On brane cosmological solutions

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Abstract

This paper studies some cosmological consequences of the five dimensional, two brane Randall-Sundrum brane scenario. The radius of the compact extra dimension is taken to be time dependent. It is shown that the cosmology consistent with the two brane Randall-Sundrum model is a power law expansion of the universe, with scale factor growing as $t^{1/2}$. The two branes tend to move towards each other with time. Some comments are made on the contribution of surface terms in deriving the four dimensional effective action.

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The idea that we may be living in dimensions higher than four has been under investigation for several decades now. In fact, the original Kaluza-Klein theory is as old as Einstein’s theory of gravity. Generically, Kaluza-Klein theories (for a review see, [1]) are theories of gravitation formulated in higher dimensions, some of which are compactified. The possibility that there are additional dimensions which remain small was put forward as a means of unifying the electromagnetic and gravitational fields as components of a single higher-dimensional field. The matter fields arise out of geometry and the theory essentially is a minimal extension of Einstein’s theory of relativity to higher dimensions.

Recently there have been attempts to address the mass hierarchy problem by invoking the presence of extra dimensions. One assumes localisation of matter fields on a four-dimensional brane embedded in a $D = 4 + n$ dimensional bulk [2, 3], while gravity can propagate in all the $D$ dimensions. In these higher dimensional models, it is assumed that the geometry is a direct product of the four dimensional spacetime with an $n$-dimensional compact manifold. From the point of view of an observer on the four-dimensional brane, the Planck scale is given by $M_{Pl} = M_n + V_n$, where $V_n$ is the compact space volume and $M$ is the fundamental Planck scale. If the radius of the extra dimension $R_n$ is large, i.e. of the order of a millimeter, the fundamental Planck scale can be of the order of a TeV. The phenomenology which arises as a result is expected to be tested in the next generation of particle colliders. However the demand that the extra dimensions be large introduces an implicit hierarchy and is therefore not a satisfactory solution to the problem.

Randall and Sundrum (RS) [4] proposed a five-dimensional model based on a non-factorisable geometry. The “warp” factor, which scales the four dimensional spacetime with respect to the extra dimension, is a rapidly changing function of the extra dimension. This obviates the need that extra dimensions be large. The hierarchy between the four dimensional Planck scale and the fundamental scale of the theory is resolved because of the presence of the exponential “warp” factor. The first RS model consists of two four dimensional branes which are defects in a five dimensional anti-deSitter background. One is a positive tension Planck brane and the other is the brane on which standard model particles are confined, which has a negative tension and is called the TeV brane. A variant of this higher dimensional scenario is the one brane world, in which the Planck brane is taken to infinity [5] and one has a four-dimensional brane embedded in a five-dimensional anti-deSitter bulk.

Since this new class of models have non-factorisable warped geometry, they are fundamentally different from the usual Kaluza-Klein theories, and hence involve different phenomenological issues. One of the reasonable questions to ask is how these will affect the early universe cosmology. In fact, a large number of papers (for instance [6, 7, 8, 9]) have investigated the cosmological aspects of the scenarios
proposed by Arkani-Hamed et al. and by Randall and Sundrum. The motivation for such studies is to see if the brane scenario can provide a unified solution to the mass hierarchy problem as well as some problems in cosmology such as inflation.

Motivated by the above arguments, we study some implications of five-dimensional warped geometry on early universe cosmology. The spacetime structure considered here is the RS two-brane model. We first briefly review the model under consideration. It is shown that on integrating out the extra dimension from the action, we have an extra term not dealt with so far in papers. The main result in this paper is the universe has a power law expansion. Here we have taken radion dynamics into account; radion being the field associated with the fluctuations of the extra dimension. There is no inflation and the branes tend to move towards each other with time.

The five-dimensional spacetime is a slice of anti-deSitter geometry, where we have a negative cosmological constant. Two 3-branes are located at fixed points of orbifold $S^1/Z_2$. In other words, the extra fifth dimension is a circle with opposite points identified. Following [12] we take the two orbifold points to be situated at $y = 0$ and $y = 1/2$. The positive tension Planck brane is located at $y = 0$ and the negative tension TeV brane is situated at $y = 1/2$.

The five dimensional action for anti-deSitter spacetime is given by

$$S = 2 \int d^4x \int_0^{1/2} dy \sqrt{-G} \left( M_3^3 R - \Lambda \right) + \int d^4x \sqrt{-g^{(+)}(L^+ - V^+)} + \int d^4x \sqrt{-g^{(-)}(L^- - V^-)} \tag{1}$$

where

$$V^+ = -V^- = 12m_0 M^3, \quad \Lambda = -12m_0^2 M^3, \tag{2}$$

and where $R$ is the five dimensional Ricci scalar, the bulk cosmological constant is denoted by $\Lambda$ and $M$ is the five dimensional Planck mass. The constant $m_0$ is same as the parameter $k$ in [4, 5]. The ($+$) sign denotes the Planck brane and ($-$) sign represents the negative tension TeV brane. The matter fields on the positive and negative tension branes are $L^+$ and $L^-$ respectively, while $V^\pm$ represent the brane tensions on positive and negative tension branes respectively.

The metrics on the two four-dimensional branes are therefore given by

$$g_{\mu\nu}^{(+)} = G_{\mu\nu}(x^\mu, y = 0) \quad \text{and} \quad g_{\mu\nu}^{(-)} = G_{\mu\nu}(x^\mu, y = 1/2) \tag{3}$$

The five-dimensional Einstein equations of motion are solved by the metric

$$ds^2 = e^{-2m_0 r_c|y|} \eta_{\mu\nu} dx^\mu dx^\nu + r_c^2 dy^2 \tag{4}$$
where $\eta_{\mu\nu}$ represents the flat four dimensional 3-brane while $r_c$ is the radius of the extra dimension.

The relation between the four dimensional Planck scale $M_{Pl}$ and the fundamental scale $M$ in the theory

$$M_{Pl}^2 = \frac{M^3}{m_0} \left[ 1 - e^{-2m_0 r_c} \right]$$

Because of the exponential warp factor, to scale the hierarchy required between the two scales, we need the product $m_0 r_c$ to be of the order of 50. Therefore this provides us with an interesting approach to solve the mass hierarchy problem.

To accommodate cosmological solutions, we deviate from the RS flat branes geometry and assume the modulus $r_c$ to be time dependent [12, 15]. The five-dimensional metric ansatz is [12]

$$ds^2 = e^{-2m_0 b(t) y} g_{\mu\nu} dx^\mu dx^\nu + b^2(t) dy^2$$

with the four-dimensional spacetime being described by the spatially flat Friedmann-Robertson-Walker metric

$$g_{\mu\nu} = \text{diag}(-1, a^2(t), a^2(t), a^2(t))$$

where $a(t)$ is the scale factor. We take the same notation as in Ref. [12].

Using the above metric ansatz, we integrate over the extra dimension from the action given in Eq. (1). The four dimensional action can be written as

$$S_{eff} = -\frac{3}{k^2 m_0} \int d^4 x \left[ (1 - \Omega_b^2) \frac{\dot{a}^2}{a^2} + m_0 \Omega_b^2 \frac{\dot{b}}{a} - \frac{1}{4} m_0^2 \Omega_b^2 b^2 \right]$$

$$- \frac{1}{k^2} \int d^4 x a^3 m_0 \left( 1 - e^{-2m_0 b} \right)$$

where $\Omega_b = e^{-m_0 b(t)/2}$ and $k^2 = 1/2M^3$.

The action can further be written as

$$S_{eff} = -\frac{1}{2k^2 m_0} \int d^4 x a^3(t) \left[ (1 - \Omega_b^2) R_4 \right]$$

$$- \frac{3}{2} m_0^2 \Omega_b^2 b^2 + 16 m_0 \left( 1 - e^{-2m_0 b} \right)$$

The four dimensional Ricci scalar is denoted by $R_4$. The last ‘extra’ term in the above action gets canceled if one includes surface terms in the action. A discussion regarding cancellation of these terms is presented in Appendix A.
The cosmological equations of motion obtained from the action without the extra term are given by

\[ 3H^2 = \frac{3\Omega_b^2}{1 - \Omega_b^2} + 6H \frac{\Omega_b \dot{\Omega}_b}{1 - \Omega_b^2} \]  
\[ 2\dot{H} + 3H^2 = -\frac{\dot{\Omega}_b^2}{1 - \Omega_b^2} + 4H \frac{\Omega_b \dot{\Omega}_b}{1 - \Omega_b^2} + \frac{2\Omega_b \ddot{\Omega}_b}{1 - \Omega_b^2} \]
\[ 6\frac{\ddot{\Omega}_b}{\Omega_b} + 18H \frac{\dot{\Omega}_b}{\Omega_b} = 0 \]

which have solutions given by

\[ H = \frac{1}{2t} \]
\[ \Omega_b(t) = \pm 1 + At^{-1/2} \]

where \( H = \frac{\dot{a}(t)}{a(t)} \), \( \Omega_b(t) = e^{-m_0 b(t)/2} \) and \( A \) is a positive integration constant. Since \( \Omega_b \) is an exponential function, we need to consider only the + branch.

Here, \( \Omega_b \) approaches 1 as time \( t \) tends to a large value. In this limit \( b(t) \) approaches 0, i.e. the two branes are moving closer and closer to each other. If one considers the matter on the brane to be radiation only, one can again show (with a similar calculation) that the power law expansion as \( a(t) \sim t^{1/2} \) holds.

A mechanism to stabilise the radius of the extra dimension to its equilibrium value was proposed in Ref. [13]. Here one considers a bulk scalar field which is allowed to propagate in all the five dimensions. The scalar field, in general, has different vacuum expectation values on the two branes. The gradient created along the extra dimension and the potential energies, lead to a radion potential. This potential has the properties required for inflation. The scalar field rolls over and settles at its minimum hence stabilising the value of \( b(t) \). This brings us to an interesting question if the scalar field remains at its minimum and does not oscillate out of the potential well (work in progress and will be reported elsewhere) [14].

This paper presents some cosmological solutions allowed by the five dimensional Randall-Sundrum two brane scenario. The solutions to cosmological equations of motion, reveal that even if the extra term is absent, one obtains a power law cosmology. However, the branes tend to move towards each other with time, leading to a future collapse. The situation remains the same if one assumes the presence of radiation on the brane. Inflationary solutions can be obtained only if one assumes the presence of an external potential. Interestingly, if one makes
a variable transformation as in [15] the solutions are still power law in the new
time parameter. It is conjectured by many earlier authors that the potential
suggested by Goldberger and Wise [16] to stabilise the radion to its equilibrium
value can be the potential driving inflation. The equations of motion then have to
be solved numerically to investigate the behaviour of the cosmological equations.
Work on this is in progress. We integrate out the extra dimension to obtain
the four-dimensional effective action. The effective action obtained from such an
exercise contains an extra term which has not been considered in papers so far.
It is conjectured that this term arises because the surface terms have not been
included in the calculations. Another possible explanation could be because of
metric ansatz being put at the action level, which gives different results from
when it is put in the equations of motion. Hopefully the underlying connection
will become clear with further work.

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A Calculation of higher dimensional Ricci tensor and the
surface term

In this appendix we present the steps in the calculation of the four-dimensional
action. The five dimensional Ricci scalar is

\[
R = -20m_0^2 - e^{2m_0 b(t)y} \left\{ -6 \frac{\dot{a}^2}{a^2} - 6 \frac{\dot{a}}{a} \frac{\dot{b}}{b} - 6m_0^2 y^2 \dot{b}^2 - 6 \frac{\ddot{a}}{a} - 2 \frac{\ddot{b}}{b} \right\} - ye^{2m_0 b(t)y} \left\{ 18m_0 \frac{\dot{a}}{a} \frac{\dot{b}}{b} + 4m_0 \frac{\ddot{b}}{b^2} + 6m_0 \ddot{b} \right\}
\] (11)

We consider the first term in the five dimensional action, \textit{viz.}

\[
S_1 = 2 \int dx^4 \int_0^{1/2} \sqrt{-G} \left( M^3 R - \Lambda \right)
\]
The constant term $-20m_0^2$ combines with $\Lambda = 12m_0^2M^3$ and the first term in the five dimensional action contains

$$S_1 = 2M^3 \int d^4x \int_0^{1/2} - e^{-4m_0b(t)}a^3(t)b8m_0^2$$  \hspace{1cm} (12)$$

in addition to the other terms mentioned above.

Integration over the extra dimension gives

$$S_{1(\text{eff})} = -2M^3 \int d^4x a^3(t)2m_0 \left[ 1 - e^{-2m_0b(t)} \right]$$  \hspace{1cm} (13)$$

Combining the terms containing $V^+)$ and $V^-(t)$, we are left with a term

$$S_{(\text{eff})} = -2M^3 \int d^4x a^3(t)8m_0 \left[ 1 - e^{-2m_0b(t)} \right] \hspace{1cm} + \text{other terms}$$  \hspace{1cm} (14)$$

In general, the Ricci scalar in $d + 1$ dimensions contains a term proportional to $d(d + 1)$ and the cosmological constant has a term proportional to $d(d - 1)$. This leaves a quantity proportional to $2d^2$ on combining the above two. Hence there will remain an extra factor in $d$ dimensions, and if present may have nontrivial contribution to the solutions.

We now come to computation of the surface term. In the original Randall-Sundrum brane world, the metric is

$$ds^2 = e^{-2ky}g_{\mu\nu}dx^\mu dx^\nu + dy^2.$$  \hspace{1cm} (15)$$

The Ricci tensor and scalar in this case are

$$R_{\mu\nu} = -4k^2 g_{\mu\nu} \quad \text{and} \quad R = -20k^2$$  \hspace{1cm} (16)$$

The term $\frac{\delta R_{\mu\nu}}{\delta g_{\mu\nu}}$ is $-8k^2$ the number required for cancellation! The factor is same for the time dependent case considered above.

Since the five dimensional spacetime is finite, the surface terms (which do not contribute for an infinite spacetime) should contribute in this case. Therefore, we have to take into account the contribution from the term $\frac{\delta R_{\mu\nu}}{\delta g_{\mu\nu}}$ while calculating the equations of motion. Indeed, for the ‘static’ Randall-Sundrum case this term cancels the ‘extra’ term. One requires this term in the action (to cancel away the surface term) and integration along $y$ leads to the equations of motion as in other papers. The same factor is required for the time dependent case too.

An alternate explanation is that we have assumed the form of metric at the action level. The equations of motion obtained from this transformed action, do not give a general solution. These solutions are a subset of those obtained by solving the equations of motion obtained from the full action. Therefore, a
more general calculation does away with this term. Comparing our results with another paper dealing with similar aspects, inflationary solutions to the equations of motion are obtained in [15]. Incidentally, the potential used in the paper is same as what one would obtain if one makes the same field transformation in the ‘extra’ term.

References