An application of supersymmetric quantum mechanics to a planar physical system

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Abstract

Supersymmetry (SUSY) in non-relativistic quantum mechanics (QM) is applied to a 2-dimensional physical system: a neutron in an external magnetic field. The superpotential and the two-component wave functions of the ground state are found out.

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The algebraic technique of supersymmetry in quantum mechanics (SUSY QM) was first introduced by Witten [1]. The essential idea of this formulation is based on the Darboux procedure on second-order differential equations, which has been successfully utilized to achieve a supersymmetric generalization of the harmonic-oscillator raising and lowering operators for shape-invariant potentials [2,3]. The SUSY algebra has also been applied to construct a variety of new one-parameter families of isospectral supersymmetric partner potentials in quantum field theory [4]. The shape-invariance conditions in SUSY have been independently generalized for systems described by two-component wave functions [5]. Recently, we have found a two-by-two matrix superpotential associated to the linear classical stability from the static solutions for a system of two coupled real scalar fields in (1+1)-dimensions [6].

We also presented an integral representation for the momentum space Green’s function for a neutron in interaction with a static magnetic field of a straight current carrying wire, which is also described by two-component wave functions [7]. The SUSY QM formalism was also applied to this planar physical system in the momentum [8] and coordinate [9] representations.

In this letter, we consider the notation of Ref. [8]. However, according to our developments, we can realize the supersymmetric algebra in coordinate representation, introducing some transformations in the original system corresponding to a neutron interacting with the magnetic field of a linear current carrying conductor, so that we are able to implement a comparison with both superpotentials for the cases corresponding to currents located along $x$ and $z$ directions.

Now, let us consider an electrically neutral spin-$\frac{1}{2}$ particle of mass $M = 1$ and magnetic moment $\mu \vec{\sigma}$ (a neutron) interacting with an infinite straight wire carrying a current $I$ and located along the $z$-axis. The magnetic field generated by the wire is given by (we use units with $c = \hbar = 1$)

$$\vec{B} = 2I \frac{(-y, x, 0)}{(x^2 + y^2)} ,$$

where $x$ and $y$ are Cartesian coordinates of the plane perpendicular to the wire.

The Hamiltonian associated with the physical system is given by

$$H = \frac{\vec{\mathbf{p}}^2}{2M} + \mu \vec{\sigma} \cdot \vec{B} = \frac{\vec{\mathbf{p}}^2}{2} + 2I \mu \frac{(-y \sigma_1 + x \sigma_2)}{(x^2 + y^2)} ,$$

where $\vec{\mathbf{p}}$ is the momentum of the particle.
where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices. The motion along the $z$-axis is free and will be ignored in what follows and in this way we get a two-dimensional problem.

Due to the translational symmetry in the $z$-direction, the two-component wave function $\psi(\rho, k)$ can be written as

$$\psi^{(n_\rho)}(\rho, k) = \frac{1}{\sqrt{4\pi L}} \begin{pmatrix} \tilde{\psi}_1^{(n_\rho, m)}(\rho, k) e^{im\phi} \\ \tilde{\psi}_2^{(n_\rho, m)}(\rho, k) e^{i(m+1)\phi} \end{pmatrix} e^{i\frac{2\pi}{L} k z}$$

$$\equiv \begin{pmatrix} \psi_1^{(n_\rho)}(\rho, k) \\ \psi_2^{(n_\rho)}(\rho, k) \end{pmatrix},$$

where $n_\rho = 0, 1, 2, \cdots$ is the radial quantum number; $k = 0, 1, 2, \cdots$; $m = 0, \pm 1, \pm 2, \cdots$; $\rho, \phi, z$ are the usual cylindrical coordinates and the parameter $L$ is the macroscopic length of the conductor.

Therefore, the Schrödinger equation splits up into a system of two coupled second order differential equations as follows

$$\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{d}{d\rho} \tilde{\psi}_1^{(n_\rho, m)} \right) - \frac{m^2}{\rho^2} \tilde{\psi}_1^{(n_\rho, m)} + 2\tilde{E} \tilde{\psi}_1^{(n_\rho, m)} + \frac{2F}{\rho} \tilde{\psi}_2^{(n_\rho, m)} = 0,$$

$$\frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{d}{d\rho} \tilde{\psi}_2^{(n_\rho, m)} \right) - \frac{(m+1)^2}{\rho^2} \tilde{\psi}_2^{(n_\rho, m)} + 2\tilde{E} \tilde{\psi}_2^{(n_\rho, m)} + \frac{2F}{\rho} \tilde{\psi}_1^{(n_\rho, m)} = 0,$$

where

$$F = -\frac{\mu_0 \mu I}{2\pi}$$

and

$$\tilde{E} = E - \frac{2\pi k^2}{L^2}.$$  

Note that Eq. (4) is exactly Eq. (2.8) given in [8]. Now, using the relation

$$\tilde{\psi}_i^{(n_\rho, m)} = \rho^{-\frac{1}{2}} \phi_i^{(n_\rho, m)} \quad (i = 1, 2),$$

we can write the system in (4) in the matrix form as

$$\begin{pmatrix} -\frac{d^2}{d\rho^2} + \frac{m^2-1}{\rho^2} - 2\tilde{E} & -\frac{2F}{\rho} \\ \frac{2F}{\rho} & -\frac{d^2}{d\rho^2} + \frac{(m+1)^2-1}{\rho^2} - 2\tilde{E} \end{pmatrix} \begin{pmatrix} \phi_1^{(n_\rho, m)} \\ \phi_2^{(n_\rho, m)} \end{pmatrix} = 0,$$

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which corresponds to a one-dimensional Schrödinger-like equation associated with the two-component wave function. Therefore, we get the eigenvalue equations

\[ H_1 \Phi_1^{(n,\rho,m)} = \tilde{E}_1^{(n,\rho,m)} \Phi_1^{(n,\rho,m)}, \quad E_1^{(n,\rho,m)} = 2\tilde{E}^{(n,\rho,m)}, \]  

(9)

where

\[ \Phi_1^{(n,\rho,m)} = \Phi_1^{(n,\rho,m)}(\rho,k) = \begin{pmatrix} \phi_1^{(n,\rho,m)}(\rho,k) \\ \phi_2^{(n,\rho,m)}(\rho,k) \end{pmatrix} \]  

(10)

and

\[ H_1 = -1 \frac{d^2}{d\rho^2} + \begin{pmatrix} \frac{m^2 - \frac{1}{4}}{\rho^2} & \frac{-2F}{\rho} \\ \frac{-2F}{\rho} & \frac{(m+1)^2 - \frac{1}{4}}{\rho^2} \end{pmatrix}. \]  

(11)

Defining

\[ H_1 \equiv A^+ A^- + 1\tilde{E}_1^{(0)}, \quad A^\pm = \pm \frac{d}{d\rho} + W(\rho), \]  

(12)

we obtain the following Riccati equation in matrix form

\[ W'(\rho) + W^2(\rho) + 1\tilde{E}_1^{(0)} = \begin{pmatrix} \frac{m^2 - \frac{1}{4}}{\rho^2} & \frac{-2F}{\rho} \\ \frac{-2F}{\rho} & \frac{(m+1)^2 - \frac{1}{4}}{\rho^2} \end{pmatrix}, \]  

(13)

where \( W(\rho) \) is a two-by-two superpotential matrix. The hermiticity condition allows us to write

\[ W = W^\dagger = \begin{pmatrix} f(\rho) & g(\rho) \\ g(\rho) & h(\rho) \end{pmatrix}, \]  

(14)

where \( f, g \) and \( h \) are real functions and satisfy the nonlinear system of differential equations

\[
\begin{aligned}
f' + f^2 + g^2 + E_1^{(0)} &= \frac{m^2 - \frac{1}{4}}{\rho^2} \\
g f + h g + g' &= \frac{-2F}{\rho} \\
h' + h^2 + g^2 + E_1^{(0)} &= \frac{(m+1)^2 - \frac{1}{4}}{\rho^2}.
\end{aligned}
\]  

(15)

Now, let us try a solution for equation (15) assuming that \( g \) is constant. Then, we have

\[ f + h = \frac{-2F}{g\rho}, \]  

(16)

which gives
\[ f' - h' - \frac{2F}{g\rho} (f - h) + \frac{2m + 1}{\rho^2} = 0. \]  

(17)

Solving the last equation and imposing finiteness condition on the solutions, we get

\[ f(\rho) = \frac{b}{\rho}, \]

\[ h(\rho) = \frac{c}{\rho}, \]

(18)

where \( b \) and \( c \) are arbitrary constants. Substituting these solutions into the system (15), we find that a consistent solution is possible only if

\[ g = \frac{-F}{m + 1} \]

(19)

where \( F \) is defined in Eq. (5). Then, turning to Eq. (17) and substituting Eqs. (18) and (19) we find constants \( b \) and \( c \). Putting these results back into Eq. (18), we have that

\[ f(\rho) = \frac{m + \frac{1}{2}}{\rho}, \]

\[ h(\rho) = \frac{m + \frac{3}{2}}{\rho}. \]

(20)

In this case, the two almost isospectral Hamiltonians are given by

\[ H_1 = A^+ A^- - \frac{F^2}{2(m + 1)^2} \mathbf{1}, \]

\[ H_2 = A^- A^+ - \frac{F^2}{2(m + 1)^2} \mathbf{1}. \]

(21)

(22)

Since \( A^+ A^- \) is positive semidefinite, according to (12) and (21) the energy eigenvalue of the ground state is

\[ \tilde{E}^{(0)} = -\frac{F^2}{2(m + 1)^2}; \]

(23)

with the annihilation conditions

\[ A^- \Phi_1^{(0)} = 0 \]

(24)

and

\[ A^+ \Phi_2^{(0)} = 0 \]

(25)

and the new superpotential
The energy eigenvalues of magnetically bound excited states in terms of the radial quantum number $n_{\rho}$, for $m \geq m_0$ becomes

$$\tilde{E}^{(n_{\rho})} = -\frac{F^2}{2(n_{\rho} + m_0 + 1)^2}. \quad (27)$$

Now let us to determine the eigenfunction associated with the ground state given by Eq.(24). To do this let us consider the transformations

$$\phi(\rho) = \chi^{(0)} \rho^{m + \frac{1}{2}}, \quad \rho = 2(m + 1)\eta, \quad F = -\frac{1}{2}, \quad (28)$$

which implies that Eq. (24) turns into the following matrix differential equation

$$\frac{1}{d\eta} \chi(\eta) = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{\eta} \end{pmatrix} \chi^{(0)}(\eta), \quad \chi^{(0)}(\eta) = \begin{pmatrix} \chi_1^{(0)} \\ \chi_2^{(0)} \end{pmatrix}, \quad (29)$$

so that we obtain the following equations for the components $\chi_1^{(0)}$ and $\chi_2^{(0)}$:

$$\frac{d}{d\eta} \chi_1^{(0)}(\eta) = \chi_2^{(0)}(\eta),$$

$$\frac{d}{d\eta} \chi_2^{(0)}(\eta) = \chi_1^{(0)}(\eta) + \frac{1}{\eta} \chi_2^{(0)}(\eta), \quad (30)$$

which leads us a second-order differential equation for $\chi_2^{(0)}(\eta)$, viz.,

$$\frac{d^2}{d\eta^2} \chi_2^{(0)}(\eta) - \frac{1}{\eta} \frac{d}{d\eta} \chi_2^{(0)}(\eta) + \left( \frac{1}{\eta^2} - 1 \right) \chi_2^{(0)}(\eta) = 0. \quad (31)$$

From equations (3), (30) and (31) we obtain the $m$-dependent normalizable ground state

$$\Psi^{(0)}(\rho) = C_m \rho^{m+1} \left( e^{i(m+1)\phi} K_0 \left( \frac{\rho}{2m+2} \right) \right) e^{i\frac{2\pi}{L} k z} \quad (32)$$

where $C_m$ is the normalization constant, and $K_1 \left( \frac{\rho}{2m+2} \right)$ and $K_0 \left( \frac{\rho}{2m+2} \right)$ are the modified Bessel functions. The eigenfunction $\Psi^{(0)}(\rho)$ is in accord with the result found via momentum representation in Ref. [8]. Note that the complete solution of Eq. (31)

$$\chi_2^{(0)}(\eta) = \eta (C_1 K_0(\eta) + C_2 I_0(\eta)), \quad (33)$$
where $C_1$ and $C_2$ are arbitrary non-normalizable constants. Therefore, in order to get a normalizable solution, we choose $c_2 = 0$ and in this way we drop $I_0(\eta)$ which is divergent when $\eta \to \infty$.

It is worthy noticing that under a unitary transformation, $U \mathbf{W}_m U^{-1} = \tilde{\mathbf{W}}_m$, this superpotential, together with the interchange of $m$ by $m + \frac{1}{2}$, and taking $F = -\frac{1}{2}$ becomes that superpotential matrix ($\mathbf{W}_{LJM}$) shown in [9], viz., $\tilde{\mathbf{W}}_{m+\frac{1}{2}} = -\mathbf{W}_{LJM}$. This minus sign that connects $\tilde{\mathbf{W}}_{m+\frac{1}{2}}$ and $\mathbf{W}_{LJM}$ is associated to the fact that we have chosen the first-order differential operator $A^-$ with the opposite sign in the derivative term of the operator $A_m$ considered in Ref. [9].

Using the coordinate representation, we investigate the SUSY in non-relativistic quantum mechanics with two-component eigenfunctions and find a new realization of supersymmetry in a planar physical system of a neutron in interaction with a straight current-carrying wire.

The $N=2$–SUSY superalgebra has the following representation

$$ H_{SUSY} = [Q_-, Q_+]_+ = \begin{pmatrix} A^+ A^- & 0 \\ 0 & A^- A^+ \end{pmatrix}_{4 \times 4} = \begin{pmatrix} H_- & H_0 \\ 0 & H_+ \end{pmatrix}, $$

where the supersymmetric partners are given by $H_- = H_1 - 1E_1^{(0)}$, $H_+ = H_2 - 1E_1^{(0)}$ and the supercharges $Q_\pm$ are 4 by 4 matrix differential operators of first order and can be given by

$$ Q_- = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}_{4 \times 4}, \quad Q_+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}_{4 \times 4}. $$

We have seen that, in non-relativistic quantum mechanics applied to two-component eigenfunctions, if $\Phi_1^{(0)}$ is a normalizable two-component eigenfunction, one cannot write $\Phi_2^{(0)}$ in terms of $\Phi_1^{(0)}$, as in the case of ordinary supersymmetric quantum mechanics. This may be shown in the system considered here of a neutron interacting with an external magnetic field with the current of the conductor in the $z$ direction. Only in the case of 1-component wave functions one may write the superpotential as $W(x) = \frac{d}{dx} \ell n(\psi_0(x))$.

The hermiticity condition satisfied by the superpotential, in the general case, leads us to a method that permits to solve the matrix Riccati equation. As a final remark, we would like to draw the attention to the fact that our result, for a superpotential corresponding to
a neutron in an external magnetic field in the coordinate representation, is related by the following unitary transformation, \( U = \frac{1}{\sqrt{2}} (\sigma_1 + \sigma_3) \), where \( \sigma_1 \) and \( \sigma_3 \) are the Pauli matrix, with a new superpotential so that, after the substitution \( m \) by \( m + \frac{1}{2} \) (the total angular momentum along the wire direction) it reduces to the superpotential recently found in [9], where a current \( I \) along the \( x \) axis of a Cartesian system is considered.

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