentum must be accompanied by a constraint on the eigenvalues of the covariant derivative. It will be shown that this constraint leads to a quartic dependence of an action on unitary matrix instead of a quadratic dependence discussed in previous considerations. In the strong coupling limit one can expect that the constraint can be omit and one lefts with an action which is quadratic on the unitary matrix V. In this case we can identify the translation invariant master field with a reduced field without quenching, i.e. without constraints. The relation of the translation invariant master field with the reduced field without quenching was noted by Makeenko [7]. However one cannot ignore the constraint in all regions of the coupling constant. This drops a hint that the approximation of the constant master field does not fully describes an asymptotic of the KM model at the large N limit. This constraint makes the KM model not exactly solvable for the weak coupling. We calculate the integral over unitary matrix in the semiclassical approximation, i.e. at small coupling. By the analogy with the Gross-Witten [21] and the Brezin-Gross [20] models it is natural to expect a phase transition for a theory with an action being a polynomial on the one link unitary , in particular, for the quartic action. In the framework of some approximation procedure scheme this phase transition gives rise to a phase transition for the KM model.

The paper is organized as follows. In Section 2 the general quenched momentum prescription with constraints is applied to the KM model and a reduced action with quartic dependence on the gauge field is presented. Then, in Section 3 we integrate out the gauge fields in the semiclassical approximation. In Section 4 we use an approximation scheme to evaluate the remaining integrals over the quenched momentum and the eigenvalues of the scalar field. The concluding remarks are collected in the last Section.

2 Quenched Momentum Prescription for the KM model

The KM lattice gauge model is defined by the partition function

\[ Z_{KM} = \int \prod_{x,\mu} dU_\mu(x) \prod_x d\Phi(x) e^{\sum_x N tr \left( -V[\Phi(x)] + \beta \sum_\mu \Phi(x) U_\mu(x) \Phi(x+\mu) U_\mu^\dagger(x) \right)} . \]  

Here the field \( \Phi(x) \) takes values in the adjoint representation of the gauge group \( SU(N) \) and the link variable \( U_\mu(x) \) is an element of the group, \( \mu = 1, \ldots, d \).

The well known procedure for treating the large N limit of \( N \times N \) matrix models consists in application of a quenched momentum prescription. This prescription for treating the limit of infinite N of theories with a global, or local gauge, \( U(N) \) symmetry, both on the lattice and in the continuum, has been obtained more than ten years ago by Eguchi...
and Kawai [11], Bhanot, Heller and Neuberger [12], Parisi [13] and Gross and Kitazawa [14] (see also [16]-[17]). Generally speaking, according to this prescription a reduced model can be described as containing just one side (one space-time point) and one scalar field $\Phi$ on this side, or $d$ links attached to one side and $d$ fields for vector field. To get the reduced action one should replace a matrix field $\Phi(x)$ with $D(x)\Phi D^\dagger(x)$, where $D(x) = e^{i\mathbf{p}_\mu x^\mu}$, and $\mathbf{p}_\mu$ is the diagonal matrix with matrix elements $p^i_\mu, i = 1, \ldots, N$. Then, one gets the vacuum energy in the large $N$ limit by integration the free energy of obtained action over $\mathbf{p}_\mu$. It has been shown that quenched theory produces the standard Feynman diagrams for invariant Green functions in all orders in perturbation theory.

In the case of gauge theories the quenching of the momentum must be accompanied by a constraint on the eigenvalues of the covariant derivative. Without this constraint one gets naive reduced model without quenching. In the case of the Wilson gauge theory the naive reduced model describes correctly the theory at large $N$ only in the strong coupling regime.

Let us apply the quenched momentum prescription to the KM gauge theory (1). If we were to follow the quenched momentum prescription we would replace $\Phi(x + \mu)$ with $D_\mu \Phi(x) D^\dagger_\mu$, where $D_\mu = e^{i\mathbf{p}_\mu a},$ and $\mathbf{p}_\mu$ is the standard diagonal matrix whose elements are $|p^i_\mu| < \pi/a$, $a$ is the lattice spacing. The quenched model will then have an action

$$\bar{S}[\Phi, U_\mu] = (a)^d[-\text{tr} V(\Phi) + \frac{1}{g^2} \sum_{\mu > 0} \text{tr} \Phi U_\mu D_\mu \Phi D^\dagger_\mu U^\dagger_\mu],$$

and the vacuum energy

$$\bar{F}(p) = \ln \left[ \int d\Phi d\mathbf{U}_\mu \exp\left(-\bar{S}(\Psi, U_\mu, p_\mu)\right) \right].$$

The vacuum energy per unit volume at the large $N$ limit is then obtained by integrating $\bar{F}(p_\mu)$ over all values of $p_\mu$

$$F = \lim_{N \to \infty} \int dp_\mu \bar{F}(p_\mu).$$

The integration over $p_\mu$ is normalized to unity for all values of $N$

$$\int dp = \prod_{\mu=1}^d \int_{-\pi/2a}^{\pi/2a} \prod_{i=1}^N \left(\frac{d^d p_i}{(\pi/a)^d}\right) \left(\frac{1}{2a}d\right).$$

If we were used the standard Haar measure, $dU_\mu$, then by a change of variables $U_\mu \to U_\mu D_\mu$ we would eliminate $D_\mu$ from the action (2). Thus no quenching would have occurred, and we would recover the KM model in the translationally invariant master field approximation,

$$F = \ln \left[ \int d\Phi dU_\mu \exp(a^d N[-\text{tr} V(\Phi) + \beta \sum_{\mu > 0} \text{tr} \Phi U_\mu U^\dagger_\mu]) \right].$$

The same procedure being applied to the standard Wilson theory yields to the Eguchi-Kawai reduced model without quenching, which is known to be necessary to obtain the correct results in weak coupling. The exit from this problem is known, one has to introduce a gauge invariant constraint on the $U_\mu$’s [14]. The suitable constraint restricts the eigenvalues of $U_\mu D_\mu$ to be equal to $D_\mu$. Note that in continuum case the analog of these constraints yield the correct coupling constant renormalization [14]. We are going to use
these constraints for the reduced model (2). This means that we have to take the measure $d\mu(U)$ in (3) to be
\[ d\mu(U) = \prod dU_\mu C(U_\mu, D_\mu), \]
where $dU_\mu$ is the Haar measure on $SU(N)$ and
\[ C(U_\mu, D_\mu) = \prod \int dV_\mu \Delta(D_\mu) \delta(U_\mu - V_\mu D_\mu V_\mu^\dagger D_\mu^\dagger) \]
\[ \Delta(D_\mu) = \prod_{i<j} \sin^2 \left( \frac{p^i_\mu - p^j_\mu}{2a} \right). \]
For the moment we omit the Faddeev-Popov determinant which should be taken into account if one wants to get a correct result in weak coupling. If we integrate out the $U_\mu$, setting $U_\mu = V_\mu D_\mu V_\mu^\dagger D_\mu^\dagger$ we obtain
\[ \exp(\tilde{F}(p)) = \int d\Phi \prod dV_\mu \exp \{ a^d N \left[ -\text{tr} V(\Phi) + \beta \sum_{\mu>0} \text{tr} \left( \Phi V_\mu D_\mu V_\mu^\dagger \Phi V_\mu D_\mu^\dagger V_\mu^\dagger \right) \right] \}. \]
To find the vacuum energy at the large $N$ limit one should integrate $\tilde{F}(p)$ over $d\mu(p)$. In fact we will use the formula
\[ F = \int d\mu(p) \tilde{F}(p), \]
where
\[ d\mu(p) = \prod_i \frac{d^d p_i}{(\sqrt{\pi} a)^d} \exp(-p_i^2 a^2). \]
In comparison with equation (6), where we have the integral from the exponent containing the quadratic dependence from the unitary matrix $U$, in (9) we have to integrate the exponent with the quartic dependence from the unitary matrix.

### 3 Weak Coupling for the Reduced KM Model with Quenching

The action in (9) contains the quartic interaction between the unitary matrix $V$ and the hermitian matrix $\Phi$, and the model does not look the exactly solvable. It is clear that the integral $I(\Phi, D)$
\[ I(\Phi, D) = \int dV \exp[\beta \text{tr} \left( \Phi V D V^\dagger \Phi V D^\dagger V^\dagger \right)] \]
depends only on eigenvalues of the hermitian matrix $\Phi$, $\phi_i$.

We are going to calculate the integral over $V$‘s in (9) in the semiclassical approximation. To this end let us note that the corresponding classical equation has a form
\[ DV_\mu^\dagger \phi V_\mu D_\mu^\dagger V_\mu^\dagger \phi - V_\mu^\dagger \phi V_\mu D_\mu^\dagger V_\mu^\dagger \phi V_\mu D_\mu^\dagger + \]
\[ D_\mu^\dagger V_\mu^\dagger \phi V_\mu D_\mu^\dagger V_\mu^\dagger \phi - V_\mu^\dagger \phi V_\mu D_\mu^\dagger \phi V_\mu D_\mu^\dagger V_\mu^\dagger = 0 \]
Here $(\phi)_{ij} = \delta_{ij}\phi_i$. The solutions of this equation are
\[ V_0 = \Upsilon P, \]
where $\Upsilon$ is any diagonal unitary matrix and $P$ is any $N \times N$ permutation matrix which, when applied to an $N$-vector $\psi$, gives

$$P_{ij}\psi_j = \psi_{P(i)}.$$  \hspace{1cm} (15)

Thus, for each permutation $P$, one has a solution of equation (13). So, for each $P$, we start with the decomposition of $V$ around $\Upsilon P$,

$$V_\mu = V_0 \U P_\mu,$$  \hspace{1cm} (16)

where

$${\U} = 1 + i\lambda^\alpha \xi_\alpha - \frac{1}{2} \lambda^\alpha \lambda^\beta \xi_\alpha \xi_\beta + \mathcal{O}(\xi^3)$$  \hspace{1cm} (17)

To get the complete answer in the semiclassical approximation, we have to sum over all saddle points

$$Z(\phi) = \prod_\mu \int dV_\mu e^{i\delta(\Phi,V_\mu)} \approx \prod_\mu \sum_P \int_{\text{around } P} d\mathcal{U} e^{i\delta N S(\phi,D_\mu,P,\mathcal{U})}$$  \hspace{1cm} (18)

The effective action $S(\phi,D_\mu,P,\mathcal{U})$ up to quadratic terms on $\xi$ has the form

$$S(\phi_i, D_\mu, P, \mathcal{U}) = \sum \phi_{P(i)}^2 +$$

$$2 \text{tr} (\lambda \cdot \xi) \phi_\mu D_\mu \phi \phi \phi_\mu + \text{tr} (\lambda \cdot \xi) \phi_\mu D_\mu \phi_\mu (\lambda \cdot \xi) D_\mu^\dagger$$

$$+ \text{tr} (\lambda \cdot \xi) D_\mu (\lambda \cdot \xi) \phi_\mu D_\mu^\dagger \phi_\mu - 2 \text{tr} (\lambda \cdot \xi) \phi_\mu D_\mu (\lambda \cdot \xi) D_\mu^\dagger - 2 \text{tr} (\lambda \cdot \xi)^2 (\phi_\mu)^2$$  \hspace{1cm} (19)

where $\phi_\mu$ is a diagonal matrix $\text{diag} \phi_\mu = \phi_{(1)}, \phi_{(2)}, \ldots \phi_{(\mu)}$; $\xi_{ij}$ are the components of $(\lambda \cdot \xi)$ along of generators of $U(N)$ (from this place for simplicity we deal with $U(N)$)

$$(\lambda_{+ij})_{ij} = (\delta_i^j, \delta_j^i, - \delta_j^i \delta_i^j); \quad (\lambda_{-ij})_{ij} = i(\delta_i^j \delta_j^i + \delta_j^i \delta_i^j);$$  \hspace{1cm} (20)

(the components corresponding to generators being diagonal matrices do not make a contribution in (21)). Taking into account the formula

$$\text{tr} [A(\lambda \cdot \xi) B(\lambda \cdot \xi) - A B(\lambda \cdot \xi)(\lambda \cdot \xi)] = - \sum_{i<j} [A_i B_j + A_j B_i - A_i B_i - A_j B_j][(\xi_{+ij})^2 + (\xi_{-ij})^2],$$  \hspace{1cm} (21)

we get

$$S(\phi_i, D_\mu, P, \mathcal{U}) = \sum \phi_{P(i)}^2 + 2 \sum_{i<j} (\phi_{P(i)} - \phi_{P(j)})^2 D_{ij} [(\xi_{+ij})^2 + (\xi_{-ij})^2]$$  \hspace{1cm} (22)

where

$$D_{ij} = (D_{\mu i} - D_{\mu j})(D_{\mu i}^\dagger - D_{\mu j}^\dagger).$$  \hspace{1cm} (23)

In the basis (20), the measure for $U(N)$ takes the form

$$dV = \prod_{i<j} d\xi_{+ij} d\xi_{-ij} \exp\left[-\frac{1}{6} N((\xi_{+ij})^2 + (\xi_{-ij})^2)\right] \prod_{i=1}^N d\xi_i \exp\left[-\frac{1}{6} N\xi_i^2 + \frac{1}{6} (\sum_{i=1}^N \xi_i)^2\right].$$  \hspace{1cm} (24)
Performing the gaussian integration over $\xi^{\pm,ij}$, we get

$$
\prod_\mu \left( \sum_P e^{a^d N \beta \sum_i \phi^2_P(i)} \prod_{i<j} \frac{1}{a^d N \beta (\phi_P(i) - \phi_P(j))^2 D_{\mu ij}} \right). \tag{25}
$$

Having in mind that the distribution of the external momentum $p_i$ is invariant under permutation $i \rightarrow P(i)$ one can for any given $P$ renumber the momentum and therefore one can claim that all permutations give the same contribution, so the final answer in semiclassical approximation up to some normalization factor has a form

$$
\exp(a^d N \beta \sum_i \phi^2_i) \prod_\mu \prod_{i<j} \frac{1}{a^d N (\phi_i - \phi_j)^2 D_{\mu ij} + \frac{N}{63}}. \tag{26}
$$

Including the contribution of the factor (24) in (25) we mix the orders in the perturbation theory and the $O(g^2)$ term ($g^2 = \beta^{-1}$) is not correct. To get the correct $O(g^2)$ answer the two-loop corrections should be computed. Note that the expression (26) is well defined for some equal eigenvalues of $\Phi$, that is in accordance with the well defined integral (10) over compact manifold, and only singular point for finite $N$ is the point $\beta = \infty$, i.e. one cannot neglect the contribution of the second term in the denominator of (26). Of course, for infinite $N$ the integral (10) can have a singular points (compare with phase transitions for the Gross-Witten [21] and the Brezin-Gross [20] models). A hint to a phase transition for $N = \infty$ gives a zero of the denominator.

Let us to compare the integral (10) with the Itzykson-Zuber integral [22], where one integrates the exponent of a quadratic form on $V$. If we calculate the Itzykson-Zuber integral using the semiclassical approximation, then the corresponding classical solutions have the same form, but the contributions around given $P$ depend explicitly on $P$. Summing over $P$ provides a compensation of the singularities coming from coinciding eigenvalues of matrix $\phi$, and in this case there is no reason to mix the orders in the perturbation theory, and, moreover, the remarkable fact is that summing over $P$ the leading terms of the semiclassical approximation gives the exact expression found by different methods [22, 23].

Substituting (26) in (4) we get the vacuum energy as the large $N$ limit of the following expression

$$
\frac{F}{(Vol)} = \int dp_\mu \ln \left[ \int d\phi e^{N a^d (m_0^2 + \beta) \sum_i \phi^2_i} \prod_\mu \prod_{i<j} \frac{\sin^2 \left( \frac{p^\mu_i - p^\mu_j}{\sqrt{a^d}} \right)}{a^d N (\phi_i - \phi_j)^2 D_{\mu ij} + \frac{N}{63}} \right]. \tag{27}
$$

The free energy (27) can be rewritten using the replica trick [24]. The replica trick allows us to estimate the free energy of the quenched system as the analytic continuation of the free energy in a class of annealed systems with the action being the copy of n’s actions (27). We have

$$
\mathcal{F} = \lim_{n \rightarrow 0} \frac{1}{n} \left( \int dp \prod_{\alpha=1}^n d\phi^\alpha e^{N a^d (m_0^2 + \beta) \sum_i \phi^2_\alpha} \prod_\mu \prod_{i<j} \frac{1}{a^d (\phi^\alpha_i - \phi^\alpha_j)^2 D_{\mu ij} + \frac{N}{63}} - 1 \right) + \text{const.} \tag{28}
$$

Equation (27) points that there is a phase transition for some $\beta = m_0^2 \beta_{cr}$. The reason is the following. For $\beta = \infty$ one has not the second term in the denominator of (27) and...
we left with the answer which was expected in the naive continuous limit and which is unstable, and for $\beta = 0$ we get a stable model. So, it is natural to expect that the model exhibits a phase transition.

To estimate the order of $\beta_{cr}$ one can roughly estimate (28) as

$$\mathcal{F} = \ln \int d\phi_i e^{Na^d(-m_0^2+\beta d)\sum \phi_i^2} \Delta^2(\phi) \prod_{\mu} \prod_{i<j} a^d \beta <(\phi_i - \phi_j)^2> <D_{\mu ij}> + \frac{1}{6}, \tag{29}$$

where $<(\phi_i - \phi_j)^2>$ means the average with semi-circular distributions of eigenvalues $\phi_i$ given by $u(\phi) = \frac{m_{eff}}{2\pi}\sqrt{4 - m_{eff}^2}\phi^2$, i.e.

$$\frac{1}{N^2} \lim_{N \to \infty} \sum_{i<j} <(\phi_i - \phi_j)^2> = \frac{1}{2} \int (\phi - \phi')^2 u(\phi)u(\phi')d\phi d\phi' = \frac{1}{m_{eff}^2} = \frac{1}{m_{eff}^2}. \tag{30}$$

In our case $m_{eff}^2 = a^d(m_0^2 - d\beta) = a^d m_0^2(1-d\beta_0)$, $\beta_0$ is a dimensionless constant.

$<D_{\mu ij}>$ means the average of (23) with the measure (11), so

$$<D_{\mu ij}> = 2(1-1/\sqrt{e}).$$

Therefore, the average of the first term in the denominator is equal to

$$\frac{2(1-1/\sqrt{e})\beta_0}{1-d\beta_0}.$$

So, for $\beta_0 = 1/d$ the contribution of the first term becomes infinite and we can neglect the second term and we get unstable model what is in accordance with expected phase transition for $\beta_0 = 1/d$.

The region $\beta_0 > 1/d$ is forbidden, since the action becomes unbounded from below. Let us examine the region $\beta_0 < 1/d$ including the negative $\beta_0$. In this case the action is bounded from below, but still there is a reason for a phase transition due to zero of denominator in (29) and

$$\beta_{cr} = \frac{1}{d-4.8}. \tag{31}$$

This phase transition occurs only for $d < 4.8$.

As for numerical factors we would like to note that they may be changed by a factor $1/J$ which takes into account the two-loop corrections, i.e. in (26) must be $\frac{1}{6}$ instead of $\frac{1}{6}$. If $J$ is such that $0, 8J > d$ then we can expect a phase transition. For negative $J$ the above estimation gives a phase transition without any restrictions on $d$.

## 4 Concluding Remarks

In this paper an application of the quenched procedure to the KM model has been considered. It has been noted that using the quenched procedure with constraints one obtains the one side model with the quartic interaction. This model has been investigated in the semiclassical approximation and the phase transition at $\beta_{cr} = \frac{1}{d-4.8}$ has been found.

It will be shown in forthcoming paper [25] how it is possible to modify the initial model to get the reduced model with quenching which admits an analytical treating at the large $N$ limit using the technique of the orthogonal polynomials.
References


