Coasting Beam in HERA-\textit{p} Ring

S. Ivanov, O. Lebedev
on behalf of the IHEP$^1$ Task Team

$^1$Institute for High Energy Physics
Protvino, Moscow Region, 142281, Russia

Abstract

This document is a report on studies of production/transport mechanisms resulting in and of cures against a coasting beam halo observed at the HERA proton ring of DESY. Noises in the 208 MHz RF system, drift beyond RF buckets due to energy loss through synchrotron radiation of the high-energy protons and effect of adding the 52 MHz RF voltage are claimed to account for the diverse observations over the halation effect available by now.

The work was performed under Attachment #63 to the Agreement between DESY (Hamburg, Germany) and IHEP (Protvino, Russia), re: Coasting Beam in HERA-\textit{p} Ring, June 2000.

\textit{Deutsches Elektronen-Synchrotron DESY, Hamburg, Germany}
# Contents

1 Introduction ................................................................................................................................. 1

2 Motion under a Single-RF System ............................................................................................... 3
   2.1 Potential Well and Phase-Space Plane .................................................................................. 3
   2.2 Effects of Synchrotron Radiation on Beam Dynamics .......................................................... 5
      2.2.1 Energy Losses and Drift along $J$ beyond RF Buckets ................................................... 5
      2.2.2 Radiation Damping ......................................................................................................... 6
      2.2.3 Quantum Excitation ....................................................................................................... 6
   2.3 Conclusion ............................................................................................................................... 7

3 Diffusion Equation, Generalities ................................................................................................. 8
   3.1 Distribution Function ............................................................................................................. 8
   3.2 Diffusion Equation .................................................................................................................. 8
   3.3 2D Image of Target Wire ....................................................................................................... 9
      3.3.1 Branch $A_i \leq |x|$ ........................................................................................................... 9
      3.3.2 Branch $A_i > |x|$ ........................................................................................................ 9
   3.4 Initial and Boundary-Value Conditions ................................................................................... 10
   3.5 An Illustration to Statement of the Problem ......................................................................... 11
   3.6 Diffusion versus Drift: a Rough Estimate .......................................................................... 12
   3.7 The Suspected Noise Source ............................................................................................... 13
   3.8 Conclusion ............................................................................................................................... 14

4 Diffusion under External Noise .................................................................................................. 15
   4.1 Generalities ............................................................................................................................. 15
   4.2 Diffusion Coefficient inside RF Buckets ............................................................................. 15
   4.3 Diffusion Coefficient beyond RF Buckets ......................................................................... 16
   4.4 Numerical Solution of a Diffusion Equation ....................................................................... 17
   4.5 Conclusion ............................................................................................................................... 19

5 Low-Pass Filtered Noise Source ............................................................................................... 20
   5.1 Input Noise Source ................................................................................................................ 20
   5.2 Digital Filtering and Output Noise ....................................................................................... 20
      5.2.1 A Low-Pass Filter $F(\omega)$ ......................................................................................... 21
      5.2.2 A High-Pass Filter $T(\omega)$ ......................................................................................... 23
   5.3 Conclusion ............................................................................................................................... 25

6 Longitudinal Tracking Code ....................................................................................................... 26
   6.1 General Description ............................................................................................................... 26
      6.1.1 Tracking Procedure ....................................................................................................... 26
      6.1.2 Initial Distribution ......................................................................................................... 27
      6.1.3 Accumulation of Dimensionless Times and Noise-to-Drift Ratio .................................. 28
   6.2 Tracking versus a Diffusion Equation .................................................................................... 29
      6.2.1 Case #1, without Drift in a Halo, Phase Noise ............................................................... 31
      6.2.2 Case #1, without Drift in a Halo, Amplitude Noise ....................................................... 32
      6.2.3 Case #2, Inward Drifts in a Halo, Phase Noise ............................................................... 32
      6.2.4 Case #3, Outward Drifts in a Halo, Phase Noise ........................................................... 33
   6.3 Conclusion ............................................................................................................................... 34

7 Beam Halo Effect, an Example of ............................................................................................. 35
   7.1 Case #4, Inward/Outward Drifts in a Halo, Phase Noise ....................................................... 35
   7.2 Conclusion ............................................................................................................................... 36

8 Beam Halo Effect in the HERA-$p$ Ring .................................................................................... 38
8.1 Phase Noise .................................................................................................................... 38
  8.1.1 Zero Boundary-Value at Separatrix................................................................. 38
  8.1.2 Zero Boundary-Value at Aperture, Diffusion Equation & Tracking............. 41
8.2 Amplitude Noise......................................................................................................... 43
  8.2.1 Zero Boundary-Value at Separatrix................................................................. 43
8.3 Elementary Theory of Halo Accumulation: the Drift-Dominated Case .............. 44
8.4 Conclusion................................................................................................................. 45

9 Comparison with Beam Observations................................................................. 46
  9.1 Sustained Length of a Bunch.................................................................................. 46
  9.2 Allan’s Variance of the 208 MHz Radio-Frequency Gap Voltage ................. 48
  9.3 Motion under a Double-RF System................................................................. 50
    9.3.1 Tracking Algorithm....................................................................................... 50
    9.3.2 Decay of the Bunched Core......................................................................... 51
    9.3.3 Accumulation of Coasting Beam Halo....................................................... 52
    9.3.4 Effect of the Two RF systems: the Conclusion.......................................... 53
  9.4 Conclusion........................................................................................................... 53

10 Experimental Verification and Cures (Proposals)........................................... 54
  10.1 Experimental Tests ............................................................................................. 54
  10.2 A New Diagnostic Tool...................................................................................... 54
  10.3 Gap Voltage Measurements............................................................................ 54
  10.4 Beam Measurements....................................................................................... 55
  10.5 Closing Beam Feedback Circuit.................................................................... 55
  10.6 Longitudinal Cleaning..................................................................................... 56

11 Summary.................................................................................................................. 57

12 Acknowledgements ............................................................................................... 57

13 References.............................................................................................................. 58
1 Introduction

During a 10 hr long operation of the HERA-p ring, about (1–2)% of the protons are lost out of RF buckets and populate a coasting-beam halo. On interacting with internal targets of the HERA–B experiment, these protons contribute to background readouts that are not correlated with bunch crossing signals acquired by the HERA–B trigger system. To this end, there is a demand in studies of production/transport mechanisms resulting in and of cures against the coasting beam halo observed at the HERA-p ring.

Basic parameters of the HERA proton ring relevant to the problem in question are specified by [1] and listed in Table 1.

<table>
<thead>
<tr>
<th>Table 1: The HERA-p ring, list of parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top momentum (protons)</td>
</tr>
<tr>
<td>DC current of beam</td>
</tr>
<tr>
<td>Orbit length</td>
</tr>
<tr>
<td>Bending radius of orbit curvature in dipoles</td>
</tr>
<tr>
<td>Transition energy, $\gamma$</td>
</tr>
<tr>
<td>Harmonic number of 52 MHz RF system</td>
</tr>
<tr>
<td>Harmonic number of 208 MHz RF system</td>
</tr>
<tr>
<td>Top number allowed of equidistant bunches</td>
</tr>
<tr>
<td>Number of bunches in a luminosity run $^*$</td>
</tr>
<tr>
<td>Design peak voltage of 52 MHz RF system</td>
</tr>
<tr>
<td>Design peak voltage of 208 MHz RF system</td>
</tr>
<tr>
<td>Operating voltage of 52 MHz RF system at flat top</td>
</tr>
<tr>
<td>Operating voltage of 208 MHz RF system at flat top</td>
</tr>
<tr>
<td>Under the standard accelerating voltage:</td>
</tr>
<tr>
<td>momentum spread of a bucket</td>
</tr>
<tr>
<td>momentum spread of a bunch @ 1$\sigma$</td>
</tr>
<tr>
<td>half-length of a bunch @ 1$\sigma$</td>
</tr>
<tr>
<td>Betatron tunes:</td>
</tr>
<tr>
<td>vertical 32.298</td>
</tr>
<tr>
<td>Transverse emittance @ 1$\sigma$, normalised $^{**}$</td>
</tr>
<tr>
<td>Average value of $\beta$-function, either direction</td>
</tr>
<tr>
<td>in focusing quads</td>
</tr>
<tr>
<td>in dipoles</td>
</tr>
<tr>
<td>Optical functions @ HERA–B target:</td>
</tr>
<tr>
<td>horizontal dispersion function</td>
</tr>
</tbody>
</table>

$^*$) Filling pattern of RF buckets is 3 fills with 6 $\times$ 10 bunch trains + a kicker gap.

$^{**}$) Emittance is a product of semi-axes of a phase ellipse.

Phenomenological picture of the coasting-beam halo effect is summarised in references [2], [8]. The significant features of the coasting-beam halo to be accounted for are:

1. About a 5-to-1 asymmetry between, respectively, outer and inner target wires with respect to readout rate of interactions non-correlated with bunch crossing signals.
2. Scale observed of a transport rate into half-plane of lower momentum constitutes around $10^{-3}$ in fractional momentum off-set per 40 min of beam circulation.

3. A typical lifetime of bunched-beam core itself in a HERA–B run exceeds 100 hr (exponential, in a transverse direction), and about 10 hr (initial instantaneous bunch length doubling, in longitudinal direction).

This report contains the study of longitudinal stochastic mechanisms of the halo production and transport (noises in RF system, ripple in dipole power supply, etc.). Noise-driven longitudinal diffusion is treated as a dominant effect to originate the coasting beam halo. Still, the important role is shown to be played by a drift transport mechanism beyond RF buckets due to energy losses via synchrotron radiation of the high-energy protons.

Effects of single-particle dynamics in a single-harmonic accelerating field are elaborated in more detail. They are expected account for all the essential features of the beam halation effect in the HERA-$p$ ring in a plain and transparent way. It allows for a straightforward comparison between analytical calculations and tracking — calculations via Fokker-Planck (diffusion) equation are confirmed and verified with a longitudinal multi-turn Monte-Carlo tracking of macro-particles.

Effects of beam dynamics in a double-harmonic accelerating field are addressed to in the aftermath of the single-RF studies with the tracking code.

Levels of noise power densities and shapes of their spectra which are consistent with the diverse halo observations available are determined.
2 Motion under a Single-RF System

On the one hand, it is clear intuitively that all essential features of the beam halation effect in the HERA-\(p\) ring must reveal themselves in case of a circulating beam governed by a single-RF system.

On the other hand, beam dynamics in a single-harmonic potential well is conveniently parameterised in terms of elliptic functions. This knowledge is to save a lot of efforts both in tracking procedures and in solving the diffusion equation. Consequently, more attention could be devoted to better understanding the physics of the problem.

To this end, the single-RF assumption is followed throughout major bulk of the report. Effects of employing a double-harmonic accelerating field are addressed in Section 9.3 with the tracking code that is first thoroughly tested in the single-RF case.

2.1 Potential Well and Phase-Space Plane

Let phase \(\phi \propto h(\Theta - \omega_0 t)\) in RF radians be the longitudinal co-ordinate. Here, \(h = 4400\) is RF harmonic number (208 MHz), \(\Theta\) is generalised azimuth of a rotating proton, \(\omega_0 = 2\pi \times 47.317\) kHz is angular rotation frequency, \(t\) is time. A reference particle has \(\phi = 0\) (mod 2\(\pi\)). Conjugated momentum is taken as \(\xi = d\phi/dt\), and variables \((\phi, \xi)\) constitute the longitudinal phase-space plane.

Assume no energy loss or gain. Then, stable phase angle is \(\phi_s = 0^\circ\), and the potential well of longitudinal motion is

\[
U(\phi) = \Omega_0^2 \cdot (1 - \cos \phi) \to \Omega_0^2 \cdot \phi^2/2 \quad @ \quad \phi \to 0.
\]

Here, \(\Omega_0 \equiv 2\pi \times 40\) Hz is a circular frequency of small-amplitude longitudinal oscillations,

\[
\Omega_0^2 = \frac{\omega_0^2 h V_{RF} \cos \phi_s}{2\pi \beta^2 \gamma E_0 / e}
\]

where \(V_{RF}\) is peak RF voltage of around 0.6–0.7 MV at beam, \(\eta = \alpha - 1/\gamma^2\) is frequency slippage factor, \(\alpha = 1.285 \times 10^{-3}\) is momentum compaction factor, \(E_0 = 938.3\) MeV and \(e = 1.602 \times 10^{-19}\) Coul are rest energy and charge of a proton, \(\beta\) and \(\gamma\) are Lorenz factors.

Hamiltonian of the longitudinal motion is

\[
H(\phi, \xi) = U(\phi) + \xi^2/2,
\]

and \(E = H(\phi, \xi)\) is energy of oscillations along the phase-plane trajectory drawn through the point \((\phi, \xi)\) of the phase-space plane.

Let \((\psi, J)\) denote angle and action variables conjugated to \((\phi, \xi)\). By definition,

\[
J(E) = \frac{1}{\pi} \int_{\phi_1(E)}^{\phi_2(E)} \frac{1}{\sqrt{2(E - U(\phi))}} d\phi
\]

where \(\phi_2(E) > \phi_1(E)\) are co-ordinates of return points of the phase-plane trajectory \(H(\phi, \xi) = E\), \(\phi_{2,1}\) being equal to \(\pm \pi\) beyond RF buckets. Function \(J(E)\) is assumed to continue smoothly beyond the RF buckets where it represents the non-trapped, orbital motion.

Energy value \(E = 2\Omega_0^2\) corresponds to RF separatrix whose action variable is denoted with

\[
J_s = 8\Omega_0/\pi,
\]

and is used as a natural unit to measure the distance along \(J\)-axis.

The key role in description of the motion is played by complete elliptic integrals of the 1\(^{st}\) and 2\(^{nd}\) kinds, respectively,
\[ K(k) = \frac{\pi^2}{2} \frac{1}{\sqrt{1-k^2 \sin^2 y}} dy, \quad E(k) = \frac{\pi^2}{2} \sqrt{1-k^2 \sin^2 y} dy. \]  

Table 2 lists important dynamical functions as parametric dependencies versus parameter \( k \) of (6). There, \( A_\phi \leq \pi \) denotes amplitude of oscillations along \( \phi \), frequency \( \Omega_s(k) \) of non-linear oscillations complies with its canonical definition \( \Omega_s = \partial E / \partial J \) where \( E = E(J) \) is the function inverse to (4).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Trapped motion:</th>
<th>Orbital motion:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>( k = \sin(A_\phi/2) ), 0 \leq k \leq 1</td>
<td>1 \leq k &lt; \infty</td>
</tr>
<tr>
<td>Frequency</td>
<td>( E(k) = \Omega_0^2 \cdot 2k^2 )</td>
<td>( E(k) = \Omega_0^2 \cdot 2k^2 )</td>
</tr>
<tr>
<td>Action</td>
<td>( \Omega_s(k) = \Omega_0 \cdot \frac{\pi}{2K(k)} )</td>
<td>( \Omega_s(k) = \Omega_0 \cdot \frac{\pi k}{2K(1/k)} )</td>
</tr>
<tr>
<td>Action @ separatrix</td>
<td>( J(k) = J_S \cdot (E(k) - (1-k^2) \cdot K(k)) )</td>
<td>( J(k) = J_S \cdot k E(1/k) )</td>
</tr>
</tbody>
</table>

The HERA-\( p \) ring has a positive frequency slippage factor \( \eta \) and negative horizontal dispersion \( D_x \) at the HERA–B target location. To this end, in case of vanishing betatron amplitudes, the upper/lower half-plane of \((\phi, \xi)\) would have hosted, respectively, the outer/inner horizontal beam-halo tail at target.

Schematic view of the longitudinal phase-space plane is shown in Fig. 1. Particles rotate in the clock-wise direction. Phase-plane trajectory with \( J = J_S/2 \) sketches bunch size (@ base of distribution) at start of a typical HERA-\( p \) run. Trajectories with \( J = 2J_S \) are, roughly, a boundary where behaviour of an off-set particle looses most of specifics peculiar to bunched motion (refer to Fig. 2 and Fig. 9).

Fig. 2 shows the plot of synchrotron frequency \( \Omega_s(J) \). It merges with the coasting beam asymptote at about \( J > 2J_S \). Inverse function \( J(\Omega_s) \) is double-valued for \( 0 \leq \Omega_s \leq \Omega_0 \). In this range of arguments, it has two branches running within \( J_S \geq J \geq 0 \) (inside RF bucket) and \( J_S \leq J \leq 1.62J_S \) (beyond RF bucket). To this end, any technique dealing with a resonant excitation of the bunch core, say, in a dipole side-band of incoherent frequencies would inevitably
affect halo particles in a certain outer vicinity of separatrix. In other words, it is not possible to suppress noise-driven dilution of the bunch core without retarding the diffusive transport of particles just beyond the RF bucket.

![Fig. 2: Frequency of synchrotron oscillations.](image1)

![Fig. 3: Normalised momentum off-set.](image2)

Fig. 3 shows a plot of extremes in oscillations of the conjugated momentum ($|\xi|_m \leq |\xi| \leq |\xi|_M$) versus $J$. At separatrix, $|\xi|_M = 2\Omega_0$ and the fractional momentum spread is

$$\left|\frac{\Delta p}{p_0}\right| = 2 \frac{\Omega_0}{\hbar \omega_n} \leq 3 \cdot 10^{-4}$$

where $\Delta p = p - p_0$ is off-set in momentum $p$ about its reference value $p_0$. The upper curve of Fig. 3 has a zero slope at point $J = J_S$.

### 2.2 Effects of Synchrotron Radiation on Beam Dynamics

In total, there are three effects of synchrotron radiation (SR) that are discussed below in a decreasing order of importance for the HERA-\(p\) ring.

#### 2.2.1 Energy Losses and Drift along $J$ beyond RF Buckets

Energy loss per turn of a proton due to synchrotron radiation in bending dipoles is

$$eV_{SR} = \frac{2}{3} r_0 E_0 \gamma^4 \cdot I_2,$$

where $r_0 = 1.535 \cdot 10^{-18}$ m is classical proton radius, $I_2$ is the 2-nd radiation integral, $\rho = 584.2$ m is bending radius of orbit curvature in dipoles. These losses amount to $eV_{SR} = 9.54(6.02)$ eV at beam energies $\gamma E_0 = 920(820)$ GeV, respectively.

Accelerating system drives peak RF voltage $V_{RF}$ of about 0.6–0.7 MV. Stable phase angle $\phi_s$ is very close to $0^\circ$, namely,

$$\sin \phi_s = \frac{V_{SR}}{V_{RF}} = \frac{1}{q} \ll 1$$

where $q$ is a huge over-voltage factor with a magnitude of about $10^5$.

Potential well (1) of longitudinal oscillations now acquires a small stationary additive term

$$\delta U(\phi) = -\Omega_0^2 \cdot \tan \phi_s \cdot (\phi - \sin \phi).$$

Dynamical consequences of adding such a $\delta U(\phi)$ can be estimated with a perturbation theory, up to the first order in $\delta U(\phi) \propto 1/q$. 

5
For a proton trapped into RF bucket energy losses (8) are fully recovered by accelerating system. Due to the cyclic way of motion, there would be no systematic change in action variable \( J \) in a time scale \( t \gg T_s \). Thus, on average,

\[
\frac{1}{T_s(J)} \int_t^{t+T_s(J)} \frac{dl}{dt} dt = -\frac{1}{2\pi} (\delta U (\phi(J, \psi + 2\pi)) - \delta U (\phi(J, \psi))) = 0.
\]

Here, \( T_s(J) = 2\pi/\Omega_s(J) \) is a period of non-linear oscillations and \( d\psi = \Omega_s(J) dt \).

On the contrary, due to the non-vanishing value of (9), protons that constitute beam halo beyond RF buckets are slowly crabbed towards lower values of \( \delta p \) where \( \delta p = p - p_0 \) is off-set in momentum \( p \) about its reference value \( p_0 \). The relevant velocity \( V(J) \) of the systematic drift along the longitudinal action co-ordinate \( J \) can be found as a time average

\[
V(J > J_s) = \frac{1}{\Delta t(J)} \int_t^{t+\Delta t(J)} \frac{dl}{dt} dt = -\frac{1}{2\pi} (\delta U (\phi(J, \psi + \pi)) - \delta U (\phi(J, \psi))) \neq 0
\]

where \( \Delta t(J) \) is the time interval required for a halo proton to cover a \( 2\pi \) distance along \( \phi \), either downstream (for \( \xi > 0 \)) or upstream (\( \xi < 0 \)) of beam motion.

Thus, the resultant drift velocity is step-wise constant with \( J \) and can be put as

\[
V(J) = \begin{cases} 0, & J \leq J_s; \\ \text{sgn} \xi V, & J > J_s; \end{cases}
\]

with

\[
V = 2\Omega_0^2 \cdot \tan \varphi_s = \frac{\pi}{4} \Omega_0 J_s \cdot \tan \varphi_s > 0.
\]

Here, \( \tan \varphi_s \) can be substituted by \( V_{SR}/V_{RF} \ll 1 \), and \( \text{sgn} \xi = -\text{sgn} \delta p \) above transition.

The drift velocity \( V \) is 9.5(6.3) \( J_s/\text{hr} \) at beam energies \( \gamma E_0 = 920(820) \) GeV, respectively. Momentum acceptance of the machine of around \( \pm 1 \cdot 10^{-3} \) in fractional momentum off-set corresponds to an aperture limitation at \( J_A = (4.5-5.1)J_s \). Hence, the energy loss due to the SR is a well noticeable effect in a 10 hr time scale of a typical HERA-\( p \) run.

### 2.2.2 Radiation Damping

Exponential damping constant of longitudinal oscillations (\( \propto \exp(-\Gamma t) \)) is

\[
\Gamma = \frac{2\Omega_0}{4\pi \gamma E_0/e} P_s, \quad P_s = 2 + d
\]

where \( P_s \) is a longitudinal partition number, \( d \) is a damping parameter. For isomagnetic and separated-function lattice of the HERA-\( p \) ring,

\[
d = \alpha \frac{\Pi}{2\pi \rho} = 2.22 \cdot 10^{-3}
\]

where \( \Pi = 6335.82 \) m is the orbit length.

Damping time of longitudinal oscillations is \( 1/\Gamma = 23.6(33.3) \) day at beam energies \( \gamma E_0 = 920(820) \) GeV, respectively. Hence, radiation damping is not an effect of concern in the time scale involved.

### 2.2.3 Quantum Excitation

In the HERA-\( p \) ring, natural r.m.s. fractional momentum spread of a bunch under effect of synchrotron radiation might have established itself at

\[
\left( \frac{\sigma_{p_0}}{p_0} \right)_{SR} = \sqrt{\frac{C_q}{2} \frac{l_3}{12 P_s}}, \quad l_3 = \frac{2\pi}{P_s}\rho^2.
\]

Here, \( l_3 \) is the 3-nd radiation integral, and
\[ C_q = \frac{55}{32\sqrt{3}} \frac{\hbar c}{2\pi E_0} = 2.087 \cdot 10^{-16} \text{m} \]

where \( c = 0.2998 \cdot 10^9 \text{ m/sec} \) is velocity of light, \( \hbar = 6.626 \cdot 10^{-34} \text{ J-sec} \) is Planck’s constant.

Ratio of \( (\sigma/p_0)_{SR} \) to inherent momentum spread of a proton bunch (at 1\( \sigma \)) is around \( 3\cdot10^{-4} \). Therefore, effect of quantum excitation can be ignored, all the more that it could have settled itself at this negligible level only in a time scale of \( t > 1/\Gamma \).

### 2.3 Conclusion

Circulation of a bunched proton beam under a single-RF system and a stable phase angle \( \varphi_s = 0^\circ \) is well understood. It can be treated in terms of elliptic functions. Loss of energy due to SR is a non-negligible effect for the HERA-\( p \) ring. Still, rate of this loss is too small to justify involvement into study of dynamics along unbounded phase-plane trajectories peculiar to a conventional accelerating regime with \( \varphi_s \neq 0^\circ \). More appropriate to employ a perturbation theory, up to the first order in tan\( \varphi_s \neq 0 \). From this view-point, energy loss is reducible, on average, to a slow drift along longitudinal action co-ordinate \( J \) beyond RF buckets, upwards in a phase plane (\( \varphi, \xi = d\varphi/dt \)) above transition.
3 Diffusion Equation, Generalities

In this Section, a few topics related to general statement of the diffusion problem at issue are discussed, prior to specification of the diffusion coefficient $D(J)$ imposed by external noises, see Section 4.

3.1 Distribution Function

A bunch is described by a 2D distribution function $f(J, A_x, t)$ where $J$ is longitudinal action variable, $A_x > 0$ is horizontal betatron amplitude, $t$ is time. This function is normalised to 1 at the initial instant $t = 0$,

$$\int \int f(J, A_x, t) \, dJ \, dA_x = 1. \tag{19}$$

For $t > 0$, integral (19) would eventually acquire values $< 1$ which is due to loss of protons on a target or an aperture.

To simplify the matters, let us ignore any dilution of a bunch in the betatron phase space $(x, x')$. This assumption is supported by observation [1] that, normally, transverse lifetime of the HERA-$p$ beam exceeds 100 hr, while longitudinal degrade of beam core occurs in about 10 hr. Under such an assumption, $f(J, A_x, t)$ can be factored out as

$$f(J, A_x, t) = f(J,t|A_x) \cdot f(A_x). \tag{20}$$

Assume a Gaussian bunch in $(x, x')$. Distribution over betatron amplitudes $f(A_x)$ is then the Rayleigh function

$$f(A_x) = \frac{A_x}{\sigma_x^2} \exp \left( -\frac{A_x^2}{2\sigma_x^2} \right), \quad \int f(A_x) \, dA_x = 1. \tag{21}$$

Here, $\sigma_x$ is r.m.s. horizontal half-size of the bunch

$$\sigma_x = \sqrt{\frac{\beta_x \varepsilon_x}{\beta \gamma}} \tag{22}$$

where $\beta_x$ is horizontal $\beta$-function, $\varepsilon_x$ is normalised horizontal emittance at 1$\sigma$ level, $\beta$ and $\gamma$ are Lorenz factors.

(Since there is also a residual vertical dispersion at the HERA–B target, vertical oscillations can be treated in a similar way with a replacement of $x$ by the vertical co-ordinate $z$.)

3.2 Diffusion Equation

Longitudinal distribution function $f(J, t|A_x)$ — a slice of (20) for the amplitude $A_x$ — is governed by a diffusion equation that reads

$$\frac{\partial f(J,t|A_x)}{\partial t} = -\frac{\partial}{\partial J} \left( V(J) \cdot f(J,t|A_x) \right) + \frac{\partial}{\partial J} \left( D(J) \frac{\partial f(J,t|A_x)}{\partial J} \right). \tag{23}$$

It is nothing but a continuity equation

$$\frac{\partial f(J,t|A_x)}{\partial t} = -\frac{\partial}{\partial J} \left( Q(J,t|A_x) \right) \tag{24}$$

in terms of the sum of drift and diffusive fluxes of particles along $J$

$$Q(J,t|A_x) = V(J) \cdot f(J,t|A_x) - D(J) \frac{\partial f(J,t|A_x)}{\partial J}. \tag{25}$$

Drift velocity $V(J)$ is given by (13). Diffusion coefficient $D(J)$ might be built up due to amplitude or phase (frequency) noises in RF system and noisy ripple of bending field $B$ in dipoles. The explicit expressions for $D(J)$ are given in Section 4.
3.3 2D Image of Target Wire

The HERA–B target wire positioned at a horizontal co-ordinate \( x_t \) — domain of the 2D distribution at issue — by a curve \( J = J_w(A_x, |x_t|) \). It represents the locus of points where the particles are lost (the scraping or absorbing “Wall”). A hard-edge model of the target is assumed for simplicity: no survival is presumed for particles whose peak excursion from the on-momentum closed orbit at the target location ever exceeds \( x_t \).

The particles with either sign of their momentum off-set \( \delta p \) are treated in terms of the action variable \( J \) that does not respect the sign of \( \delta p \). Therefore, the curve \( J = J_w(A_x, |x_t|) \) has two branches, one per each half-plane of \((\phi, \delta p)\).

3.3.1 Branch \( A_x \leq |x_t| \)

This, descending branch of function \( J_w(A_x, |x_t|) \) is related to the bunched core itself \((J \leq J_S)\), and to coasting beam halo \((J > J_S)\) from the “proper” half-plane of \((\phi, \delta p)\) whose

\[
\text{sgn } \delta p = \text{sgn}(D_x \cdot x_t) \tag{26}
\]

and where diffusion along \( J \) drags protons towards the target. Here, \( J_w = J_w(A_x, |x_t|) \) is introduced formally as the function yielding the value of action variable \( J \) at which

\[
\max \left| \frac{\delta p}{p_0} \right| = \frac{|x_t| - A_x}{|D_x|} \tag{27}
\]

where \( D_x \) is the value of horizontal dispersion function at the HERA–B wire target.

In a single-RF (208 MHz) system, \( J_w \) can be calculated as

\[
J_w = J_S \cdot \left[ \frac{\xi}{\Omega} \right]^{-1}_{0|M} \left\{ 2 \frac{|x_t| - A_x}{|D_x|} \left| \frac{\Delta p}{p_0} \right|^{-1}_{S} \right\} \tag{28}
\]

where \( \left[ \frac{\xi}{\Omega} \right]^{-1}_{0|M}(\ldots) \) is the inverse function to \( \left[ \frac{\xi}{\Omega} \right]_{0|M}(J/J_s) \) plotted in Fig. 3, the upper curve. Plot of function (28) is shown in Fig. 4 by the curve ABD, in arbitrary units. Segment AB stands for the bunched core region.

3.3.2 Branch \( A_x > |x_t| \)

This, ascending branch of function \( J_w(A_x, |x_t|) \) accounts for cross-talk in between the two halo regions at the wire target when away-from-the-target diffusion is overridden by larger betatron amplitudes. It is related to the coasting beam halo \((J > J_S)\) from the “improper” half-plane of \((\phi, \delta p)\) whose

\[
\text{sgn } \delta p = -\text{sgn}(D_x \cdot x_t). \tag{29}
\]

Now, \( J_w = J_w(A_x, |x_t|) \) is introduced as the function yielding the value of action variable \( J \) at which

\[
\min \left| \frac{\delta p}{p_0} \right| = \frac{A_x - |x_t|}{|D_x|} \tag{30}
\]

In a single-RF (208 MHz) system, \( J_w \) can be calculated as

\[
J_w = J_S \cdot \left[ \frac{\xi}{\Omega} \right]^{-1}_{0|M} \left\{ 2 \frac{A_x - |x_t|}{|D_x|} \left| \frac{\Delta p}{p_0} \right|^{-1}_{S} \right\} \tag{31}
\]

where \( \left[ \frac{\xi}{\Omega} \right]^{-1}_{0|M}(\ldots) \) is the inverse function to \( \left[ \frac{\xi}{\Omega} \right]_{0|M}(J/J_s) \) plotted in Fig. 3, the lower curve. Plot of (31) is shown in Fig. 4 by the curve EF. Beam scraping in the entire half-plane (29), bunched core included, would proceed along the saw-toothed curve ABEF.
3.4 Initial and Boundary-Value Conditions

Initial longitudinal profile of a bunch is conventionally assumed to be Gaussian in variables $\varphi$, $\xi \propto \delta \rho$. In this case the initial condition for (23) reads

$$f(J, t = 0 | A_x) = \begin{cases} \frac{1}{2J_\sigma} \exp\left(-\frac{J}{2J_\sigma}\right) & J < J_w(A_x, |x|); \\ 0, & J \geq J_w(A_x, |x|); \end{cases}$$

(32)

where $J_\sigma$ stands for action variable of the phase-plane trajectory with $1\sigma$ oscillation amplitudes. This function is normalised to 1 when $J_w >> J_\sigma$ (i.e. up to scraping effect of the wire).

Boundary-value condition at the axis $J = 0$ is simply

$$Q(J = 0, t | A_x) = 0$$

(33)

since there is no influx of particles in the bunch centre irrespective of the betatron amplitude $A_x$.

There are two ways to impose the second boundary-value condition.

1. Assume a disabled wire position control. In this case of a fixed wire, the boundary-value condition in question is

$$f(J = J_w, t | A_x) = 0.$$  

(34)

Flux of particles hitting the absorbing “wall” is produced by a natural beam diffusion and amounts to

$$Q_1(t) = \int Q(J_w(A_x), t | A_x) \cdot f(A_x) \, dA_x$$

(35)

where $Q(J, t | A_x)$ is given by (25).

2. In case of the wire position feedback switched on, the target rather behaves as a scraper and leaves no time for beam distribution to settle to comply with (34). The target would shave the distribution established earlier under the condition

$$f(J = J_A, t | A_x) = 0$$

(36)

where $J_A$ is aperture limit of the ring. Scraping rate by the wire moving all way towards the beam centre can be estimated as

$$Q_2(t) = -\frac{d|x|}{dt} \int f(J_w(A_x, |x|), t | A_x) \cdot \frac{dj_w(A_x, |x|)}{d|x|} \cdot f(A_x) \, dA_x.$$  

(37)
Given boundary-values (33), (34)/(36) and initial conditions (32), diffusion equation (23) becomes a correctly formulated problem.

Net interaction rate at the HERA–B target is proportional to (35) or (37)

\[
\frac{dN}{dt} = \varepsilon_T \cdot N_0 \cdot Q_{12}(t)
\]

where \(N_0\) is initial beam population (\(10^{13}\) p.p.p. per 76 mA of beam DC current), \(\varepsilon_T\) is target efficiency (\(\geq 50\%\), according to [3]).

According to [4], in a conventional operation of the HERA–B target, the major bulk of readouts is generated due to the scraping effect, (37). Normally, \(Q_2\) exceeds \(Q_1\) by more than an order of magnitude. To this end, of practical interest are the studies of bunch longitudinal profiles \(f(J, A_x)\) rather than that of fluxes \(Q(J, t|A_x)\) through the boundary.

3.5 An Illustration to Statement of the Problem

To illustrate formal statement of the diffusion problem, Fig. 5 sketches a surface plot of distribution \(f(J, A_x, t)\) (20) at \(t = 0\). Partial distributions are given by (21) and (32) with their parameters specified in Table 3.

**Table 3: Flat-top parameters at start of the HERA-p run**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal (\beta)-function at HERA–B target, (\beta_x)</td>
<td>37 m</td>
</tr>
<tr>
<td>Normalised transverse emittance at 1(\sigma) level, (\varepsilon_x)</td>
<td>4 mm-mrad</td>
</tr>
<tr>
<td>Horizontal dispersion function, (D_x)</td>
<td>-0.5 m</td>
</tr>
<tr>
<td>Initial r.m.s. half-length of a bunch, (\sigma_s)</td>
<td>13 cm</td>
</tr>
</tbody>
</table>

**Fig. 5: Surface plot of the 2D distribution \(f(J, A_x, t = 0)\).**

**Fig. 6: Plan view of the plane \((A_x, J)\).**

Fig. 6 offers a top plan view over the plane \((A_x, J)\) of Fig. 5. For convenience, 1D projections of \(f(J, A_x, t)\) are shown folded onto the plane. Footprints of absorbing “walls” are plotted by solid lines in compliance with realistic target positions at 5.0\(\sigma_x\), 4.5\(\sigma_x\), and 4.0\(\sigma_x\), respectively. A tentative position of collimator/aperture shadow at target is set to 4.1\(\sigma_x\) for definiteness. This shadow spans parallel to \(J\)-axis since there are no momentum collimators in the HERA-p for the time being. Particles diffuse upward along \(J\)-axis and face either head (\(\delta p > 0\)) or tail (\(\delta p < 0\)) “wind” (13), (14) in the half-plane \(J \geq J_S\).
Composition of (37) by a number of particles coming from the bunched core or the coasting halo regions depends upon the target position. It ranges from bunch-crossing non-correlated readouts in case of target position #1 to a strong dominance of bunch-crossing correlated contributions in position #3 of Fig. 6. Outer target position #2 is a characteristic point at which a pure coasting beam halo regime of interaction turns into the bunched-beam dominated mode.

The asymmetry observed of coasting beam readouts between inner/outer wires is presumed to occur due to the two reasons:
1. a toggling drift component in flux (25) that affects density profiles in the lower/upper half-planes of \((\phi, \xi)\), and
2. characteristically different 2D images of the inner/outer wires in \((A_x, J)\)-domain of interest, Fig. 4 and Fig. 6.

3.6 Diffusion versus Drift: a Rough Estimate

Here, the preliminary estimates are accomplished so as to comply roughly with beam observations in the HERA-p ring.

Usually, a phase/frequency noise in RF system (or, may be, a noise in guide field \(B\) of dipole magnets) shows up as a dominant cause of beam longitudinal dilution. For these noises and flat power spectra use can be made of a linear approximation

\[
D(J) = D_1 \cdot J
\]

that holds in the centre of a bucket, at about \(J \leq J_s/2\).

On the other hand, explicit dependence of longitudinal distribution \(f(J, t|A_x)\) upon transverse amplitude \(A_x\) might enter via the boundary condition (34) only. Normally, it takes a long time for particles to arrive either at this boundary, or at the aperture limit \(J_A\) (36). Until that time, one can employ approximation \(J_W, J_A >> J_\sigma\) and consider a distribution \(f(J, t)\) flat versus parameter \(A_x\).

Average value of action variable

\[
\int_{0}^{\infty} \int J \cdot f(J, t) \, dJ
\]

can be interpreted as a properly normalised longitudinal emittance of the beam.

Insert \(V(J) = 0\) and (39) into equation (23) and multiply both its sides by \(J\). Integration by parts in the right hand side and use of definition (40) results in the well-known law of beam evolution

\[
\frac{d\overline{J}(t)}{dt} = D_1 \quad \text{and} \quad \overline{J}(t) = \overline{J}(0) + D_1 \cdot t
\]

that implies a conventional square-root lengthening of \(\sigma_\phi(t) \propto (C_1 + C_2 \cdot t)^{1/2}\).

The second of equations (41) allows to estimate initial emittance doubling time \(\tau_{J2}\) as

\[
\tau_{J2} = \frac{\overline{J}(0)}{D_1}.
\]

Specification [1] sets initial instantaneous bunch-length doubling time \(\tau_{\sigma2}\) to about 10 hr. From a direct proportionality of \(\overline{J}(t)\) to \(\sigma_\phi^2(t)\) it follows that

\[
\tau_{J2} = \frac{1}{2} \tau_{\sigma2} = 5 \text{ hr}.
\]

Take the initial distribution (32) with its \(\overline{J}(0) = 2J_\sigma\)

and
Recalling (42), one can now estimate diffusion flux inside bucket (at \( t = 0 \)) as

\[
Q(J, t) = D_J \cdot \frac{f(J, t)}{\tau_{j2}} = \frac{J}{\tau_{j2}} f(J, t)
\]

where ratio \( J/\tau_{j2} \) can be referred to as a local speed of diffusion \( V_{D}(J) \). On extrapolating its value up to separatrix, one gets \( V_{D}(J_S - 0) = 0.2 J_S/hr \) which sets a scale of diffusion rate consistent with bunched core life-time (43) observed.

On the other hand, according to Section 2.2.1, drift velocity encountered beyond RF bucket due to SR losses does not depend upon \( J \). Its magnitude is \( V = 9.5(6.3) \ J_S/hr \) at beam energies \( \gamma E_0 = 920(820) \) GeV, respectively. Therefore, the HERA-\( p \) ring represents a drift-dominated case with a diffusion-to-drift ratio \( V_D/V \cong 1/50-1/30 \). To accumulate the alleged coasting beam halo in the lower half-plane of \((\phi, \xi)\) one has to provide the diffusive transport well competitive against such a strong drift beyond RF buckets.

Both the demands — of a good life-time of bunched core and of a strong enough diffusion beyond RF bucket — can only be met by a noise with a power density \( P(\Omega) \) that

1. nearly vanishes for frequencies around \( 0-\Omega_0 \), and
2. rapidly increases beyond this range.

In other words, \( P(\Omega) \) with a DC notch is the most probable candidate to comply with the coasting beam halo observations in the HERA-\( p \) ring.

### 3.7 The Suspected Noise Source

Origin of a DC notch in \( P(\Omega) \) can be traced with a simple scheme shown in Fig. 7. It sketches entry points of phase and/or frequency errors in the RF phase circuit of the 208 MHz RF system of the HERA-\( p \) ring. Specification of these errors is listed in Table 4.

![Fig. 7: Schematic diagram of phase/frequency errors in the 208 MHz RF system.](image)

Table 4: A list of tentative phase/frequency errors

| \( \phi_1 \) | VCO noise |
| \( \phi_2 \) | Phase noise of phase FB circuitry |
| \( \phi_3 \) | Phase noise of RF power amplifier and cavity reduced to accelerating gap |
| \( \phi_4 \) | \( B \)-field ripple reduced to equivalent RF phase/frequency noise |
| \( \phi_5 \) | Resultant noise @ beam |

Closing the feedback loop yields the in-out transfer equation
\[ \varphi_5 = \frac{\varphi_1 - G \cdot (\varphi_2 - \varphi_1) + \varphi_3 + \varphi_4}{1 + G} \quad (47) \]

where \( G \) is a gain of the DC-coupled RF phase feedback. According to (47), VCO signal is used as a reference for cavity phase measurements, and thus

\[ \varphi_5 = \varphi_1 - \frac{G \cdot \varphi_2 + \varphi_3 + \varphi_4}{1 + G}. \quad (48) \]

Since \(|G| >> 1\), the only input signal that can yield an apparent DC notch at output \( \varphi_5 \) is \( \varphi_3 \) (shaded in Table 4). A high-quality VCO is expected to inject a very low noise \( \varphi_1 \). Error \( \varphi_2 \) of phase detection just reoccurs at gap with the opposite sign, without any pronounced suppression of DC. Power spectrum of \( B \)-field ripple equivalent error \( \varphi_3 \) is anticipated to be smooth and, may be, increasing with \( \Omega \to 0 \) because of (i) an inherent low-\( Q \) response of the magnet chain, and (ii) integration in conversion from \( B \)-field error \( \delta B \propto \delta \omega_{RF} \) to phase error \( \varphi_4 \) resulting in an extra factor \( 1/\Omega^2 \) in \( P(\Omega) \), refer to (54).

In this context, we suspect **phase noise of RF power amplifier and cavity reduced to accelerating gap** to be the major cause of beam dilution in the HERA-\( p \) ring. Consequently, in compliance with (48), noises from the family of

\[ \varphi_{\text{OUT}}(\Omega) \equiv \frac{\varphi_{\text{IN}}(\Omega)}{1 + G(\Omega)}, \quad P_{\text{OUT}}(\Omega) \equiv \frac{P_{\text{IN}}(\Omega)}{(1 + G(\Omega))^2} \quad (49) \]

are mostly involved into further studies.

### 3.8 Conclusion

Interaction rate at the HERA–B target depends upon a 2D particle density profile \( f(J, A_x, t) \). As a starting point, transverse component of this distribution is taken to be constant in \( t \). Evolution in \( t \) of distribution along longitudinal action \( J \) is assumed to be governed by a noise-driven diffusion/transport mechanism plus a systematic drift due to SR losses beyond RF buckets. The HERA-\( p \) ring is expected to represent a drift-dominated case at its top energies. In this context, for the diffusive transport to ever get a chance to compete against the drift (to build the alleged coasting beam halo in both the half-planes) while ensuring a good longitudinal life-time of the bunched core, a DC notch in an apparent noise power spectrum is likely to be involved.
4.1 Generalities

Introduce a stationary, in a wide sense, random voltage \( u(t) \) with the zero expectation value \( \langle u(t) \rangle = 0 \) and auto-correlation function \( \langle u(t+\tau)\cdot u(t) \rangle = \langle u^2(\tau) \rangle \). Here, angular brackets \( \langle ... \rangle \) denote ensemble averaging, and \( \langle u^2(-\tau) \rangle = \langle u^2(\tau) \rangle \). Use is made of a double-sided spectral power density

\[
P_u(\Omega) = \int_{-\infty}^{\infty} \langle u^2(\tau) \rangle \exp(i\Omega \tau) d\tau = 2\int_{0}^{\infty} \langle u^2(\tau) \rangle \cos \Omega \tau d\tau.
\]

Let \( P_u(\Omega) \) be a base-band function with a limited band-width \(< \omega_0 \) the angular rotation frequency.

Unperturbed motion proceeds under the stable phase angle \( \varphi_s = 0^\circ \). Consider main RF voltage at accelerating gap as a reference for I/Q decomposition of gap errors. Then, the quadrature voltage error is seen in the beam frame as

\[
u(t) \cdot \cos \varphi
\]

and stands for a phase noise with the \( \varphi \)-error signal \( \delta \varphi(t) \equiv u(t)/V_{RF} \). Accelerating frequency error \( \delta \omega_{RF}(t) \) is equivalent to \( \delta \varphi(t) = \int \delta \omega_{RF}(t') dt' \).

On the contrary, the in-phase voltage error

\[
u(t) \cdot \sin \varphi
\]

represents an amplitude noise with the \( a \)-error signal \( \delta V_{RF}(t) = u(t) \).

Error \( \delta B(t) \) in magnetic guide field \( B \) varies the angular rotation frequency \( \omega_0 \) of a reference particle. Externally, \( \delta B(t) \) has the same dynamical consequences as the RF frequency error \( \delta \omega_{RF}(t) \) with

\[
\delta \omega_{RF} = -\alpha \frac{\delta B}{B}
\]

where \( \alpha \) is momentum compaction factor. Therefore, noise in \( B \)-field can be treated in terms of an equivalent quadrature voltage error (51) with the power spectrum

\[
P_u(\Omega) = \frac{\omega_0^2}{\Omega^2} P_{\delta B}(\Omega) .
\]

4.2 Diffusion Coefficient inside RF Buckets

Diffusion coefficient \( D(J) \) depends upon I/Q polarisation (51), (52) of the perturbation. It is a sum over multipolar excitations of the bunch driven by noise through a set of incoherent side-bands of synchrotron frequency \( \Omega_s(J) \),

\[
D^{(\varphi,a)}(J) \|_{J \leq J_s} = \frac{1}{2} \left( \frac{\Omega_0^2}{V_{RF}} \right)^2 \sum_{j=-\infty}^{\infty} W_{m}^{(\varphi,a)}(J) \cdot P_u(m \Omega_s(J)),
\]

\[m^{(\varphi)} = 2j + 1, \quad m^{(a)} = 2j\]
where $j$ is an integer. In a symmetrical potential well (1), $\varphi$-noise drives odd multipoles $m$ while $a$-noise — only even $m$.

In a smooth approximation, noisy perturbation is seen as a plane wave of random accelerating field propagating around the orbit. On the other hand, the inherent dynamics of a bunch naturally proceeds in a periodic way that is described by a cyclic variable $\psi$.

To this end, a key component to calculate weight functions $W_m^{(\varphi,a)}(J)$ is a decomposition coefficient of a plane wave into Fourier series over multipoles,

$$l_{nk}^*(J) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(i \frac{k}{\hbar} \varphi(J,\psi) - i m \psi\right) d\psi.$$

(56)

In these terms,

$$W_m^{(\varphi,a)}(J) = m^2 l_{mk}(J)^2 \times \frac{1}{4} (1 \mp (-1)^m)^2.$$

(57)

The weights can be eventually calculated versus a parameter $0 \leq k \leq 1$ of (6),

$$W_m^{(\varphi,a)}(k) = \left(1 \mp (-1)^m\right)^2 \left(\frac{\pi}{2K(k)} \right)^4 \times \left\{ \frac{\cosh^{-2}\left(\frac{\pi m}{2K(k)} \sqrt{1-k^2}\right)}{\sinh^{-2}\left(\frac{\pi m}{2K(k)} \right)} \right\}.$$

(58)

and function $W_m^{(\varphi,a)}(J)$ is then recovered parametrically through $J = J(k)$ taken from Table 2.

Fig. 8 shows a plot of the weight functions in question. For a short bunch ($J \rightarrow 0$) the dominant excitations are driven either by a $\varphi$-noise at the first ($m = \pm 1$) harmonic of $\Omega_s(J)$ with $W_{\pm 1}^{(\varphi)}(J) \propto J$, or by an $a$-noise at the second ($m = \pm 2$) harmonic of $\Omega_s(J)$ with $W_{\pm 2}^{(a)}(J) \propto J^2$.

Fig. 8: Weight functions inside RF bucket.

4.3 Diffusion Coefficient beyond RF Buckets

In case of $\varphi_s = 0^\circ$, motion of an off-set particle beyond RF buckets is still bounded, but in a scale of the entire ring. It can be described in terms of the appropriate angle and action variables $(\psi_1, J_1)$, one turn around in beam frame corresponding to $\psi_1 \rightarrow \psi_1 + 2\pi$. From this viewpoint, orbital motion beyond RF buckets about the centre of the ring is physically equivalent to a trapped motion inside an RF bucket where it also proceeds along a closed phase trajectory about the stable fixed point.

Still, due to external RF voltage and random perturbations (51), (52) that are both periodic with $2\pi$ in $\varphi \propto \hbar \Theta$, orbital motion has a sub-period $2\pi/\hbar$ in $\psi_1$. Therefore, $(\psi_1, J_1)$ and the other parameters of orbital motion can be scaled down to the phase portrait of Fig. 1 so as
to ensure a continuous matching of dynamic parameters across the RF separatrix (the last column of Table 2). Motion along the orbital phase trajectories in Fig. 1 is assumed to obey a periodical boundary condition at $\varphi = \pm \pi$.

Diffusion coefficient $D(J)$ is again a sum over multipolar excitations of the motion in halo driven by noise through incoherent side-bands of “synchrotron” frequency $\Omega_s(J \geq J_S)$

$$D^{(\varphi,a)}(J)_{\varphi \geq J_S} = \frac{1}{2} \left( \frac{\Omega_s^2}{V_{RF}} \right)^2 \sum_{m=-\infty}^{\infty} W_m^{(\varphi,a)}(J) \cdot P_a(m\Omega_s(J)),$$

$$n_{(\varphi,a)} = 2j.$$

Contrary to (55), multipoles $m$ driven are now even for both $\varphi$- and $a$-noises. In a full analogy to (56) and (57),

$$I_{m}^{*}(J) = \frac{1}{\pi} \int_{0}^{\varphi} \exp \left( \frac{i k}{\hbar} \varphi(j, \psi) - i m \psi \right) d\psi$$

and

$$W_m^{(\varphi,a)}(J) = \frac{m^2}{4} \left| I_{m}^{*}(J) \pm I_{m-n}^{*}(J) \right|^2.$$ 

Weights can be calculated versus a parameter $1 \leq k < \infty$ of (6),

$$W_m^{(\varphi,a)}(k) = \left( 1 + (-1)^n \right)^2 \left( \frac{\pi m k}{2K(1/k)} \right)^4 \times \left( \frac{\cosh^2}{\sinh^2} \left( \frac{\pi m}{2K(1/k)} k \left( 1 - 1/k^2 \right) \right) \right) (\varphi)$$

with $J = J(k)$ taken from Table 2.

Fig. 9 shows a plot of functions (62). At about $J > 2J_S$, the motion proceeds as if it were in a coasting beam. In this case, modulation of velocities due to external RF voltage is too low for a halo particle to distinguish between I/Q polarisation of a random signal, (51) or (52). The surviving $2W_{\pm 2}^{(\varphi,a)}(J)$ merge at a constant value of 2, the coasting-beam limit.

![Fig. 9: Weight functions beyond RF bucket.](image)

### 4.4 Numerical Solution of a Diffusion Equation

Spatial interval $J = [0, J_W]$ or $J_A$ under study is broken by $L+1$ nodes $J_l (l = 0, 1, \ldots, L)$ into $L$ segments of length $\Delta J_l = J_l - J_{l-1} (l > 0)$. Boundary nodes are $J_0 = 0$ and $J_L = J_{WA}$. One of the inner nodes corresponds to separatrix $J_S$ exactly.

**Spatial discretisation** of equation (23) is accomplished via a Finite Element Technique. To this end, $f(J, t)$ is approximated by a piece-wise linear law
$$f(J, t) \equiv \sum_{l=0}^{L} f_l(t) \cdot \Phi_l(J)$$

where $\Phi_l(J)$ constitutes a set of piece-wise linear basic functions ($\Phi_l(J) = 1$ in the $l$-th node, and is 0 in all the rest nodes); $f_l(J, t) \equiv f_l(t)$.

Use of a weak formulation of Galerkin’s weighted residual technique yields a system of linear homogeneous equations

$$\hat{A} \frac{d}{dt} \bar{f}(t) + (\hat{\beta} + \hat{V}) \bar{f}(t) = 0$$

for an unknown column-vector

$$\bar{f}(t) = (f_0(t), f_1(t), \ldots, f_L(t))^T$$

with symmetrical three-diagonal matrices

$$\hat{A} = \begin{pmatrix}
A_{00} = \frac{\Delta J_0}{3} & A_{01} = \frac{\Delta J_0}{6} & & \\
A_{10} = \frac{\Delta J_1}{6} & A_{11} = \frac{\Delta J_1 + \Delta J_{1+1}}{3} & & \\
& \ddots & \ddots & \\
A_{L-1,0} = \frac{\Delta J_{L-1}}{6} & A_{L-1,1} = \frac{\Delta J_{L-1} + \Delta J_L}{3} & A_{L,1} = \frac{\Delta J_L}{6}
\end{pmatrix}$$

$$B = \begin{pmatrix}
B_{00} = -\frac{D_0}{\Delta J_0} & B_{01} = \frac{D_0}{\Delta J_0} & & \\
B_{10} = -\frac{D_1}{\Delta J_1} & B_{11} = \frac{D_1 + D_{1+1}}{\Delta J_{1+1}} & & \\
& \ddots & \ddots & \\
B_{L-1,0} = -\frac{D_{L-1}}{\Delta J_{L-1}} & B_{L-1,1} = -\frac{D_{L-1} + D_L}{\Delta J_L} & B_{L,1} = \frac{D_L}{\Delta J_L}
\end{pmatrix}$$

$$\hat{V} = \begin{pmatrix}
V_{00} = -\frac{V_0}{2} & V_{01} = \frac{V_1}{2} & & \\
V_{10} = \frac{V_1}{2} & V_{11} = \frac{V_1 - V_{1+1}}{2} & & \\
& \ddots & \ddots & \\
V_{L-1,0} = \frac{V_{L-1}}{2} & V_{L-1,1} = \frac{V_{L-1} + V_L}{2} & V_{L,1} = \frac{V_L}{2}
\end{pmatrix}$$

Here, $\bar{D}_l$ denotes diffusion coefficient $D(J)$ averaged over $[J_{l-1}, J_l]$, $V_l$ is a drift velocity, if any, in the $l$-th node. The last component of (65) is trivial: $f_L(t) = 0$ due to boundary-value condition (34) or (36), and dimension of problem (64) is $L \times L$ de facto.

**Temporal discretisation** of vector equation (64) is performed with an equidistant time grid $\bar{t}^{(n)} = n\Delta t$, $n = 0, 1, \ldots$ where $\Delta t$ is increment in time. Use is made of a central finite difference approximation of a derivative in $t$

$$\frac{d}{dt} \bar{f}^{(n+\frac{1}{2})} \approx \frac{1}{\Delta t} \left( \bar{f}^{(n+1)} - \bar{f}^{(n)} \right) + O(\Delta t^2),$$

and of linear interpolation to the intermediate time instant

$$\bar{f}^{(n+\frac{1}{2})} \approx \frac{1}{2} \left( \bar{f}^{(n+1)} + \bar{f}^{(n)} \right) + O(\Delta t^2).$$

Here, $\bar{f}^{(n)}$ denotes the nodal value of $\bar{f}(t = t_n)$.

Inserting these two equations into (64) written for a central time $\bar{t}^{(n+1/2)} = (n+1/2)\Delta t$ results in a two-layer time-domain integration algorithm known as Crank-Nicolson’s numerical scheme,
\[
\left( \hat{A} + \frac{\Delta t}{2} (B + V) \right) f^{(n+1)} = \left( \hat{A} - \frac{\Delta t}{2} (B + V) \right) f^{(n)}. \quad (71)
\]

This iterative process is

1. unconditionally stable and
2. free of unwanted oscillations for an appropriately small \( \Delta t \propto \Delta J^2 / \overline{D} \).

It is used to solve the diffusion equation in question with a dedicated computer code.

**4.5 Conclusion**

A full inventory of research tools is available to study longitudinal diffusion caused by external noise in a single-RF system operated under \( \varphi_s = 0^\circ \). The treatment with a diffusion equation is well extendable beyond RF buckets to study diffusive transport in a halo region. No unnecessary simplifications are involved in taking into account the inherent non-linearity of synchrotron motion.
5 Low-Pass Filtered Noise Source

This Section presents a numerical procedure to generate low-pass filtered noise samples with a prescribed spectral power density employed in tracking simulations.

5.1 Input Noise Source

The primary source noise signal $v(t)$ is assumed to be piecewise constant in time $t$ during a given turn around orbit, and non-correlated on a turn-by-turn basis. To this end,

$$v(t) = \sum_n \zeta_n \left[ \Phi(t - nT_0) - \Phi(t - (n + 1)T_0) \right],$$  \hspace{1cm} (72)

where $\Phi(t)$ is Heaviside step function (= 1 when $t \geq 0$ and 0 otherwise), $T_0 = 2\pi/\omega_0$ is rotation period, $\omega_0$ is angular rotation frequency, $\zeta_n$ is a random sequence with a zero cross-correlation $\langle \zeta_n \zeta_m \rangle$ for $n \neq m$. Angular brackets $\langle \ldots \rangle$ denote ensemble averaging. Random variable $\zeta_n$ has the uniform probability density in a range $[-R, R]$. Therefore, the expectation values $\langle \zeta_n \rangle = \langle v(t) \rangle = 0$, and the variances are

$$\langle \zeta_n^2 \rangle = \langle v^2 \rangle(0) = \frac{1}{3} R^2.$$  \hspace{1cm} (73)

Auto-correlation function $\langle v(t+\tau) \cdot v(t) \rangle \equiv \langle v^2 \rangle(\tau)$ has a value of (73) at $\tau = 0$, is piecewise linear for $|\tau| \leq T_0$, and zero for $|\tau| \geq T_0$ (see broken line between bullets in Fig. 11).

Making use of the double-sided spectral power density (50), one gets

$$P_v(\omega) = P_{v0} \cdot \left( \frac{\sin(\omega T_0/2)}{\omega T_0/2} \right)^2 \hspace{1cm} \text{where} \hspace{0.5cm} P_{v0} = \frac{1}{3} R^2 T_0.$$  \hspace{1cm} (74)

At lower frequencies $|\omega| << \omega_0$ of interest, signal $v(t)$ behaves as a white noise with a constant spectral power density $P_{v0}$.

Fig. 10 shows a sample of noise $v(t)$. Its auto-correlation function, as recovered numerically from 2000 samples, is plotted in Fig. 11.

5.2 Digital Filtering and Output Noise

To obtain output noise $u(t)$, $\langle u(t) \rangle = 0$ with controlled spectral properties, primary noise signal $v(t)$ (72) is filtered through a linear digital network with a transfer function $G(\omega)$ that yields
\[ u(\omega) = G(\omega)v(\omega), \quad P_u(\omega) = |G(\omega)|^2 P_v(\omega). \quad (75) \]

The problem in question demands for noise spectrum shapes that ensure a control over, at least, three parameters of interest in a resultant \( P_u(\omega) \):

1. higher cut-off frequency \( \omega_H \) of the spectrum, if required,
2. a fractional depth \( g \) and
3. bandwidth \( \omega_L \) of a DC notch.

This goal is achieved by a cascade connection of the two filters: a low-pass \( F(\omega) \) followed by a high-pass \( T(\omega) \). Their cut-off and roll-off frequencies are set to \( \omega_H \) and \( \omega_L \), respectively. Hence

\[ G(\omega) = T(\omega) \cdot F(\omega) \quad (76) \]

where \( F(0) = 1, |T(0)|^2 = g, F(\infty) = 0 \) and \( T(\infty) = 1 \).

In what follows, these filters and digital algorithms to implement them are discussed. Either filter is treated and tested individually.

### 5.2.1 A Low-Pass Filter \( F(\omega) \)

Use is made of a conventional Butterworth filter of the 4-th order whose

\[ F(\omega) = \frac{\omega_H^4}{\prod \left( \omega^2 + 2i\omega\omega_h \sin \left( \frac{\pi}{8} \right) - \omega^2 \right)}, \quad |F(\omega)|^2 = \frac{\omega_H^8}{\omega_H^8 + \omega^8} \quad (77) \]

where \( \omega_H \) is a cut-off frequency. This filter offers a good approximation of a perfect low-pass stepwise window, refer to Fig. 12 which plots the second of Eqs.(77).

![Fig. 12: Basic low-pass filtering.](image)

Under assumption \( P_v(\omega) = P_{v,0} \) of a white input noise, auto-correlation function of the output noise \( u(t) \) at exit from \( F(\omega) \) can be recovered as the inverse Fourier transform of (77)

\[ \langle u^2 \rangle(\tau) = P_{v,0} \cdot \frac{1}{2\pi} \int |F(\omega)|^2 \exp(-i\omega\tau) d\omega = \]

\[ = P_{v,0} \cdot \frac{1}{4} \left( Re \left( b \omega_H \exp(-b\omega_H|\tau|) \right) + Im \left( b \omega_H \exp(ib\omega_H|\tau|) \right) \right), \quad b = \exp \left( i \frac{\pi}{8} \right) \]

Therefore, variance of the noise on passing through \( F(\omega) \) reduces to

\[ \langle u^2 \rangle(0) = \frac{\omega_H T_0}{8 \sin(\pi/8)} \langle v^2 \rangle(0). \quad (79) \]
Digital filtering algorithm is obtained by inserting
\[ \omega = \frac{2}{T_0} \frac{z - 1}{z + 1} \]
into (77) with \( z \) being a parameter of the \( z \)-transform. As a result, one gets
\[ F(z) = \frac{\Theta^4 \cdot (1 + 4z + 6z^2 + 4z^3 + z^4)}{a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4} \]
where
\[ \Theta = \frac{1}{2} \omega H T_0, \]
\[ a_{0,4} = (1 + \Theta^2)^2 \pm 2\Theta(1 + \Theta^2) \left( \cos \frac{\pi}{8} + \sin \frac{\pi}{8} \right) + 2\Theta^2 \sin \frac{\pi}{4}, \]
\[ a_{1,3} = -4(1 - \Theta^2) \mp 4\Theta(1 - \Theta^2) \left( \cos \frac{\pi}{8} + \sin \frac{\pi}{8} \right), \]
\[ a_2 = 4(1 - \Theta^2)^2 + 2(1 + \Theta^2)^2 - 4\Theta^2 \sin \frac{\pi}{4}. \]

As it follows from (81), digital filtering algorithm to reproduce transfer function (77) now reads
\[ u_n = \frac{1}{a_0} \left( \Theta^4 \cdot (v_0 + 4v_{-1} + 6v_{-2} + 4v_{-3} + v_{-4}) - \right. \]
\[ \left. - a_1u_{-1} - a_2u_{-2} - a_3u_{-3} - a_4u_{-4} \right) \]
where \( v_{-n} \) and \( u_{-n} \) denote readouts of I/O signals from the \( n \)-th preceding turn, the current one being subscribed with 0. The first and second lines of (82) represent, respectively, non-recursive and recursive parts of the digital filter involved.

Fig. 13 demonstrates a sample of noise \( u(t) \) at exit from filter (77) with a cut-off frequency \( \omega / \omega_0 = 1/20 \). Its auto-correlation function, as recovered numerically from 2000 samples, is plotted in Fig. 14 by a solid line. Thin curve plots a function given by (78).
5.2.2 A High-Pass Filter \( T(\omega) \)

Use is made of a filter comprising a dummy network section with the unit transfer function encircled by a feedback loop equipped with an integrator \( H(\omega) \) and a perfect amplifier with a gain factor \( K \), refer to Fig. 15. This scheme represents DC-coupled feedback loops employed in 208 MHz RF system of the HERA-p ring (in both, phase and amplitude contours). Similar block diagrams were dealt with earlier in Fig. 7 and (49).

![Fig. 15: Schematic diagram of high-pass filtering.](image)

The in-out transfer function of such a filter is

\[
T(\omega) = \frac{1}{1 + K \cdot H(\omega)},
\]

\[
H(\omega) = \frac{\omega_{L1}}{\omega_{L1} - i\omega},
\]

where \( \omega_{L1} \) is a roll-off frequency of integrator \( H(\omega) \). As a result, \( T(\omega) \) can be put as

\[
T(\omega) = \frac{\omega_{L1} - i\omega}{\omega_{L} - i\omega},
\]

\[
|T(\omega)|^2 = \frac{\omega_{L1}^2 + \omega^2}{\omega_{L1}^2 + \omega^2}.
\]

Bandwidth \( \omega_L \) (the higher roll-off frequency of \( |T(\omega)|^2 \)) and fractional depth \( g \) of a DC notch are found as

\[
\omega_L = (K + 1) \cdot \omega_{L1},
\]

\[
g = \frac{1}{(K + 1)^2}.
\]

As an example, Fig. 16 plots the second of Eqs. (84) for \( g = 1/2000 \). Frequency \( \omega_{L1} \) of \( H(\omega) \) exhibits itself as a width of flat-bottom (the lower roll-off frequency) in \( |T(\omega)|^2 \).

![Fig. 16: High-pass filtering, a DC notch.](image)

Under assumption \( P_v(\omega) = P_v0 \) of a white input noise, auto-correlation function of the output noise \( u(t) \) at exit from \( T(\omega) \) can be recovered as the inverse Fourier transform of (84):

23
\[
\langle u^2 \rangle(\tau) = P_{\omega} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} |T(\omega)|^2 \exp(-i\omega\tau) d\omega =
\]

\[
= P_{\omega} \left( \delta(\tau) - \frac{K(K + 2)}{(K + 1)^2} \cdot \frac{\omega_L T_0}{2} \cdot \exp(-\omega_L |\tau|) \right).
\]

Suppression of noise variance on passing through \( T(\omega) \) is

\[
\langle u^2 \rangle(0) \approx \left( 1 - \frac{K(K + 2)}{(K + 1)^2} \cdot \frac{\omega_L T_0}{2} \right) \langle v^2 \rangle(0)
\]

which estimate holds for \( \omega_L T_0 \ll \pi \). Indeed, to justify the white noise approximation in question, bandwidth \( \omega_L \) of the DC notch must extend over only a small fraction of flat-top in the centre of the source spectrum \( P_v(\omega) \), (74).

The \( z \)-transform approximation of transfer function (84) is

\[
T(z) = \frac{b_0 + b_1 z}{a_0 + a_1 z}
\]

where

\[
\Theta = \frac{1}{2} \omega_L T_0, \quad a_{0,1} = (K + 1) \Theta \pm 1, \quad b_{0,1} = \Theta \pm 1.
\]

Hence, the digital filtering algorithm reads

\[
u_t = \frac{1}{a_0}(b_1 v_{t-1} + b_2 v_{t-2} - a_1 u_{t-1}).
\]

Fig. 17 demonstrates a sample of noise \( u(t) \) at exit from filter (84) with a lower roll-off frequency \( \omega_L / \omega_0 = 1/1000 \) and gain \( K = 20 \). Its auto-correlation function, as recovered numerically from 2000 samples, is plotted in Fig. 18 by a solid line. Thin curve plots a function given by (86) where a product \( \delta(\tau) \cdot T_0 \) is replaced with a finite function \( \langle v^2 \rangle(\tau)/\langle v^2 \rangle(0) \) shown as a broken line between bullets in Fig. 11.

In practice, gains \( K \) as high as \( 10^3 \text{–} 10^5 \) are employed. Therefore, factors \((K+1)\) and \((K+2)\) of Eqs.(85), (86), (87), etc can be safely replaced with \( K \).
5.3 Conclusion

Plots shown in Fig. 11, Fig. 14 and Fig. 18 demonstrate that prescribed correlation/spectral properties of the noise signal are reproduced adequately in a time domain. A cascade connection (76) of two filters fed by a noise source (72) offers a realisable family of noise power spectrums with four control entries \((g, \omega_L, \omega_H \text{ and } P_v0)\). This set meets the demand of a tracking code since the longitudinal diffusion per se is expected to be a robust process that is well insensitive to finer details of the noise spectrum.
6 Longitudinal Tracking Code

This Section describes a code for multi-turn longitudinal tracking of macro-particles subjected to phase or amplitude noises. Results of this tracking are benchmarked against the relevant solutions of a diffusion equation.

6.1 General Description

6.1.1 Tracking Procedure

Use is made of a longitudinal co-ordinate (phase in RF rads) \( \phi \propto h(\Theta - \omega_0 t) \) where \( h \) is RF harmonic number (208 MHz), \( \Theta \) is generalised azimuth of a rotating proton, \( \omega_0 \) is revolution frequency, \( t \) is time. A reference particle has \( \phi = 0 \) (mod \( 2\pi \)). Conjugated momentum is \( \xi = d\phi/dt \), and variables \((\phi, \xi)\) constitute the longitudinal phase-space plane.

Fig. 19: Longitudinal map decomposition.

A full one-turn transform \((0 \rightarrow 2\), see Fig. 19\) is a product of the two area-preserving maps: of a one-turn drift \((0 \rightarrow 1)\) followed by a non-linear and noisy kick in a thin RF cavity \((1 \rightarrow 2)\). To this end,

\[
\begin{align*}
\varphi_1 &= \varphi_0 + T_0 \cdot \xi_0, \\
\xi_1 &= \xi_0,
\end{align*}
\]

and, then, for a phase noise,

\[
\begin{align*}
\varphi_2 &= \varphi_1, \\
\xi_2 &= \xi_1 - T_0 \cdot \Omega_0^2 \cdot \sin \varphi_1 + \\
&+ T_0 \cdot \Omega_0^2 \cdot (u_1/V_{RF}) \cdot \cos \varphi_1 + \\
&+ T_0 \cdot \Omega_0^2 \cdot \tan \varphi_1 \cdot (1 - \cos \varphi_1).
\end{align*}
\]

Here, \( T_0 = 2\pi/\omega_0 \) is a rotation period, \( \Omega_0 \) is a circular frequency of small-amplitude longitudinal oscillations \((2)\), \( \varphi_s \) is a stable phase angle that depends upon energy losses due to synchrotron radiation \((9)\), \( V_{RF} \) is peak RF voltage per turn of beam around orbit. Phase \( \varphi \) of a macro-particle is treated in mod \( 2\pi \) and is reduced back to interval \( (-\pi, +\pi] \) if occurs beyond.

\( u_1 \) is a random voltage that is presumed to be constant during the given turn \#1. Prescribed spectral properties of \( u \) are tailored out according to Section 5 on a turn-by-turn basis. Care is taken to ensure \( |u| \ll V_{RF} \) so as to keep the noise in frames of the small-signal approximation. Ratio \( u_1/V_{RF} \) can be interpreted as a phase error whose effect is reduced to an additive quadrature voltage term \( \propto \cos \varphi_1 \) in line \( (i) \) of equations \((91)\). (On the contrary, studies of an amplitude noise request for an in-phase voltage term \( \propto (u_1/V_{RF}) \cdot \sin \varphi_1 \) there.)

In the code itself, tracking proceeds in terms of the normalised variables \((\phi, \xi/\Omega_0)\) use of which reveal the product \((\Omega_0 T_0) \ll 2\pi \) to be a natural small parameter of the mapping with \((90), (91)\).
Post-processing of raw data and internal branching in the tracking algorithm relies on analytical formulae. They are available for a single-RF system in a smooth \((\Omega_0 T_0 \rightarrow 0)\) approximation in terms of elliptic functions (refer to Section 2.1). Parameter \(k\) of these functions is recovered from co-ordinates of a macro-particle as

\[
k^2 = \left(\frac{\sin \phi}{2}\right)^2 + \left(\frac{\xi}{2\Omega_0}\right)^2. \tag{92}\]

### 6.1.2 Initial Distribution

Initial longitudinal profile of a bunch is conventionally assumed to be Gaussian in variables \(\phi, \xi\). Therefore, tracking is launched with a matched exponential distribution

\[
f(J, \psi, t)|_{t=0} = \frac{1}{4\pi J_\sigma} \exp\left(-\frac{J}{2J_\sigma}\right) \tag{93}\]

where \(\psi\) and \(J\) are longitudinal phase and action variables, \(J_\sigma\) stands for a value of \(J\) at the phase-plane trajectory with \(1\sigma\) oscillation amplitudes.

Initial distribution of particles over \(J\) is

\[
f(J, t) = \frac{1}{2\pi} \int_0^{2\pi} f(J, \psi, t) d\psi|_{t=0} = \frac{1}{2J_\sigma} \exp\left(-\frac{J}{2J_\sigma}\right) \tag{94}\]

Both functions, (93) and (94), are normalised to 1.

Distribution (93) is modelled with a set of macro-particles injected onto an equidistant rectangular 2D grid in variables \((A\phi, \psi)\) where \(A\phi \leq \pi\) is amplitude of oscillations along \(\phi\). Cartesian phase-plane co-ordinates \((\phi, \xi)\) and weight \(W\) of a macro-particle occurring in the node \((A\phi, \psi)\) are recovered in compliance with analytical formulae available for a single-RF system:

\[
k = \sin \frac{A\phi}{2}, \tag{95}\]

\[
\phi = 2\arcsin \left(k \cdot \text{sn} \left(\frac{2K(k)}{\pi} \psi, k\right)\right), \tag{96}\]

\[
\xi = 2\Omega_0 \cdot k \cdot \text{cn} \left(\frac{2K(k)}{\pi} \psi, k\right), \tag{97}\]

\[
W = f(J(k), \psi, t=0) \cdot \Omega_0 \cdot \frac{2K(k)}{\pi} \cdot \sin A\phi \cdot \Delta A\phi \Delta \psi, \tag{98}\]

\[
J(k) = J_S \cdot \left(E(k) - (1 - k^2) \cdot K(k)\right). \tag{99}\]

Here, \(K(k)\) and \(E(k)\) are complete elliptic integrals of the 1\(^{st}\) and 2\(^{nd}\) kinds \((6)\), \(k\) is their parameter, \(\text{sn}(x, k)\) and \(\text{cn}(x, k)\) are Jacobi elliptic functions, \(\Delta A\phi\) and \(\Delta \psi\) are grid periods.

Phase-plane portraits of a bunch are shown in the top row of Fig. 20. This figure is an example of noise-free tracking over 25,000 turns. Mapping parameter \(\Omega_0 T_0 = 2\pi/50\) and \(\tan \phi_S = 0\). In total, 1500 macro-particles are arranged into \(25 \times 60\) grid in \((A\phi, \psi)\). Parameter \(J_\sigma\) of distribution (93) is set to \(0.062 J_S\) where \(J_S\) is action variable at separatrix, (5). Such a \(J_\sigma\) corresponds to r.m.s. half-length of a bunch \(\sigma_\phi = (0.18–0.19)\pi\) — equivalent to \(\sigma_s = 13\) cm @ \(\lambda_{RF} = 1.44\) m (208 MHz) which is bunch size at start of a HERA-\(p\) run \([1]\).

Histograms in lower plots of Fig. 20 reconstruct distribution \(f(J, t)\) of particles over \(J\). Smooth lines show the anticipated law (94). Longitudinal profile of a bunch is thus transported without a noticeable distortion. Integrals over distribution (such as bunch population \(N_B\) and average emittance \(\bar{\mathcal{J}}\) of a bunch, \((106)\)) are kept constant in \(t\).
6.1.3 Accumulation of Dimensionless Times and Noise-to-Drift Ratio

There are two inherent time scales imposed by the dynamical effects under study — diffusion and drift. Hence, depending on the context, use is made of the two dimensionless time variables: *diffusive* time $t_1$ and *drift* time $t_2$. They are defined, respectively, by

$$t_1 = t \cdot \frac{\Omega^2 \phi_{0}}{V_{RF}^2},$$

$$t_2 = t \cdot \frac{V}{J} = t \cdot \frac{\pi}{4} \frac{\Omega_0 \tan \phi_s}{},$$

where $V$ is amplitude of drift velocity along $J$ of protons in beam halo, (14). These definitions are dictated by a diffusion equation. Contrary to yet uncertain level $P_{\phi0}$ of external white noise, drift velocity $V$ is well known a priory for the HERA-$p$ ring (refer to Section 2.2.1), and use of $t_2$ is thus preferable.

Results of tracking must be compared to those obtained by numerical solution of diffusion equation at the identical instants of $t_1$ or $t_2$. These times are incremented with a turn number $n$ of tracking according to

---

Fig. 20: Tracking without noise and losses due to SR.
\[ t_1 = n \cdot (\Omega_0 T_0)^2 \frac{\langle v^2 \rangle(0)}{V_{RF}^2}, \]  
\[ t_2 = n \cdot (\Omega_0 T_0) \frac{\pi}{4} \Omega_0 \tan \phi_s. \]  

In the latter case, level required of a noise in the system is adjusted by setting a proper excess of noise source variance \( \langle v^2 \rangle(0)/V_{RF}^2 \ll 1 \) with respect to a given value of its drift counterpart \( \frac{\pi}{4} \Omega_0 \tan \phi_s. \)

To parameterise relative level of external noise contamination, use is made of a Noise-to-Drift Ratio that reads

\[ \text{NDR} = \frac{\Omega_0^2 P_0}{V_{RF}^2} \frac{J_S}{V} = \frac{t_1}{t_2}. \]  

Still, from a viewpoint of adverse effect on the beam, more representative is a value of NDR reduced to the apparent-to-beam noise \( u(t) \) at the first (dipole) harmonic of synchrotron frequency

\[ \text{NDR}_1 = \frac{\Omega_0^2 P_0(\Omega_0)}{V_{RF}^2} \frac{J_S}{V} \]  

which value will also be mentioned for reference.

### 6.2 Tracking versus a Diffusion Equation

Here, tracking data is subjected to tests versus results yielded by solving a diffusion equation. A one-to-one agreement between outcomes of tracking and 1D diffusion equation can only be expected in case of a perfect symmetry between upper and lower half-planes of phase plane \((\phi, \xi)\). The latter implies:

1. equal distances between top/bottom absorbing walls and bunch centre,
2. either absence altogether of drift fluxes in a halo region,
3. or presence of both inward or both outward drifts there.

These conditions are sketched as cases #1, 2 and 3 in Fig. 21. Case #1 can be readily modelled with tracking by taking \( \tan \phi_s = 0 \). On the contrary, cases #2, 3 cannot be encountered in a conventional accelerator regime represented by a straightforward tracking algorithm (90), (91). Still, the drift fluxes can be reversed artificially, in a computer code, by a mere toggling sign of \( \tan \phi_s \neq 0 \) in line (i) of equations (91), according to the prescription:

<table>
<thead>
<tr>
<th>Case #2:</th>
<th>( \tan \phi_s &lt; 0 ) for particles with ( J &gt; J_S ) &amp; ( \xi &gt; 0 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case #3:</td>
<td>( \tan \phi_s &lt; 0 ) for particles with ( J &gt; J_S ) &amp; ( \xi &lt; 0 ).</td>
</tr>
</tbody>
</table>

Input parameters the for benchmark calculations are taken such as to retain the most essential features of the effect under study:

1. a slow degrade of the bunched beam core,
2. crossing the RF separatrix,
3. diffusive transport beyond buckets,
4. concurrent drift due to energy losses in a halo region,
5. ultimate loss of particles at an aperture limit.

In what follows, for demonstration purposes, spectrum of noise is taken as \( P_n(\Omega) = |G(\Omega)|^2 P_{v0} \) where \( G(\Omega) = T(\Omega) \cdot F(\Omega) \) is a transfer function of low-pass and high-pass filters in
series (refer to Section 5 for details). Lower and higher roll-off frequencies are $\Omega_L/\Omega_0 = 0.05$ and $\Omega_H/\Omega_0 = 5$. Relative depth of DC notch is $g = 1/2000$. Fig. 22 plots spectral power density $P_u(\Omega)$ with $P_u(\Omega_0) \equiv P_u(6)$. The resultant diffusion coefficients $D(J)$ are shown in Fig. 23 for phase (solid line) and amplitude (dashed line) noises. Phase noise is more dangerous for bunch core, while amplitude noise has a stronger effect on the tails of distribution.

Absorbing wall is located outside of RF buckets at $J_w = 2J_S$. In total, 6,000 macro-particles are involved into tracking. At injection, they are arranged into $50 \times 120$ grid in $(A_\phi, \psi)$ so as to reproduce distribution shown in Fig. 20 by a smooth line.

Fig. 21: Classification of test problems.

Fig. 22: Noise power spectrum, tests.

Fig. 23: Diffusion coefficient, tests.
6.2.1 Case #1, without Drift in a Halo, Phase Noise

Energy loss is ignored, and \( \tan \phi_s = 0 \). Diffusion is followed for an interval of time \( t_1 = 0–25 \). To accumulate this time, about 130,000 map iterations are required. Prior to plotting, all tracking results are averaged over 5 realisations (initial seeds) of a random process \( u(t) \) to accomplish a reasonable statistics. Thick solid lines plot solutions of diffusion equation while thin lines are the tracking results.

Fig. 24 shows evolution of beam population inside bucket and aperture.

Fig. 25 illustrates degrade of beam core. It plots (effective) beam size that is recovered from an integral over distribution:

\[
\sigma_{\phi_1}(t) = \left[ \frac{\int \langle f \rangle(t)}{\Omega_0} \right]^{1/2} \quad \text{with} \quad \langle f \rangle(t) = \frac{\int_0^{J_s} f(J,t) \, dj}{\int_0^{J_s} f(J,t) \, dj}.
\]

Value of \( \sigma_{\phi_1} \) coincides exactly with a standard r.m.s. half-length \( \sigma_{\phi} \) for a short Gaussian bunch maintained in a purely parabolic longitudinal potential well \( U(\phi) = \Omega_0^2 \phi^2/2 \).

Histograms in Fig. 26 reconstruct distribution of particles along \( J \) by the end of a run, at \( t_1 = 25 \). Fig. 27 zooms the halo region and also reveals a partition (close to 50/50) of particles between lower and upper half-planes.

![Fig. 24: Beam population, case #1, \( \phi \)-noise.](image)

![Fig. 25: Bunch size, case #1, \( \phi \)-noise.](image)

![Fig. 26: Longitudinal distribution, case #1, \( \phi \)-noise.](image)

![Fig. 27: Beam halo region, case #1, \( \phi \)-noise.](image)
6.2.2 Case #1, without Drift in a Halo, Amplitude Noise

This case demonstrates that both the codes (to solve diffusion equation and to track macro-particles) do properly distinguish between phase and amplitude noises. Results are plotted in Fig. 28–Fig. 31. Due to a large beam size at injection (about half of RF bucket at base of distribution), adverse effects of phase and amplitude noises are comparable quantitatively. Such integrated observables as rate of population decay and accumulation of particles in halo region are well insensitive to the nature of noise, either phase or amplitude. The only distinct signature of amplitude noise available is a long-time persistence of the pronounced beam core (compare Fig. 20, Fig. 26 and Fig. 30). Eventually, it exhibits itself in a different scenarios of bunch elongation, see Fig. 25 and Fig. 29.

6.2.3 Case #2, Inward Drifts in a Halo, Phase Noise

Now, tan\(\phi_s\) = 0.002, and the prescription of Table 5, the first line, is followed. Diffusion is studied for an interval of time \(t_2 = 0–12.5\). To accumulate this time, about 65,000 map iterations are required. Again, a 5-sample averaging of random process is performed prior to plotting. Noise-to-Drift Ratios (104), (105) are \(NDR = 2\) and \(NDR_1 \approx 1/3\).
Tracking results for \( t_2 = 12.5 \) are shown in Fig. 32–Fig. 35. The major bulk of particles is confined close to RF bucket since diffusion is overridden by drift. There is only a negligible loss of particles at aperture.

[Fig. 32: Beam population, case #2, \( \varphi \)-noise.]

[Fig. 33: Bunch size, case #2, \( \varphi \)-noise.]

[Fig. 34: Longitudinal distribution, case #2, \( \varphi \)-noise.]

[Fig. 35: Beam halo region, case #2, \( \varphi \)-noise.]

6.2.4 Case #3, Outward Drifts in a Halo, Phase Noise

Now, the second line of Table 5 is active. Tracking results for \( t_2 = 12.5 \) are shown in Fig. 36–Fig. 39. Particles are quickly transported by diffusion plus drift mechanisms through the halo region of \( J = (1–2)J_3 \). Therefore, plots of beam population inside bucket \( N_b(t_2) \) and aperture \( N_a(t_2) \) virtually repeat one another, up to an estimated time lag \( \Delta t_2 \approx 0.9 \), Fig. 36. Halo population \( N_a(t_2) - N_b(t_2) \) reaches a level of around 5% of injected particles at \( t_2 \approx 6 \) (refer to the solid line of Fig. 44 below).
6.3 Conclusion

A proper solver for diffusion equation with a drift term and a correct tracking procedure are available. They yield coincident results in the areas of mutual applicability (cases #1, 2 and 3 of Fig. 21).

The important features of diffusion equation solver thus verified comprise:

1. Adequate handling of separatrix crossing and of an integrable divergence in diffusion coefficient $D(J)$ at $J = J_s$.

2. Proper treatment of diffusion transport beyond RF buckets in a halo region where analytical formulae in terms of elliptic functions for a single-RF system are available.

3. Systematic effect of SR losses is confirmed to be reducible to a slow drift along $J$ with a constant velocity $V$ beyond RF buckets.

Adequate synthesis of noise samples in time domain for tracking is provided. Prescribed spectral properties and magnitudes of noise are reproduced correctly.

Phase or amplitude noises of the same power can be distinguished by different laws of bunch lengthening versus time. Longer bunches are obtained with a phase noise.
7 Beam Halo Effect, an Example of

In Section 6.2, tracking code is benchmarked against solutions of diffusion equation which properly treats three qualitatively distinct cases #1, 2 and 3 in Fig. 21. Here, the thus verified code is converted and applied to case #4 of practical interest. This conversion is accomplished by commenting out a single line in the program source text. Such a minor update of the code enhances confidence into tracking results.

Case #4 of Fig. 21 is amenable to tracking studies only. There exists a plain physical analogue for this problem — dissolving a warm spinning bit of a solid aquarelle colour immersed into a laminar flow of a hot water.

It should be emphasised that it is a conventional effect of bunch dilution inside RF bucket that fails to obey rigorously a 1D diffusion equation in variable $J$ for $\tan \phi_s \neq 0$ rather than motion in beam halo per se. The latter is well governed by 1D diffusion plus drift that are fed by a proper influx through the emitting boundary.

Study of the neighbouring cases #1, 2 and 3 of Fig. 21 with a diffusion equation provides an useful insight into case #4 in question. Say, it has already allowed to recover natural time scales of the problem (refer to (102) and (103)). Solving the diffusion equation is a powerful and time-saving procedure. Naturally, the question arises what is the correspondence between easily available solutions for cases #1, 2 and 3 and those for case #4. Here, it this question that is dealt with.

7.1 Case #4, Inward/Outward Drifts in a Halo, Phase Noise

All the parameters of tracking procedure are retained from Section 6.2. Still, prescription of Table 5 is now discarded.

Fig. 40 shows evolution in time $t_2$ of beam population inside bucket (lower curves, $N_B(t_2)$) and aperture (upper curves, $N_A(t_2)$). Thick lines plot solutions of a diffusion equation. Thin lines are recovered from tracking studies of case #4 on a 5-sample averaging. Cases #3 and #4 exhibit a very close behaviour. It tells that the particles are readily removed from the outer vicinity of separatrix, suffice it to arrange a draining drift flux even in a single half-plane of $(\phi, \xi)$.

Fig. 41 shows evolution of beam r.m.s. half-length. Tracking curve goes in between plots for cases #1 (no drift, $V = 0$) and #3 ($V > 0$). Case #2 ($V < 0$) yields an overestimated bunch size since the particles are crabbed back into RF bucket area from both half-planes which results in flat distributions with large variances in $\phi$.

Fig. 42 reconstructs longitudinal distribution function at $t_2 = 12.5$. As is expected, this distribution is continuous at separatrix. Again, case #3 provides a good approximation for the ultimate bunch and net halo profiles (population of upper plus lower half-planes).

Fig. 43 reveals composition of beam halo by particles occurring in the upper/lower half-planes. Tail of distribution in the upper half-plane, case #4, is well represented by net profile available from solution for case #3. On the other hand, solution for case #2 provides an estimate from above of beam halo extent in the lower half-plane. No particles can occur beyond beam tail width at base found with a diffusion equation for case #2 in which gradients of distribution function and diffusion fluxes to counteract drift are at their top values possible.

Similar to the HERA-$p$ ring, there is a significant drift background in the test problem under study. It tends to depopulate halo region. On crossing the separatrix, particles are quickly transported to aperture, and $N_A(t_2)$ nearly follows $N_B(t_2)$ with halo population vanishing. Therefore, solutions for a zero boundary-value problem at separatrix (@ $J_S$, $t = 0$) shown in Fig. 40–Fig. 42 would offer a very satisfactory approximation of beam parameters inside RF buckets (the bunched-core observables) for case #4 of interest.
Fig. 40: Beam population, case #4, ϕ-noise.

Fig. 41: Bunch size, case #4, ϕ-noise.

Fig. 42: Longitudinal distribution, case #4, ϕ-noise.

Fig. 43: Beam halo region, case #4, ϕ-noise.

Fig. 44 plots a difference $N_A(t_2) - N_B(t_2)$ that shows how coasting beam current is accumulated in beam halo region. Halo population $N_H(t_2)$ steadily increases to < 5% of injected intensity $N_B(0)$ until around 20% of beam is lost at an aperture limitation. Then, halo decays exponentially with the entire beam. The smaller is the NDR (104), (105), the lower is population of halo in total. Ratio of instantaneous populations $N_H(t_2)/N_B(t_2)$ shown in Fig. 45 is about to saturate at a level $\propto$ NDR with $t_2 \to \infty$. This effect is treated in Section 8.3.

7.2 Conclusion

Study of case #3 in Fig. 21 (both outward drifts in halo region) with a diffusion equation yields a very good approximation of case #4 (outward and inward drifts) that is of a practical interest, accumulation of net current in beam halo included.

Except for a plain tracking, there are no reliable tools to recover partition of halo population in between upper and lower half-planes of phase-space plane. Study of case #2 provides only the worst-case estimate of halo extent in a half-plane where the diffusion faces a head-on drift.

In the HERA-p ring where there is a significant drift background outside of RF buckets, solutions of diffusion equation with a zero boundary-value condition at separatrix would
provide a rather good description of evolution of the bunched-core observables — bunch lengthening and its decay in population.

Fig. 44: Accumulation of current in beam halo.

Fig. 45: Ratio of halo to bunch populations.
8 Beam Halo Effect in the HERA-p Ring

In this Section, the tools tested and benchmarked above are applied a realistic set of parameters compliant in general with the coasting beam halo effect in the HERA-p ring.

8.1 Phase Noise

Spectrum of phase noise is taken as \( P_u(\Omega) = |G(\Omega)|^2 P_{v0} \) where \( G(\Omega) = T(\Omega) F(\Omega) \) is a transfer function of low-pass and high-pass filters in series (refer to Section 5 for details). Table 6 specifies parameters of three (quite realistic for the HERA-p ring) options studied. There, \( \Omega_L \) is a higher roll-off frequency, \( g \) is a relative depth of a DC notch, \( \Omega_{L1} \) is a width of flat-bottom near DC. Effects of the spectrum cut-off at \( \Omega_H \) are discarded for the time being. Fig. 46 plots spectral power densities \( P_u(\Omega) \). Occasionally, power spectrum \( P_{v0} \) of white noise source is referred to in terms of \( P_{v0}/10^6 \) which is a power of white noise source reduced to the reference level of Fig. 46.

The resultant diffusion coefficients \( D(J) \) for synchrotron frequency \( \Omega_0/2\pi \approx 40 \) Hz are shown in Fig. 47, difference between the noise options being barely distinguishable. Value of noise spectrum at first harmonic of synchrotron frequency that mostly determines the rate of bunch dilution is \( P_u(\Omega_0) = (8.1–7.1) \cdot P_{v0}/10^6 \).

<table>
<thead>
<tr>
<th># of ( P_u(\Omega) )</th>
<th>( g )</th>
<th>( \Omega_L/2\pi ), Hz</th>
<th>( \Omega_L/2\pi ), kHz</th>
<th>( \Omega_H/2\pi ), Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 10^{-6} ) (–60 dB)</td>
<td>15</td>
<td>15</td>
<td>( \rightarrow \infty )</td>
</tr>
<tr>
<td>2</td>
<td>( 10^{-8} ) (–80 dB)</td>
<td>1.5</td>
<td>15</td>
<td>( \rightarrow \infty )</td>
</tr>
<tr>
<td>3</td>
<td>( 10^{-10} ) (–100 dB)</td>
<td>0.15</td>
<td>15</td>
<td>( \rightarrow \infty )</td>
</tr>
</tbody>
</table>

Fig. 46: Three options of noise spectra.  
Fig. 47: Diffusion coefficient.

8.1.1 Zero Boundary-Value at Separatrix

Actually, this case presumes beam halo as having a vanishing, zero population. This assumption is expected to be very close to reality (refer to Section 7). In this context, a primary evaluation of noise contamination in the HERA-p ring is accomplished so as to comply at most with observations available.

Fig. 48 and Fig. 49 show bunch length dilution and decay of beam population found with a diffusion equation. Dimensionless time \( t_1 \) is defined in (100).
Solid line in Fig. 48 plots effective bunch half-size $\sigma_{q1}$ introduced by (106). Abscissa $t_1 \equiv 0.25 \cdot 10^6$ of a crossing point between initial slope and a level of twice the injected beam size corresponds to initial instantaneous bunch-length doubling time $\tau_{\sigma2}$. Specification [1] sets $\tau_{\sigma2}$ to about 10 hr. Hence, power density of phase noise can be estimated as

$$\frac{P_{\nu \phi}}{V_{RF}^2} = \frac{t_{x*}}{\tau_{\sigma2} \cdot \Omega_0^2} \approx 1.1 \cdot 10^{-4} \text{rad}^2/\text{Hz}, \text{ or}$$

$$\frac{P_{u}(\Omega_0)}{V_{RF}^2} \approx 1 \cdot 10^{-9} \text{rad}^2/\text{Hz}.$$  \hspace{1cm} (107)

Estimated loss of bunch population is $\leq 30\%$ by the end of 10 hr long run, Fig. 49.

![Fig. 48: Evolution of bunch size.](image)

![Fig. 49: Decay of bunch population.](image)

Fig. 50 and Fig. 51 show evolution in time $t_1$ of distribution function $f(J, t)$ and of line density $\lambda(\phi, t)$ of a bunch. The latter is found as

$$\lambda(\phi, t) = \int_{-\pi}^{\pi} \int_{-2\cos^2}^{+2\cos^2} f(J(\phi, \xi), t) \, d\xi / \Omega_0,$$ \hspace{1cm} (108)

where function $J = J(\phi, \xi)$ is recovered from $J(k)$ given in (99) and $k^2 = k^2(\phi, \xi)$ of (92). Each profile corresponds to noise option #1 of Table 6, and is shot at equidistant time intervals of $t_1/10^6 = 0.0(0.1)0.5$.

With $\lambda(\phi, t)$ available, one can restore the standard r.m.s. half-length of a bunch according to

$$\sigma^2(t) = \frac{\int_{-\pi}^{\pi} \phi^2 \cdot \lambda(\phi, t) \, d\phi}{\int_{-\pi}^{\pi} \lambda(\phi, t) \, d\phi}.$$ \hspace{1cm} (109)

Generally, definition (106) results in $\sigma_{q1} \neq \sigma_{q}$. To estimate the difference, variances $\sigma_{q}$ for line densities of Fig. 51 are plotted by bullets in Fig. 48. Effective beam size $\sigma_{q1}$ is well representative. It coincides with $\sigma_{q}$ for short bunches, and is shorter than $\sigma_{q}$ by only about 10% by end of a long run.

For reference, empty circles in Fig. 48 plot the relevant half widths at half maximums (HWHM) of a bunch that are commonly used, along with the FWHM=$2 \times$HWHM, in beam diagnostics of the HERA-$p$ ring.
Fig. 48 shows that law of $\sigma_\phi(t)$ saturates with $t_1 \rightarrow \infty$. The stationary value does not depend upon initial size of a bunch. This effect is due to the ultimate survival and dominance of a leading term in Fourier–Bessel expansion of a solution of a boundary-value problem for the diffusion equation in question. Indeed, numerical solution of the eigenvalue problem associated with (64) readily yields the dominant eigenfunction that is the only one to persist at $t_1 \rightarrow \infty$. This function is plotted in Fig. 52. It is normalised to 1 at $J = 0$, and is apparently approached to by end of the run, see Fig. 50.

Recovering line density by integrating the leading eigenfunction with (108) reveals asymptotic shape of a bunch plotted in Fig. 53. Table 7 lists anticipated size of such a bunch. Stationary value of bunch-length is $\sigma_s = 20$ cm or 1.7 nsec (FWHM). This value complies with beam observations in the HERA-$p$ ring, refer to Fig. 1 of [6].

The last column of Table 7 specifies the law of ultimate beam population decay. It is derived from computed value of the (smallest) eigenvalue corresponding to the leading eigenfunction plotted in Fig. 52. Ultimate life-time of beam population ($N_B(t) \propto \exp(-t/\tau_N)$) is estimated as $\tau_N \equiv (15–17)$ hr.
Table 7: Parameters of a dominant term in Fourier–Bessel expansion

<table>
<thead>
<tr>
<th># of $P_u(\Omega)$</th>
<th>$\sigma_{\phi_1}$, RF rad</th>
<th>$\sigma_\phi$, RF rad</th>
<th>HWHM, RF rad</th>
<th>Law of a terminal decay</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.81</td>
<td>0.89</td>
<td>1.11</td>
<td>$\propto \exp(-2.61 \cdot t_1/10^9)$</td>
</tr>
<tr>
<td>2, 3</td>
<td>0.80</td>
<td>0.88</td>
<td>1.08</td>
<td>$\propto \exp(-2.37 \cdot t_1/10^9)$</td>
</tr>
</tbody>
</table>

Contrary to beam life-times $\tau_\phi$ and $\tau_n$, asymptotic shape of the bunch and its geometrical parameters like $\sigma_{\phi_1}$, $\sigma_\phi$ and HWHM do not depend upon a level of noise in the system. Rather, they are signatures of the noise type per se (either phase or amplitude), and of a shape of a noise power spectrum $P_u(\Omega)$ seen in the reduced frequency scale $\propto \Omega/\Omega_0$. Observation of these parameters yields a valuable insight into properties of noise.

### 8.1.2 Zero Boundary-Value at Aperture, Diffusion Equation & Tracking

Now, absorbing boundary is moved to $J_A = 4.5 \cdot J_S$. Noise option #1 of Table 6 is taken. To ease accumulation of dimensionless time $t_2$ (101) in tracking computations, energy losses are enhanced artificially by adopting $\tan \phi_0 = 0.002 << 1$. Beam evolution is studied for an interval of $t_2 = 0$–100. (It stands for a 10 hr long run at $\gamma E_0 = 920$ GeV and drift velocity $V$ of around $9.5 \cdot J_S$/hr.) To accumulate this time, about 500,000 map iterations are required. A 5-sample averaging of random process is performed prior to plotting.

NDR (104) is set to 2500 which corresponds to noise level (107) recovered from beam observations, and energy losses due to SR of 920 GeV protons. (More relevant is a value 0.023 of NDR$_1$ (105) seen by beam. It indicates that HERA-$p$ ring indeed represents a drift-dominated case. An example of beam halation effect studied in Section 7 has its apparent NDR$_1$ higher by an order of magnitude, around 0.33.)

Fig. 54 shows evolution in time $t_2$ of beam population inside bucket (lower curves, $N_B(t_2)$) and aperture (upper curves, $N_A(t_2)$). Thick lines plot solutions of a diffusion equation, thin lines — those of tracking. In a whole, it is essentially the earlier Fig. 49 for a bunch population decay that is reproduced in a different time scale and range.

Fig. 55 shows evolution of beam r.m.s. half-length $\sigma_{\phi_1}$ (106). Tracking curve predicts a bit lengthier bunch than diffusion equation. (A similar situation has been encountered earlier in Fig. 41). Again, Fig. 48 plotting $\sigma_{\phi_1}(t_2)$ for zero boundary-value condition at $J = J_S$ complies with Fig. 55.
ing histograms in Fig. 58 and Fig. 59. Due to a small value of NDR, lower-half plane halo is crabbed to a very narrow outer vicinity of a separatrix. In general, halo profiles are consistent with results yielded by a diffusion equation, cases #2 and 3 of Fig. 21.

![Fig. 56: Longitudinal distribution inside RF bucket.](image1)

![Fig. 57: Longitudinal distribution beyond RF bucket.](image2)

Fig. 56: Longitudinal distribution inside RF bucket.

Fig. 57: Longitudinal distribution beyond RF bucket.

![Fig. 58: Longitudinal distribution inside RF bucket, by end of a run.](image3)

![Fig. 59: Longitudinal distribution beyond RF bucket, by end of a run.](image4)

Fig. 58: Longitudinal distribution inside RF bucket, by end of a run.

Fig. 59: Longitudinal distribution beyond RF bucket, by end of a run.

Fig. 60 shows how beam current is accumulated in beam halo region. Thick line plots a difference between two curves of Fig. 54, thin line is a result of tracking calculations. Top value of halo population is around 1.5%. Diffusion equation yields reliable estimates of the net population of beam halo.

Fig. 61 is obtained with tracking entirely. It shows that partition of particles between lower and upper half-planes in halo region is as small as 1/100 ca. Diffusion is strongly superceded by drift in the lower half-plane, and coasting-beam particles constitute only a thin layer beyond of RF buckets in a half-plane that faces a head-on crabbing (positive momentum offsets).

For a given set of parameters (the NDR) consistent with evolution of beam bunched-core observables, there must be virtually no coasting beam in the lower half of phase-space plane \((\varphi, \xi)\).
Still, beam observations in the HERA-$p$ ring show that there is an asymmetry in bunch-crossing non-correlated readouts from the HERA–B wire targets of around 5-to-1 in favour of the outer wire (refer to Fig. 4 of [2]). This effect can be explained by referring to Fig. 4. Indeed, the upper, coasting beam populated half plane of $(\phi, \xi)$ is scraped along either curve BD (image of the outer wire) or EF (the inner wire). In the latter case, the wire samples less populated tails of the beam distribution due to the larger betatron amplitudes $A_x$ involved.

![Fig. 60: Accumulation of net current in beam halo.](image)

![Fig. 61: Partition of halo population between lower and upper half-planes.](image)

8.2 Amplitude Noise

8.2.1 Zero Boundary-Value at Separatrix

Studies of phase noise indicate that assumption $f(J_S, t) = 0$ is quite representative for the HERA-$p$ ring as long as questions other than beam halation are concerned. This case is well sufficient to demonstrate that amplitude noise should be condoned as a suspected cause of the bunch dilution effect observed.

![Fig. 62: Evolution of bunch size.](image)

![Fig. 63: Line density of a bunch.](image)

Indeed, solid line in Fig. 62 plots effective bunch half-size $\sigma_{\phi_1}$ (106), bullets mark standard deviation $\sigma_{\phi}$ (109) and empty circles — HWHM of a bunch. Fig. 63 shows line den-
sity $\lambda(\varphi, t)$ of a bunch at $t_1/10^6 = 0.0(0.1)0.5$. All this data is recovered with a diffusion equation.

There are at least three features of bunch evolution under amplitude noise that are not confirmed with beam observations in the HERA-\(p\) ring:

1. Shorter (by $>10\%$) beam r.m.s. sizes at $t_1 \to \infty$.
2. Overshooting sustained levels of $\sigma_\varphi$ and $\sigma_\varphi$ and approaching them from above in a long enough run.
3. Eventual (monotonous) decrease in half widths at half maximums (HWHM) due to maintained bunched core and depopulated bunch tails.

To this end, **it is definitely not an amplitude noise** of RF system that shows itself up in the HERA-\(p\) ring, at least, for the time being.

### 8.3 Elementary Theory of Halo Accumulation: the Drift-Dominated Case

Assume a strong dominance of drift over diffusion transport in the halo region. This is the case for the HERA-\(p\) ring where NDR, NDR$_1 \ll 1$.

Net halo population $N_H(t)$ is confined, mainly, to the single half-plane (refer to Fig. 57, Fig. 59). On applying to a conservation principle of particles and a continuity principle of fluxes, one gets

$$ - \frac{dN_B(t)}{dt} = Q(J_s, t) = V \cdot f(J_s, t) $$

where $N_B(t)$ denotes population inside bunch, and the second term of (25) is ignored. The halo profile is presumed to be transported by drift mechanism with only.

$$ f(J, t) = f\left(J_s, t - \frac{J - J_s}{V}\right), \quad J_s \leq J \leq J_A, $$

where $J_A$ is aperture limitation in terms of $J$. Net population accumulated in halo region by time $t$ is

$$ N_H(t) = \int_{J_s}^{J_A} f(J, t) \, dJ. $$

On inserting (111) followed by (110), one gets

$$ N_H(t) = - \int_0^{\frac{J - J_s}{V}} \frac{dN_B}{dt}(t - t_s) \, dt_s. $$

Applying to the mean-value theorem results in

$$ N_H(t) = - \frac{dN_B}{dt}(t_0) \cdot \frac{J_A - J_s}{V}, \quad t_0 = t - \theta \cdot \frac{J_A - J_s}{V}, \quad 0 < \theta < 1. $$

Central-point approximation for the delayed time $t_0$ corresponding to the value of $\theta = 1/2$ proves to meet all the practical demands.

In the terminal, exponential phase of beam decay when $N_B(t) \propto \exp(-t/\tau_N)$, influx of particles through separatrix can be estimated as

$$ - \frac{dN_B(t)}{dt} \equiv \frac{N_B(t)}{\tau_N}. $$

On inserting this equation into (114), one finds out that ratio of halo to bunch populations would saturate with $t \to \infty$ at a level of
\[
\frac{N_H(t)}{N_B(t_0)} \simeq \frac{J_A - J_S}{V \cdot \tau_N} \propto \frac{\Omega_0^2 P_v}{V_{RF}^2} \frac{J_S}{V} \left( \frac{J_A}{J_S} - 1 \right)
\]  
(116)

The leading factor in right-hand side of this equation is the Noise-to-Drift Ratio (104) — the input parameter for both the codes, to solve diffusion equation and to track macro-particles.

Equation (116) suggests experimental technique to measure the NDR. Its denominator contains a well-defined quantity \( V \), and overall level of noise can thus be diagnosed afterwards.

According to Section 2.2.1, in the HERA-\( p \) ring drift velocity \( V \) is around 9.5(6.3) \( J_S/\text{hr} \) for beam energies \( \gamma E_0 = 920(820) \text{ GeV} \), respectively. Let, for definiteness, the aperture limitation be encountered at \( J_A = 4.5 J_S \). As it follows from Section 8.1.1, ultimate life-time \( \tau_N \) of beam population is around 16 hr. Hence, due to (116),

\[
N_H(t)/N_B(t_0) \simeq 2.3\% \ (920\text{GeV}), \quad \text{or} \quad \simeq 3.5\% \ (820\text{GeV}).
\]

Exponential phase of beam decay begins since about a 10 hr age of a fill when \( N_H(t_0)/N_B(0) \) is close to 0.7. Therefore, top value expected of the halo population is

\[
N_H(t)/N_B(0) \leq 1.6\% \ (920\text{GeV}), \quad \text{or} \quad \leq 2.4\% \ (820\text{GeV}).
\]

Fig. 60 confirms this estimate for the 920 GeV beam. Primary paper [1] also specifies \( N_H(t)/N_B(0) \) as (1–2)% ca for 920 GeV protons.

Due to the retarded drift velocity \( V \), halo region must be more populated for 820 than for 920 GeV protons in the HERA-\( p \) ring, other conditions being equal. This fact is confirmed by observations as well, refer to pp. 92, 96 of [6].

8.4 Conclusion

As far as the HERA-\( p \) ring is concerned:

1. Phase noise with a spectrum \( P_\phi(\Omega) \propto \Omega^2 \) and a level of about \( 10^{-9} \text{ rad}^2/\text{Hz} \) at frequency \( \Omega_0/2\pi = 40 \text{ Hz} \) plus a drift beyond RF buckets due to synchrotron radiation losses can account for most of the coasting beam halo phenomenology available.

2. With a small value of Noise-to-Drift Ratio (NDR\( \simeq 1/40 \)) in question, studies of the bunched core evolution can be reliably accomplished by solving a conventional 1D diffusion equation in action variable with a zero boundary-value condition at separatrix.

3. Net accumulated coasting beam current is directly proportional to a value of NDR.

4. For a value of the NDR in question, partition number of coasting beam component in between upper and lower half-plane is only around 1/100. To this end, there must be virtually no coasting beam halo in the lower half-plane of \( (\phi, \xi) \).
9 Comparison with Beam Observations

This Section ascertains the RF noise model in question of coasting beam halo effect in the HERA-\(p\) ring so as to account for finer details of beam observations. It takes into account the comments and suggestions put forward by the DESY accelerator experts in course of the approval period for this study (May–August, 2001).

9.1 Sustained Length of a Bunch

Fig. 64 plots measured \(\rho\)-bunch length (FWHM) in a typical run of the HERA-\(p\) ring. It is a citation from \([6]\), \([7]\). The 22 hr long run starts with FWHM = 1.4 nsec, ultimate FWHM being 1.85 nsec.

The latter figure exceeds by about 10\% the stationary value of FWHM = 1.7 nsec obtained in Section 8. There, the assumption of higher frequency cut-off \(\Omega_H \to \infty\) in noise power spectrum is followed, refer to Table 6. However, even in that (worst) case one fails to ensure a diffusion transport beyond RF buckets sufficient to overcome drift due to the SR so as to accumulate a noticeable beam halo in the lower half-plane of \((\phi, \xi)\). To this end, the goal of keeping the suspected noise variance as large as possible may no longer be pursued, and adjusting the higher cut-off \(\Omega_H\) of the noise power spectrum model (Section 5) can be employed to match exactly the stationary bunch size to beam observations.

Fig. 64: Measured \(\rho\)-bunch length versus time in a typical run.

Fig. 65: Noise spectra with finite higher frequency cut-offs.

Fig. 66: Diffusion coefficients.
Decreasing $\Omega_H$ does not only hamper diffusion beyond RF buckets, but also reduces contribution of higher multipoles to series (55). Therefore, diffusion at the outskirts of bunch is suppressed which would result in lengthier bunches. Fig. 65 plots spectral power densities $P_u(\Omega)$ with a DC level $g = 10^{-6}$ and cut-off frequencies $\Omega_H/\Omega_0 = 2.0(0.5)4.0$. The resultant diffusion coefficients $D(J)$ are plotted in Fig. 66. Fig. 67 shows how FWHM of a bunch at $t \to \infty$ depends upon higher cut-off frequency of the noise power spectrum. To get the observed ultimate FWHM = 1.85 nsec, one has to assume

$$\Omega_H/\Omega_0 \equiv 2.3.$$  

(117)

Fig. 67: Value of bunch FWHM at $t \to \infty$ versus cut-off frequency of noise spectra.

Horizontal lines in Fig. 68 reproduce the experimental data borrowed from Fig. 64. Data points show results of calculations performed with the diffusion equation under zero boundary-value condition at separatrix. Crosses present calculated evolution of a Gaussian bunch launched with the distribution (32) whose parameter $J_d/J_S = 0.12$ corresponds to the initial FWHM = 1.4 nsec. There is a noticeable discrepancy between calculations and beam observations for the intermediate $t \leq 12$ hr. All the more, Fig. 69 (lower curve) shows that decay of bunched core in this case starts abruptly at $t = 0$ without a flat-top plateau, that is not confirmed by beam observations as well.

The culprit to be blamed might be an inconsistent assumption about the distribution function of a bunch at $t = 0$. Indeed, function $f(J, t = 0)$ from (32) has a too extended tail. Another candidate for beam distribution at $t = 0$ is a bunch with a parabolic shape in terms of variables $\varphi, \xi \propto \delta p$. This hard-edge profile is commonly encountered in proton beams. It corresponds to the square-root function

$$f(J, t = 0) = \begin{cases} \frac{3}{2J_0} \sqrt{1 - \frac{J}{J_0}}, & J < J_0; \\ 0, & J \geq J_0; \end{cases}$$

(118)

where $J_0$ stands for action variable of the phase-plane trajectory at foot of the distribution. This function is normalised to 1. Parameter $J_d/J_S = 0.32$ yields the initial FWHM = 1.4 nsec. Data points marked with bullets in Fig. 68 show elongation of the bunch (118) that now perfectly fits the bunch lengthening observations. In its turn, decay of the bunched core proceeds along a more familiar pattern, along the upper solid curve in Fig. 69.

Still, there remains a point at which the plot for a parabolic bunch in Fig. 69 is not fully confirmed by observations: the total $p$-loss in a 10 hr run is normally around 5–10% [4], rather than 18% as in Fig. 69. These losses might have been decreased to some 15% (see
dashed curve in Fig. 69) if the run were launched with a shorter bunch of FWHM = 1.2 nsec or $J_0/J_S = 0.24$, the latter bunch length being mentioned in [11]. Switching on the 52 MHz RF system at the levels of $\geq 2 \times 20$ kV is shown in Section 9.3 to be responsible for the further decrease of the bunched current losses down to the observed values of 5–10% per 10 hr.

Fig. 68: Lengthening of a bunch, experimental against calculated data.

Fig. 69: Decay of $p$-bunched current.

### 9.2 Allan’s Variance of the 208 MHz Radio-Frequency Gap Voltage

Consider a stationary random process $w(t)$ which is an instantaneous fractional deviation of the radio-frequency,

$$ w(t) = \frac{\delta \omega_{\text{RF}} (t)}{\omega_{\text{RF}}} = \frac{d \delta \varphi(t)}{\omega_{\text{RF}} dt}. \quad (119) $$

In compliance with the model of a noise source employed in this paper (refer to Section 5), spectral power density of $w(t)$ reads

$$ P_w(\omega) = \frac{P_v}{V_{\text{RF}}^2} \cdot \frac{\omega^2}{\omega_{\text{RF}}^2} \cdot \frac{\omega_H^8}{\omega_H^8 + \omega^8} \cdot \frac{\omega_{L1}^2 + \omega^2}{\omega_L^2 + \omega^2}. \quad (120) $$

Here, the first factor converts a quadrature voltage error $v(t)$ into a phase error, the second factor accounts for the derivative $d/\omega_{\text{RF}} dt$ in (119), and the two trailing factors stand for the cascade connection of (77) and (84).

Auto-correlation function of $w(t)$ can be recovered as the inverse Fourier transform of (120) and reads

$$ \langle w^2(\tau) \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_w(\omega) \exp(-i\omega \tau) d\omega = \frac{P_v}{V_{\text{RF}}^2 \omega_{\text{RF}}^2} \times $$

$$ \times \left[ \frac{1}{2} \omega_L \cdot \left( \omega_L^2 - \omega_{L1}^2 \right) \frac{\omega_H^8}{\omega_H^8 + \omega_L^8} \cdot \exp(-\omega_L |\tau|) - \right. $$

$$ - \frac{1}{4} \text{Re} \left( b^3 \omega_H^3 \cdot \frac{\omega_{L1}^2 - b^2 \omega_H^2}{\omega_L^2 - b^2 \omega_H^2} \cdot \exp(-b \omega_H |\tau|) \right) + $$

$$ + \frac{1}{4} \text{Im} \left( b^3 \omega_H^3 \cdot \frac{\omega_{L1}^2 + b^2 \omega_H^2}{\omega_L^2 + b^2 \omega_H^2} \cdot \exp(i b \omega_H |\tau|) \right), \quad b = \exp \left( i \frac{\pi}{8} \right) $$

Since $P_w(0) = 0$, function $\langle w^2(\tau) \rangle$ must have zero average value.

Instead of $w(t)$, time-domain measurements of frequency stability involve the time-averaged random process $y(t)$
where $T$ is averaging (gate) interval.

The qualitative measure of frequency stability registered via counter method with the zero dead time is the so called Allan’s variance [13]. It is defined as

$$
\sigma_y^2(T) = \frac{1}{2} \langle (\Delta y)^2 \rangle = \frac{1}{2} \left\langle (y(t+T) - y(t))^2 \right\rangle
$$

and is normally measured for a set of different $T$'s.

Due to stationarity of $y(t)$, the Allan’s variance is readily reducible to the auto-correlation function of $y(t)$

$$
\sigma_y^2(T) = \langle y^2 \rangle(0) - \langle y^2 \rangle(T).
$$

Inserting the primary random process $w(t)$ from (122) and rearranging the limits of 2D integration results in

$$
\sigma_y^2(T) = \frac{1}{T^2} \int_0^T dt_1 \int_{-T}^{+T} dt_2 \left( \langle w^2 \rangle(\tau_2) - \langle w^2 \rangle(\tau_2 + T) \right).
$$

Now, one has only to double-integrate the (complex) exponents in (121) according to the prescription of (125) to obtain the analytical expression for Allan’s variance relevant to the noise model (120):

$$
\sigma_y^2(T) = \frac{P_{v0}}{V_{RF}^2 \omega_{RF}^2} \times
$$

$$
\left\{ \frac{1}{2} \omega_l \left[ (\omega_l^2 - \omega_{\Omega_0}^2) \cdot (\omega_{\Omega_0}^2 + \omega_l^2) \right] F(\omega_l T) -
\right.
$$

$$
- \frac{1}{4} \Re \left[ b^2 \omega_l^3 \cdot (\omega_{\Omega_0}^2 - b^2 \omega_l^2) F(b \omega_{\Omega_0} T) \right] +
$$

$$
+ \frac{1}{4} \left[ b^2 \omega_l^2 \cdot (\omega_{\Omega_0}^2 + b^2 \omega_l^2) F(-i b \omega_{\Omega_0} T) \right], \quad b = \exp \left( i \frac{\pi}{8} \right)
$$

with

$$
F(z) = \frac{2}{z} \left[ \exp(-z) - 1 + z \right] - \frac{1}{z^2} \exp(-z) \left[ \exp(z) + \exp(-z) - 2 \right],
$$

$$
F(0) = 0, \quad F(z) \rightarrow \frac{2}{z} \text{ as } \text{Re} z \rightarrow \infty.
$$

The suspected noise data compliant with the first line of Table 6, level (107) and the higher cut-off (117) is specified in Table 8. Fig. 70 shows the relevant plot of $\sigma_y(T)$ for all the four 208 MHz cavities in sum (solid line), and per one cavity on average (dashed line). The gap phase noise is assumed to be statistically non-correlated from cavity to cavity (hence, there is a factor of $1/\sqrt{4}$ between the total and per-cavity sigma’s). Data points connected by broken lines show results of the preliminary measurements of the HERA 208 MHz cavities [14]. There is a fair agreement between experimental and calculated data in time domain of $T = 1$–$100$ msec representing frequency domain of 500–5 Hz, respectively, where the bunch is prone to resonant excitation by the gap noise.

**Table 8: Parameters of a tentative noise power spectrum**

<table>
<thead>
<tr>
<th>$P_{v0}/V_{RF}^2$, rad²/Hz</th>
<th>$P_{v0}(\Omega_0)/V_{RF}^2$, rad²/Hz</th>
<th>$g$</th>
<th>$\Omega_{L_1}/\Omega_0$</th>
<th>$\Omega_{L_2}/\Omega_0$</th>
<th>$\Omega_{L_3}/\Omega_0$</th>
<th>$\Omega_0/2\pi$, Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1 $\cdot 10^{-4}$</td>
<td>$\approx 1 \cdot 10^{-9}$</td>
<td>$10^{-6}$</td>
<td>0.4</td>
<td>400</td>
<td>2.3</td>
<td>$\approx 40$</td>
</tr>
</tbody>
</table>
9.3 Motion under a Double-RF System

This case is amenable to a straightforward study with the tracking code.

9.3.1 Tracking Algorithm

The 52 MHz RF system of the HERA-\(p\) ring is operated at voltages \(V_{RF52} = 2 \times (20–100)\) kV per turn, nominal voltages being 2\(\times\)70 kV [7]. Denote ratio of the two RF voltages as

\[
\kappa = \frac{V_{RF52}}{V_{RF208}}.
\]

(128)

Both the RF harmonics are sampled by a reference particle at stable-phase angle \(\vartheta_s = \vartheta_s(\kappa)\) with \(\vartheta_s(0) \equiv \varphi_s\) from (9). Losses due to the SR are small, and the stable-phase angle is very close to 0°. To this end, tracking algorithm (90), (91) is readily upgraded to the double-RF case by adding the 1/4-th sub-harmonic voltage

\[
\sin \varphi_1 \rightarrow \sin \varphi_s + \kappa \cdot \sin \left(\frac{\varphi_1}{4}\right)
\]

(129)

\[
\tan \varphi_s \cdot (1 - \cos \varphi_1) \rightarrow \tan \varphi_s \cdot (1 - \cos \varphi_s) + \kappa \cdot \tan \left(\frac{\varphi_s}{4}\right) \left(1 - \cos \left(\frac{\varphi_1}{4}\right)\right)
\]

with \(\varphi_s \approx \frac{\varphi_0}{1 + \kappa/4} << 1\).

Average drift velocity \(V\) along \(J\) beyond the outer RF bucket is kept unchanged in a time scale required for a halo particle to cover ±8\(\pi\) distance along \(\varphi\). Mapping parameters (\(\Omega_0, T_0, \tan \varphi_s,\) NDR (104), \(t_2\) (103)) are set with respect the former, single-RF case. Action variable \(J\) is measured in units of \(J_S\), the latter still being phase-plane area of the pure 208 MHz separatrix, (5).

The initial distribution is launched through 6,000 macro-particles according to (95)–(99) for \(\kappa = 0\). To ensure matching \(f(J, t)\) to the new potential well and eliminate coherent transients, parameter \(\kappa\) is switched on slowly from zero to (128) during some 100 noise-free synchrotron oscillations. The injected distribution is given by (118) with its size at foot \(J_0/J_S = 0.24\) corresponding to bunch FWHM = 1.2 nsec. Noise parameters are tabulated in Table 8, and the noise is applied to the 208 MHz RF harmonic only.
Aperture limitation is moved to $J_A = 10J_S$. It corresponds to momentum acceptance of about $\pm 2 \cdot 10^{-3}$ in fractional momentum off-set (paper [2] allows for up to $\pm 5 \cdot 10^{-3}$). It is this value of momentum acceptance that, to the final end, proves to be compliant with observed accumulation of $\leq (1–2)\%$ of injected particles in the coasting beam halo under the standard operating voltages $V_{RF52} = 2 \times (50–70)$ kV, see Fig. 74 below.

Fig. 71: Coasting beam halo effect in a double-RF system.

Fig. 71 presents snapshots of phase-space plane at start and by the end of a 20 hr long run (at dimensionless times $t_2 = 0$ and 200) for a representative sample of the random process and the nominal $V_{RF52} = 2 \times 70$ kV. The lost macro-particles are stuck to the absorbing boundary at $J_A = 10J_S$ once they hit it, and are out of processing afterwards. There is no coasting beam halo in the lower half-plane of $(\phi, \xi)$ since here the noise variance is further reduced by assuming the finite cut-off (117) in the power spectrum. The new underlying mechanisms involved are: 1. trapping the particles inside the 52 MHz (outer) bucket, 2. re-filling the empty 208 MHz side-bunches via return diffusive fluxes from the area between the 208 and 52 MHz separatrices, and 3. a modified susceptibility of bunches to noise due to the potential-well distortion.

9.3.2 Decay of the Bunched Core

Fig. 72 shows evolution of beam population in the bunched core (inside the outer 52 MHz RF bucket, the lower curves) and within the entire aperture (DC current, the upper curves) for various values of $\kappa$. A 10-sample averaging is accomplished prior to plotting to minimise shot ripple of the tracking model arising due to the finite number of macro-particles involved. For reference, dashed line shows the solution obtained for $\kappa = 0$ with the diffusion equation under zero boundary-value condition at separatrix.

Lower curves in Fig. 72 indicate that the dominant apparent effect of adding the 52 MHz RF harmonic to the main 208 MHz accelerating field is to slow down diffusion of particles through the outer RF bucket.

Indeed, Fig. 73 shows dimensionless ($\propto t_2$ of (101)) expectation times $\tau_3$ for a certain, 1–10% decay of the bunched core versus $\kappa$. These plots are recovered from the tracking data of Fig. 72 and allow to estimate the phenomenological slow-down factor of diffusion due to the second RF system as
\[
\frac{\tau_2(\kappa)}{\tau_2(0)} = 1 + (5.0 \pm 0.4) \cdot \kappa. \quad (130)
\]

In other words, the major deal of difference between the leftmost single-RF curve and the (lower) curves for all the cases with \( \kappa \neq 0 \) in Fig. 72 is reducible to a plain extension of scale along the abscissa axis by the factor of (130).

The numerical coefficient in (130) might be subject to change together with a shape of the noise spectrum.

Fig. 72: Decay of beam population. Fig. 73: Bunched-core-decay life-time.

9.3.3 Accumulation of Coasting Beam Halo

Fig. 74 shows how coasting beam halo is accumulated. It plots the differences between upper and lower curves of Fig. 72. These (differential) curves are still plagued by the shot noise of the model. Halo build-up starts roughly obeying the \( t^2 \)-law. Dashed line in Fig. 74 sketches the generic law of halo accumulation for \( \kappa = 0 \) suggested by equation (114)

\[
\frac{N_H(t)}{N_B(0)} = \frac{1}{N_B(0)} \frac{dN_B(t_{1/2})}{dt}, \quad t_{1/2} = t - \frac{1}{2} J_A - J_S
\]

where the decay rate \(-dN_B(t)/dt\) of bunch population \( N_B(t) \) is obtained via the diffusion equation as flux \( Q(J_S, t) \) through separatrix, together with the dashed curve of Fig. 72.

In the drift-dominated case under study, the law of (131) proves to be applicable to cases of \( \kappa \neq 0 \) as well. Suffice it to take into account the two factors (in priority order):

1. the suppressed influx feeding the halo — the bunch decay rate is slowed down roughly according to scaling (130) that entails

\[
\frac{dN_B}{dt}_{\parallel V_RF=0} = (t) \approx \frac{\tau_2(0)}{\tau_2(\kappa)} \frac{dN_B}{dt}_{\parallel V_RF=0} \left( t - \frac{\tau_2(0)}{\tau_2(\kappa)} \right), \quad (132)
\]

2. less phase space available to accommodate the halo particles — \( J_S \) should be replaced by about 1/3 of phase-plane area encircled by the outer separatrix, refer to the last column of Table 9.

As an example, Fig. 74 demonstrates results of applying prescription (131) to, say, \( V_{RF2} = 2 \times 50 \text{ kV} \). Now, the derivative over time \( dN_B(t)/dt \) is recovered numerically from the tracking data (Fig. 72) and is then subjected to smoothing. Closeness of the two curves plotted...
for \( V_{RF2} = 2 \times 50 \text{ kV} \) confirms that tracking data for \( \kappa \neq 0 \) follows the drift-dominated halo accumulation law (131).

![Fig. 74: Accumulation of current in beam halo.](image)

Table 9: Dimensions of the outer bucket in double-RF system

<table>
<thead>
<tr>
<th>( \kappa )</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>2×0/600 = 0</td>
<td>±1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>2×20/600 = 0.067</td>
<td>±1.11</td>
<td>3.54/3</td>
</tr>
<tr>
<td>2×50/600 = 0.167</td>
<td>±1.25</td>
<td>4.14/3</td>
</tr>
<tr>
<td>2×70/600 = 0.233</td>
<td>±1.34</td>
<td>4.48/3</td>
</tr>
<tr>
<td>2×90/600 = 0.300</td>
<td>±1.43</td>
<td>4.79/3</td>
</tr>
<tr>
<td>2×110/600 = 0.367</td>
<td>±1.51</td>
<td>5.09/3</td>
</tr>
</tbody>
</table>

A: Peak momentum spread in relative units.  
B: Phase-plane area per \( 2\pi \) of \( \phi \), in units of \( J_S \).

9.3.4 Effect of the Two RF systems: the Conclusion

Smooth behaviour of Fig. 72–Fig. 74 versus \( \kappa \) entails that the suspected interplay between the two RF systems brings about no qualitatively new features to beam halation effect in the HERA-\( p \) ring, as compared to the single-RF case.

From the view-point of the rough integrated observables — bunched, DC and coasting beam currents — switching on the second RF system is seen, mainly, as a scale-down of apparent susceptibility of bunches to noise in compliance with the phenomenological law (130). Also, less longitudinal phase space is available for halo particles to survive since this space is now demanded by increased acceptance of the outer RF buckets. Both these factors tend to decrease the coasting beam accumulated in line with (131), (132) and (130).

9.4 Conclusion

Study of the RF noise production mechanism of the coasting beam halo in the HERA-\( p \) ring offers enough free parameters. Adjusting them in feasible ranges makes the calculations compliant simultaneously with the diverse experimental data available by now.
10 Experimental Verification and Cures (Proposals)

In this Section, a tentative program is proposed to verify external RF noise as a possible source of coasting beam halo observed in the HERA-\(p\) ring of DESY. It should be accomplished under plain conditions — with the single-RF 208 MHz system, the 52 MHz system being switched off or operated at as low voltages as possible. The major bulk of the program must not intervene into the conventional operation of the machine.

New diagnostic tools to be developed and implemented are
1. mostly based on the current state of the art available in the HERA-\(p\) beam instrumentation system \([9], [10] \) and \([11]\),
2. might be subsequently integrated into feedback/compensation circuitry in the low level RF system of the HERA-\(p\) ring to cure the alleged adverse effect of noises.

10.1 Experimental Tests

Noise-to-Drift Ratio \((104), (105)\) is a key parameter of the problem in question. It can be varied by changing either its nominator or denominator.

The nominator can be affected by reshaping apparent noise power spectrum \(P_u(\Omega)\) at \(\Omega\) around \(\Omega_0\), the frequency of synchrotron oscillations. As the first step, an observable variation in \(P_u(\Omega)\) can be accomplished by re-tuning parameters (gain and bandwidth of integrating circuit) of the DC-coupled RF phase feedback circuit. Should this procedure fail to affect the diffusion, noise sources other than RF power amplifier and cavity will be inspected, Table 4.

The denominator of the NDR, the drift velocity, can be varied in a dedicated run with a lower top energy \(\leq 820\) GeV. In this case, of utmost practical importance is to keep other conditions equal.

In both the cases, a set of beam observables should be detected and processed to recover life-times \(\tau_{d}\), \(\tau_{\nu}\) of the bunched core that are both inversely proportional to \(P_u(\Omega_0)\). Asymptotic (at \(t \to \infty\)) line density of a bunch depends upon a shape of \(P_u(\Omega)\) seen versus a reduced frequency scale \(\propto \Omega/\Omega_0\). An important indicator of noise contamination is a “\(p\) DC – Bunched” signal normalised by the (properly delayed) “\(p\)-Bunched” current. It is expected to saturate with \(t\) by the end of 10 hr long run at a level proportional to the Noise-to-Drift Ratio. This level must comply with momentum acceptance of the ring \(J_A\), drift velocity \(V\) due to SR losses and exponential life-time \(\tau_{\nu}\) of the bunched core population, \((116)\). It is an important point to verify the cross-consistency of diagnosing the bunched-core and coasting-beam observables.

10.2 A New Diagnostic Tool

Since the HERA-\(p\) ring is operated with lengthy bunches occupying more than \(\frac{1}{2}\) of the 208 MHz RF bucket at base, phase and amplitude noises may be equally dangerous in principle. It demands for a balanced approach to these noises from the very beginning. Two paths of control (possibly, of unequal gains) must be foreseen. To this end, a multi-purpose diagnostic channel may be first implemented to ensure:
1. analogue measurements with a low inherent Signal-to-Noise Ratio,
2. I/Q demodulation scheme with respect to 208 MHz VCO reference, and
3. operation in a band of about \(\pm (200–500)\) Hz around the RF carrier.

10.3 Gap Voltage Measurements

With such a diagnostic tool, a direct measurement of noise power spectra at gap is possible. The rear-end electronics with a FFT Analyser may be borrowed from the Shottky monitor \([11]\) already available.
10.4 Beam Measurements

The second application of the diagnostic tool in question is to use it together with a resistive gap monitor as a beam sensor. This circuit may monitor phase of beam centre-of-mass coherent oscillations with respect to the external VCO reference (the Q-signal) and a ripple in an average length of bunches (the I-signal).

One of goals possible is to study correlation, if any, between beam centre-of-mass phase readouts and ripple either in magnet power supply bus current, or in bending field integral of the reference dipoles. Such a study would allow to decide on a feasibility of employing an open-loop compensation control scheme en route “B-field — radio-frequency” so as to minimise residual phase errors at beam.

Another goal is to monitor spectra of noise signals transferred through phase/amplitude beam transfer function around the RF frequency.

![Fig. 75: Schematic diagram of AC-coupled beam feedback circuit.](image)

10.5 Closing Beam Feedback Circuit

Irrespective of results of the preceding item, an AC-coupled beam feedback loop may be closed. It is aimed at suppressing in-phase dipole/quadrupole oscillations of bunches, and lowering apparent noise spectrum inside the dipole/quadrupole band of beam incoherent longitudinal tunes.

The prerequisite of closing this feedback is a high quality of beam signals confirmed. As an actuator, the 52 MHz cavities may be employed [10] since

1. they are almost idle at flat-top,
2. their inherent amplitude and phase control loops are open and do not interfere,
3. linear inputs are available via unused feed-forward entries.

The I/Q decomposition of feedback signals and their two-path processing is to be employed throughout though a higher priority should be granted to a phase correction. A schematic view of the beam feedback is shown in Fig. 75.
Use of beam feedback circuit to control magnitude of beam coherent response to external perturbations might raise phase-noise limited longitudinal life-time of bunches in the HERA-\(p\) ring to > 100 hr ca (refer to experience of the CERN SPS ring, [12]).

10.6 Longitudinal Cleaning

Any procedures of longitudinal coasting beam halo clean-up by injecting a band-pass limited noise seem unfeasible. There already exists an embedded cleaning mechanism via energy losses due to the synchrotron radiation of protons. It is strong enough to crab away nearly all the coasting beam component from the lower half-plane of the longitudinal phase space (positive energy off-sets). Introducing a noisy diffusive transport ample to add substantially to such a drift beyond RF buckets would rather destroy the beam bunched core altogether.
11 Summary

Phase noise in the 208 MHz RF system with a level of about 10^{-9} \text{rad}^2/\text{Hz} at frequency off-set of 40 Hz plus a drift beyond RF buckets due to synchrotron radiation losses of 920(820) GeV protons can account for most of the coasting beam halo phenomenology available in the HERA-p ring of DESY.

Beam observations explainable in frames of this assumption are:
1. Initial instantaneous bunch length doubling time \( \tau_{\sigma^2} \approx 10 \text{ hr} \), starting from a 13 cm long bunch (at 1\(\sigma\)).
2. Monotonous saturation of bunch length with \( t \to \infty \) at the same stationary level, irrespective of bunch size at start of a run.
3. The value itself of the stationary bunch length (about 1.8 nsec FWHM).
4. About 5–10% loss of initial bunched core population by the end of 10 hr long run.
5. Top value of accumulated coasting beam current of around (1.5–2.0)% of initial beam population that is normally acquired by end of a 10 hr long run.
6. More coasting beam current for 820 than for 920 GeV protons, other conditions being equal.
7. Less coasting beam halo accumulated under the higher 52 MHz RF voltages applied.

For the Noise-to-Drift Ratios attainable in the HERA-p ring and consistent with observations over the bunched core degrade, partition number of the coasting beam component in between upper and lower half-planes of the longitudinal phase-space plane constitutes only about 1/100 or less. To this end, there must be virtually no coasting beam halo in the lower half-plane of the longitudinal phase-space plane (for positive energy off-sets).

Readouts at the HERA–B inner target wire non-correlated with bunch crossing signals are most likely to come due to protons from the populated upper half-plane with the negative energy off-sets and large betatron amplitudes.

The suspected interplay between the two RF systems brings about no qualitatively new features to beam halation effect in the HERA-p ring, as compared to the single-RF case. From the view-point of the rough integrated observables — bunched, DC and coasting beam currents — switching on the second RF system is seen, mainly, as a scale-down of apparent susceptibility of bunches to the RF noise.

The best countermeasure against coasting halo background is a struggle for shorter bunches and a longer life-time of a bunch longitudinally. Introducing beam feedback to suppress dipole/quadrupole mode of beam coherent in-phase oscillations (the coupled bunch mode \( n = 0 \)) is expected to be the most promising cure.

12 Acknowledgements

This work was performed under Attachment #63 to the Agreement between DESY (Hamburg, Germany) and IHEP (Protvino, Russia), re: Coasting Beam in HERA-p Ring, June 2000.

Special thanks to Ferdinand Willeke, Georg Hoffstaetter, Klaus Ehret, Wilhelm Kriens, Elmar Vogel, Richard Wagner, Christoph Montag and other members of the DESY staff for their valuable advises and support.
13 References


