Four dimensional gravity in the low energy limit of a higher dimensional theory has been expected to be a (generalized) Brans-Dicke theory. A subtle point in brane world scenarios is that the system of four dimensional effective gravitational equations is not closed due to bulk gravitational waves and bulk scalars. Nonetheless, weak gravity on the brane can be analyzed completely. We revisit the theory of weak brane gravity using gauge-invariant gravitational and scalar perturbations around a background warped geometry with a bulk scalar between two flat branes. We show that for general potentials of the scalar field (which does or does not provide radion stabilization) and a general conformal transformation to a frame in which matter on the branes are minimally coupled to the metric, 4-dimensional Einstein gravity, not BD gravity, is restored at low energies on either brane. In contrast, in RS brane world scenario without a bulk scalar, low energy gravity is BD one. We conjecture that in general brane world scenarios with more than one scalar field, one will again encounter low energy BD gravity with the BD parameter depending on the bulk scalar sector. Equipped with the weak gravity results, we discuss the properties of 4d brane gravitational equations, in particular, the value and sign of 4d Newton’s gravitational coupling.

I. INTRODUCTION

In the recent development of string/M theory [1], branes have been playing many important roles. The idea that our universe is a brane in a higher dimensional spacetime has been attracting a great deal of interest. Although the idea of the brane world had arisen at a phenomenological level already in 1983 [2], it is perhaps the discovery of the duality between M-theory and $E_8 \times E_8$ heterotic superstring theory by Horava and Witten [3] that made it more attractive. It actually gives the brane world idea a theoretical background: by compactifying six dimensions in the 11-dimensional theory, our 4-dimensional universe may be realized as a hypersurface in 5-dimension at one of the fixed points of a $S_1/Z_2$. The 5-dimensional effective theory after compactification of the six dimensions by a Calabi-Yau manifold can be obtained, e.g. [4].

Randall and Sundrum proposed two similar but distinct phenomenological brane world scenarios [5,6]. In both scenarios the 5-dimensional spacetime is compactified on $S_1/Z_2$ and all matter fields are assumed to be confined on branes at fixed points of the $S_1/Z_2$ so that the bulk, or the spacetime region between two fixed points, is described by pure Einstein gravity with a negative cosmological constant. The first scenario [5] deals with a three-brane with negative tension as our universe. It was shown that the hierarchy problem may be solved in this scenario by a large redshift (warp) factor, between our brane and another, hidden brane. In the second scenario [6] a three-brane with positive tension is considered as our universe and four-dimensional Newtonian gravity can be realized on the brane in the limit of infinite orbifold radius.

The distance between two branes (radion) in the phenomenological RS scenario, as well as, more generally, moduli fields in superstring theory, should be stabilized. For this purpose a bulk scalar field $\Psi$ with certain potentials in the bulk and on the branes can be introduced [7]. The choice of the bulk $U(\Psi)$ and brane potentials $V_{\pm}(\Psi)$ must be consistent with the 5d warp geometry [8,9]. In the context of the supergravity realization of the RS scenario one has to consider bulk scalars of the theory with an effective scalar field potential [10].

Thus, there is a broad class of brane world scenarios with a 5d bulk between two end-of-the-world branes and a bulk scalar field with bulk and brane potentials. This is a basis for a brane cosmology construction in which our 4d expanding universe is located on the visible brane.

It is expected on the general ground that dimensional reduction of a multi-dimensional theory in the 4d low energy limit yields a (generalized) Brans-Dicke (BD) theory. Often 4d effective theory is derived by the integration of the action with respect to the extra dimension, $\int dw$, as it customary in Kaluza-Klein compactification.
In the brane world scenario one can derive effective 4d dimensional Einstein equations on the brane, starting with the full 5d bulk equations. Although at the qualitative level the expectation that you will obtain an effective BD theory is justified, many quantitative details are lost. The integration $\int dw$ must be treated with special caution in cosmological models with brane collisions [12].

Widespread expectation is that BD scalar is a manifestation of a massless bulk scalar degree of freedom. If moduli are stabilized and no massless scalars left in the bulk, a naive expectation would be that BD theory is reduced to the pure Einstein gravity on the brane. Let us inspect a resent progress in respect to these ideas.

II. EFFECTIVE EINSTEIN EQUATIONS, WEAK GRAVITY AND 4D GRAVITATIONAL COUPLING

One of the most interesting results of the brane world cosmology is the derivation of the 4d effective Einstein equations at the brane, in the scenarios with [15,16] or without [13,14] a bulk scalar field. In the models without a bulk scalar the 4d effective Einstein equations on the brane have the structure [14]

$$G_{\mu\nu} = -\Lambda_4 g_{\mu\nu} + 8\pi \tilde{G}_N \tau_{\mu\nu} + \kappa_5^4 \pi_{\mu\nu} - E_{\mu\nu} ,$$

where $\tau_{\mu\nu}$ comes from the brane matter, $\Lambda_4$ is the 4d cosmological constant defined by $\Lambda_4 = \kappa_5^4 \lambda^2 / 12 - 3/l_0^2$, the correction $\pi_{\mu\nu}$ include a bilinear combination of $\tau_{\mu\nu}$, and $E_{\mu\nu}$ is the projection of the bulk Weyl tensor. Here, $\lambda$ and $l_0$ are the brane tension and the bulk curvature scale (AdS radius). The coupling $\tilde{G}_N$ which is interpreted in [14] as the 4d effective Newton’s gravitational constant is given by

$$\tilde{G}_N = \frac{\kappa_5^4}{48\pi} \lambda .$$

Thus gravity on a positive tension brane is described by 4-dimensional Einstein gravity at low energy provided the effects of bulk gravitational waves can be neglected. On the other hand, $\tilde{G}_N$ is negative at the negative tension brane.

In the models with a bulk scalar field (1) is generalized to more complicated equations [15,16]

$$G_{\mu\nu} = -\Lambda_4 g_{\mu\nu} + 8\pi \tilde{G}_N (\Psi) \tau_{\mu\nu} + \kappa_5^4 \pi_{\mu\nu} - E_{\mu\nu} + \frac{2\kappa_5^2}{3} \tilde{T}_{\mu\nu}(\Psi) ,$$

where $\tilde{T}_{\mu\nu}(\Psi)$ is the brane contribution of the scalar field and the effective 4-dimensional cosmological constant $\Lambda_4$ now depends on the scalar field $\Psi$ in a non-trivial way. Equations (3) are not closed and must be supplemented by the bulk equations for the scalar field with proper boundary conditions. What is interpreted in [15,16] as 4d effective Newton’s gravitational constant is now expressed by the scalar field potential $V(\Psi)$ on the brane as

$$\tilde{G}_N(\Psi) = \frac{\kappa_5^4}{48\pi} V(\Psi) ,$$

which is generalization of (2). One might jump to the conclusion that there is anti-gravity on the brane with negative potential $V < 0$ [17].

To address the issues of the form of the low-energy theory and the value and sign of $\tilde{G}_N$ it is instructive to consider weak gravity on the brane in a brane world scenario with flat brane but warped 5d bulk geometry with scalar fields. We expect 4d Einstein-BD equations for the linearized metric perturbation $h_{\mu\nu}$ around a flat background $\eta_{\mu\nu}$ at the brane

$$(\Box + \Delta K) \tilde{h}_{\mu\nu} = -16\pi \tilde{G}_N \left( T_{\mu\nu} - \frac{B}{2} T \eta_{\mu\nu} \right) ,$$

where $\tilde{h}_{\mu\nu}$ is defined by $\tilde{h}_{\mu\nu} = h_{\mu\nu} - h \eta_{\mu\nu} / 2$ with the gauge condition $\partial^\nu \tilde{h}_{\mu\nu} = 0$. We have introduced a factor $B$ to describe the Brans-Dicke type correction to the Einstein tensor gravity. The usual BD parameter $\omega$ is introduced in the combination $B = \sqrt{\frac{3}{2N + 2\omega}}$. $\Delta K$ is the high-energy correction. Actually, it can be shown that the high energy correction $\Delta K$ is consistent with the so called higher derivative gravity [22]. In low energy approximation we neglect the high-energy correction, $\Delta K \approx 0$.

Weak gravity in the Randall-Sundrum brane world (without bulk scalar) was investigated by Garriga and Tanaka [18] and they showed that linearized Brans-Dicke theory is realized on the brane with

$$G_N^{\pm} = \frac{\kappa_5^2}{16\pi l_0} \frac{e^{\pm d/l_0}}{\sinh(d/l_0)} , \quad B^{\pm} = \frac{1}{2} e^{\mp 2d/l_0} ,$$

2
where $d$ is the inter-brane distance. Einstein theory is restored only in the limit of an infinite orbifold radius $d$, $\omega$ on the positive tension brane becomes infinite, $B_+ = 0$. On the other hand, formula (1) of Shiromizu, et.al [14] shows that gravity on a positive tension brane is described by 4-dimensional Einstein gravity at low energies provided the effects of bulk gravitational waves can be neglected.

In order to understand the two different-looking results (2) and (6) in a unified manner, let us briefly review linearized Einstein and Brans-Dicke theories around a 4-dimensional Minkowski background. First, because of the Poincare symmetry of the background it is possible to Fourier expand all perturbations and to classify them by irreducible representations of the little group, see Appendix for the details. There are three relevant classes: scalar, vector, and tensor perturbations. Next, we can choose a suitable gauge or construct a set of gauge-invariant variables. Third, we can write down linearized equations for each class separately. It is easily shown that vector perturbations should vanish. The characteristic feature of Einstein theory is that the coupling constant between gravity and matter is the same for scalar and tensor perturbations and given by Newton’s constant (see subsection VA). On the other hand, in Brans-Dicke theory the universality is spoiled by the existence of a scalar field included in the theory since the scalar field couples with the scalar perturbations only. The Brans-Dicke parameter determines how the scalar field couples to scalar perturbations. Hence the gravitational coupling constant for scalar perturbations is a combination of Newton’s constant and the Brans-Dicke parameter while that for tensor perturbations is Newton’s constant itself. To see this, let us set $\Delta K = 0$ in (5) to obtain the linearized equation for 4-dimensional BD theory and split $\bar{h}_{\mu\nu}$ into tensor (transverse-traceless) and scalar (trace) parts, $\bar{h}_{\mu\nu} = \bar{h}^{TT}_{\mu\nu} + \bar{h}^{LL}_{\mu\nu} - \eta_{\mu\nu}/4$, with the transversal-traceless conditions upon $\bar{h}^{TT}_{\mu\nu}$. Here, $\bar{h}^{LL}_{\mu\nu}$ is given by

$$\begin{align*}
\bar{h}^{LL}_{\mu\nu} &= \frac{1}{3\Box} \left( \partial_{\mu} \partial_{\nu} h - \frac{1}{4} \Box h_{\mu\nu} \right).
\end{align*}$$

Accordingly, equation (5) is split into two parts, tensor and scalar ones

$$\begin{align*}
\Box \bar{h}^{TT}_{\mu\nu} &= -16\pi G_N T^{TT}_{\mu\nu}, \\
\Box h &= 16\pi G_N (1 - 2B_+) T.
\end{align*}$$

Let us apply this to the issue of weak gravity on branes in the Randall-Sundrum scenario. We can easily show by using the 5-dimensional Einstein equation that there are no scalar-type perturbations in the bulk. Hence, the results (1), (2) of Shiromizu, et.al [14] without ambiguities determines the 4-dimensional gravitational coupling constant for scalar perturbations $\bar{G}_{\mu\nu} = \bar{G}_N^L (1 - 2B_+) = \pm \frac{\kappa_4}{8\pi^2}$. This follows from (6) and from (2) where one has to use flat brane tuning $\lambda = \pm \frac{\kappa_4}{2\kappa_5}$.

On the other hand, the coupling constant for tensor perturbations cannot be determined directly since there are tensor-type perturbations in the bulk manifested in $E_{\mu\nu}$. In other words, the system of equations (1) is not closed in respect tensor modes, and shall be supplemented by information about bulk Weyl tensor. Actually, the existence of tensor-type perturbations in the bulk $E_{\mu\nu}$ shifts the 4-dimensional gravitational coupling constant for tensor perturbations from $G_N$ as it defined in (1), (2), see also [19]. The gravitational coupling for tensor mode is given by $G_N$ from (6). As a result, we have different 4-dimensional gravitational coupling constants for scalar and tensor perturbations and, thus, the linearized Brans-Dicke theory.

Let us discuss more general case of the brane world scenario with bulk scalar fields.

We begin with the case of a single bulk scalar field $\Psi$. Equation (3) describes effective 4d gravity on the brane. We just learned that bulk gravity $E_{\mu\nu}$ is important to reproduce correct weak gravity at the brane. Now we have to understand the effect of bulk scalar contribution $\hat{T}_{\mu\nu}^s(\Psi)$ and $\Lambda_4(\Psi)$ to the 4d gravity in equation (3).

In the analysis of weak gravity one of the essential differences between models with and without a bulk scalar field is that there are non-trivial scalar-type perturbations in the bulk for models with a bulk scalar field, whereas these perturbations vanish for models without a bulk scalar field. Hence, the existence of the scalar field can shift the 4-dimensional gravitational coupling constant for scalar perturbations from what is given in (4). On the other hand, we already know that the existence of tensor-type perturbations in the bulk shifts the 4-dimensional gravitational coupling constant for tensor perturbations. Hence, our question now is whether these shifts in the gravitational couplings for scalar and tensor modes due to bulk gravity $E_{\mu\nu}$ and bulk scalar $\hat{T}_{\mu\nu}^s(\Psi)$ are the same or not. If they are the same then the effective gravitational theory on the brane is pure Einstein theory (with a shifted Newton’s constant, which differs from (4)). If they are not the same then the effective theory is still Brans-Dicke theory, but still shifted from (4).

The issue of weak gravity in the generalized braneworld scenario with a single scalar field was already investigated by Tanaka and Montes [20]. They obtain very intriguing result that 4-dimensional Einstein gravity is restored at low energy. In other words, according to the above arguments, their claim is that shifts in the 4-dimensional gravitational constants for scalar and tensor perturbations bring them to the same value.
The analysis of [20] can also be generalized to include non-minimal coupling (which we denote as $e^{-\alpha \pm (\Phi \pm)}$) of the brane matter to the metric in the 5-dimensional Einstein frame (to reflect the spirit of dilaton-matter coupling). This non-minimal coupling was included in the 4d Einstein equations in [11,15], where it was assumed that matter on the brane is minimally coupled not to the metric in the 5-dimensional Einstein frame but to a conformally related metric. It is evident that the induced metric on the brane in the minimally coupled frame should be considered as a physical metric on the brane.

Since the conformal factor depends on the bulk scalar field, metric perturbations in the minimally coupled frame have an extra contribution from scalar perturbations. Hence, from the above arguments about the universality of gravitational coupling constants for scalar and tensor perturbations in 4-dimensional Einstein gravity it is expected that the restoration of 4-dimensional Einstein gravity may depend on the choice of the minimally coupled frame.

In the case of a single bulk scalar field we obtain

$$G^\pm_N = \frac{k^2_e}{16\pi} \int^\pm d\omega e^{-3A(w)}, \quad B^\pm = 0. \quad (9)$$

Here, $e^{-A(w)}$ is the warp factor in the 5-dimensional Einstein frame, whereas the warp factor in the minimally coupled frame is $e^{-A \pm \alpha \pm /2}$. The value of $G^\pm_N$ is always positive and differs from (4). We conclude that the coupling with the brane energy-momentum tensor alone in the effective Einstein equation (3) cannot be interpreted as the 4d Newton’s constant. The correct value of the Newton’s constant is given by (9). In particular, there is no antigravity at the negative tension brane.

The purpose of this paper, on technical side, is to perform a complete analysis of weak gravity in the generalized braneworld world scenario in which there is a bulk scalar field. We take into account the fact that matter on the brane is not minimally coupled to the metric in the 5-dimensional Einstein frame in general and allow an arbitrary conformal transformation to the minimally coupled frame. Despite the above expectation, our conclusion is still the same as that of Tanaka and Montes: 4-dimensional Einstein gravity is restored on branes at low energy. It is notable that our result can be applied to the case in which the inter-brane distance is arbitrary, another words, is not stabilized (e.g. 5-dimensional Horava-Witten theory [4]). Here, we say radion is stabilized if the background inter-brane distance is fixed.

We formulate the necessary and sufficient condition for brane gravity at low energy to be the Einstein theory.

We also will try to to understand the connection between the bulk scalar degrees of freedom and the character of the brane gravity. Indeed Einstein, not BD, gravity at the brane is restored in the weak gravity for an arbitrary dilaton potentials in the bulk and at branes. It is rather surprising for the models with single bulk scalar even if radion is not stabilized! Moreover, this result can be extended for the second order of perturbative gravity within a certain class of symmetric perturbations [21].

It turns out that for two or more bulk scalar fields (unless their potential has special properties), the weak gravity at the brane is again BD gravity. We can summarize that for models without a bulk scalar field the parameter $B$ in the low-energy gravity theory (5) is

$$B = \frac{1}{2} e^{\mp e^{-A(w)}}. \quad (9)$$

In the case of a single bulk scalar field $B = 0$. For $n > 1$ scalars in the bulk we may again expect a nonzero $B$. It is intriguing that the BD parameter, which can be in principle constrained by observations, depend on the details of the bulk theory.

How one can understand this result? Suppose there are $n$ scalars in the bulk. In the weak gravity limit we still expect the effective gravitational theory of the form (5). In other words, we expect the BD theory with a single effective BD scalar on the brane. Thus, the effective BD scalar on the brane, in some sense, is a composite object due to superposition of contributions from the bulk scalars. For the case without scalars in the bulk, but the radion, we have the non-zero value (6) of $B$. For the case of a single bulk scalar the superposition of its contribution and the radion contribution to the brane scalar gravity are precisely canceled so that $B = 0$. In the case of more than one bulk scalars the precise cancellation is not expected.

The rest of the paper is rather technical and is organized as follows. In Sec. III we describe our model of the brane world and define the physical energy momentum tensor of matter fields on the branes. In the section we also specify the background around which we analyze perturbations in Sec. IV. In Sec. V we compare gravity in the brane world with that in 4-dimensional Einstein gravity. We show that 4-dimensional Einstein gravity is restored at low energies on both branes provided there is no tachyonic mode with a boundary condition corresponding to vanishing stress energy of matter fields on the branes. Finally, Sec. VI is devoted to a summary of the paper and discussion.

III. MODEL DESCRIPTION AND BACKGROUND

In this section we describe our brane world model and specify a background. In the next section we investigate perturbations around this background to analyze weak gravity in the brane world model.
Let us consider a 5-dimensional spacetime $\mathcal{M}$ with topology $\mathcal{M}_4 \otimes S^1/\mathbb{Z}_2$, where $\mathcal{M}_4$ represents 4-dimensional spacetime. It is easily understood that two timelike hypersurfaces $\Sigma_{\pm}$ corresponding to two fixed points of $S^1/\mathbb{Z}_2$ can be sources of gravity. In fact, because of the $\mathbb{Z}_2$ symmetry, the first derivative of the fields can be discontinuous across each of the fixed points and thus, the second derivative can include a delta-function. Since Einstein equation is a second order differential equation, inclusion of the delta-function in the second derivative of the fields implies the existence of gravitational source at each of fixed points. Since we would like to consider $\Sigma_{\pm}$ as a world-volume of a brane and would like to think of one of branes as our 4-dimensional universe, we assume that the 4-dimensional intrinsic geometry on $\Sigma_{\pm}$ is regular. On the other hand, the 5-dimensional geometry is not necessarily regular on $\Sigma_{\pm}$. The 5-dimensional region off both of $\Sigma_{\pm}$ is usually called the bulk spacetime and we denote it by $\mathcal{M}_b$. Hence,

$$\mathcal{M} = \mathcal{M}_b \cup \Sigma_+ \cup \Sigma_-.$$  \hspace{1cm} (10)

To describe the hypersurfaces $\Sigma_{\pm}$ we use the parametric equations

$$\Sigma_{\pm}: \quad x^M = Z^M_{\pm}(y^\mu_{\pm}),$$  \hspace{1cm} (11)

where $x^M (M = 0, \cdots, 4)$ are 5-dimensional coordinates in $\mathcal{M}$ and each $y_{\pm}$ denotes four parameters $\{y^\mu_{\pm}\} (\mu = 0, \cdots, 3)$ playing the role of the intrinsic coordinate system on $\Sigma_{\pm}$. See Figure 1 for illustration.

We consider a dilaton type theory with a single scalar field to describe gravity and assume that all other matter fields are confined to the hypersurfaces $\Sigma_{\pm}$. Hence, in the following arguments we assume that the system is described by the action \(^1\)

$$I_{tot} = I_{EH} + I_\Psi + I_{matter},$$  \hspace{1cm} (12)

where $I_{EH}$ is the Einstein-Hilbert action

$$I_{EH} = \frac{1}{2\kappa_5^2} \int_{\mathcal{M}} d^5x \sqrt{-g} R,$$  \hspace{1cm} (13)

$I_\Psi$ is the action of a scalar field $\Psi$

\(^1\)For simplicity we do not consider the boundary of $\mathcal{M}_4$, but it is easy to take it into account by imposing suitable boundary conditions and introducing boundary terms appropriate for these boundary conditions.
\[ I_\Psi = - \int_{M_b} d^5x \sqrt{-g} \left[ \frac{1}{2} g^{MN} \partial_M \Psi \partial_N \Psi + U(\Psi) \right] - \sum_{\sigma = \pm} \int_{\Sigma_\sigma} d^4y_\sigma \sqrt{-q_\sigma} V_\sigma(\Psi_\sigma) \]  

(14) and \( I_{\text{matter}} \) is the action of matter fields living on the branes

\[ I_{\text{matter}} = \sum_{\sigma = \pm} \int_{\Sigma_\sigma} d^4y_\sigma L_\pm[q_\pm \mu \nu, \text{matter}] \]  

(15)

Here, \( \Psi_\pm \) and \( q_\pm \mu \nu \) represent the pullback of the scalar field \( \Psi \) and the induced metric on \( \Sigma_\pm \), and \( q_\pm \mu \nu \) is the physical metric on \( \Sigma_\pm \) specified below. The physical metric \( q_\pm \mu \nu \) is actually ‘physical’ in the sense that because of the form (15) of the matter action, all matter fields living in the 4-dimensional manifolds \( \Sigma_\pm \) feel the geometry described by it.

The induced metric \( q_\pm \mu \nu \) and the pullback \( \Psi_\pm \) of \( \Psi \) on \( \Sigma_\pm \) are defined by

\[ q_\pm \mu \nu (y_\pm) = e^M_\pm e^N_\pm g_{MN} \bigg|_{x = Z_\pm (y_\pm)} \]  

\[ \Psi_\pm (y_\pm) = \Psi \bigg|_{x = Z_\pm (y_\pm)} \]  

(16) respectively, where

\[ e^M_\pm = \frac{\partial Z^M_\pm}{\partial y_\pm} \]  

(17)

We assume that the physical metric \( q_\pm \mu \nu \) is conformally equivalent to the induced metric \( q_\pm \mu \nu \): \( \tilde{q}_\pm \mu \nu = \exp[-\alpha_\pm(\Psi_\pm)] q_\pm \mu \nu \), where \( \alpha_\pm \) is a function of \( \Psi_\pm \), respectively.

As shown in ref. [25], the action above gives a consistent variational principle including variations of the positions of the hypersurfaces \( Z^M_\pm \) as well as the metric \( g_{MN} \), the scalar field \( \Psi \), and other matter fields. Specializing to our model with the \( Z_2 \)-symmetry, the equations of motion are as follows.

\[ G_{MN} = \kappa^2_T T_{MN} \]  

\[ 2K_\pm \mu \nu = -\kappa^2_T \left( S^\mu_\pm - \frac{1}{3} S_\pm q^\mu_\nu \right) \]  

(19)

and

\[ \Psi_\partial^M \mu - U'(\Psi) = 0, \]  

\[ 2\partial_\perp \Psi_\pm = V'_\perp(\Psi_\pm) + \frac{1}{2} \alpha'_\perp(\Psi_\pm) S_\pm, \]  

(20)

where the semicolon denotes the covariant derivative compatible with the metric \( g_{MN} \), \( G_{MN} \) is the Einstein tensor for the metric \( g_{MN} \), \( T_{MN} \) is the stress energy tensor of the scalar field in the bulk, and prime is a derivative with respect to \( \Psi \).

\[ T_{MN} = \partial_M \Psi \partial_N \Psi - g_{MN} \left[ \frac{1}{2} g^{M'N'} \partial_{M'} \Psi \partial_{N'} \Psi + U(\Psi) \right] \]  

(21)

\[ K_\pm \mu \nu \] and \( \partial_\perp \Psi_\pm \) are, respectively, the extrinsic curvature of \( \Sigma_\pm \) and the normal derivative of \( \Psi \) on \( \Sigma_\pm \) defined by

\[ K_\pm \mu \nu = \frac{1}{2} e^M_\pm e^N_\pm L_{\perp \pm} g_{MN} \bigg|_{x = Z_\pm (y_\pm)} \]  

\[ \partial_\perp \Psi_\pm = n^M_\perp \partial_M \Psi \bigg|_{x = Z_\pm (y_\pm)} \]  

(22)

\( n^M_\perp \) is the unit normal to the hypersurface \( \Sigma_\pm \) directed towards the bulk \( M_b \), \( S_\pm \mu \nu \) is the surface stress energy tensor given by

\[ S_\pm \mu \nu = -V_\pm(\Psi_\pm) q_\pm \mu \nu + e^{-\alpha_\pm(\Psi_\pm)} S_\perp \mu \nu, \]  

\[ \tilde{S}_\pm \mu \nu = \tilde{q}_\pm \mu \nu \tilde{q}_\pm \nu \sigma \tilde{S}_\sigma, \]  

\[ \tilde{S}_\perp \mu \nu = \frac{2}{\sqrt{-q_\perp}} \delta_{\mu \nabla} \int_{\Sigma_\perp} d^4y_\perp L_\perp[q_\perp \mu \nu, \text{matter}] \]  

(23)
\[ S_\pm \equiv q^{\mu\nu}S_{\pm\mu\nu} = -4V_\pm(\Psi_\pm) + e^{-2\alpha_\pm(\Psi_\pm)}S_\pm, \]
\[ \bar{S}_\pm \equiv \bar{q}^{\mu\nu}\bar{S}_{\pm\mu\nu}. \]

Note that \( \bar{S}_{\pm\mu\nu} \) can be called the physical surface energy momentum tensor since it is defined in terms of the matter action and the physical metric only. Hence, when we seek an effective 4-dimensional gravitational theory on the branes, we have to write the equations in terms of \( \bar{S}_{\pm\mu\nu} \) instead of \( S_{\pm\mu\nu} \).

Now let us specify the background \( g_{MN} = g_{MN}^{(0)}, \Psi = \Psi^{(0)}, Z_\pm = Z_\pm^{(0)}, \bar{T}_{\pm\mu\nu} = \bar{T}_{\pm\mu\nu}^{(0)} \) around which we shall investigate perturbations in the following sections. We consider a background with 4-dimensional Poincare symmetry:

\[ g_{MN}^{(0)} dx^M dx^N = e^{-2A(w)}(\eta_{\mu\nu} dx^\mu dx^\nu + dw^2), \]
\[ \Psi^{(0)} = \Psi^{(0)}(w), \]
\[ Z_\pm^{(0)} = y_\pm^\mu, \]
\[ Z_\pm^{(0)w} = w_\pm, \]
\[ \bar{S}_{\pm\mu\nu}^{(0)} = 0, \]

(25)

where dots denote derivative with respect to \( w \) and \( \Psi_\pm^{(0)} = \Psi^{(0)}(w_\pm) \). Note that the sign in the right hand side of (27) is due to the sign in the following expression of the unperturbed unit normal \( n_{\pm}^{(0)M} \) to \( \Sigma_{\pm}^{(0)} \) directed towards \( \mathcal{M}_b \).

\[ n_{\pm}^{(0)M} \partial_\mathcal{M} = \mp e^A \partial_w. \]

Here, we assumed that \( w_- < w_+ \) and that the bulk is the region \( w_- < w < w_+ \). The induced metric and the physical metric on \( \Sigma_{\pm} \) are

\[ g_{\pm\mu\nu}^{(0)} = e^{-2A(w_\pm)}\eta_{\mu\nu}, \]
\[ \bar{g}_{\pm\mu\nu}^{(0)} = e^{-\alpha_\pm(\Psi_\pm)}e^{-2A(w_\pm)}\eta_{\mu\nu}. \]

(29)

**IV. PERTURBATIONS**

In this section we investigate perturbations around the background specified in the previous section. Namely, we consider

\[ 2 \text{This is possible because of the Poincare symmetry.} \]
\[
g_{MN} = g_{MN}^{(0)} + \delta g_{MN},
\]
\[
\Psi = \Psi^{(0)} + \delta \Psi,
\]
\[
Z_{\pm}^M = Z_{\pm}^{(0)M} + \delta Z_{\pm}^M,
\]
\[
S_{\pm\mu\nu} = S_{\pm\mu\nu}^{(0)} + \delta S_{\pm\mu\nu}.
\]

The unperturbed functions \( \{Z_{\pm}^{(0)M}\} \) specify unperturbed hypersurfaces \( \Sigma_{\pm}^{(0)} \) and the perturbed functions \( \{Z_{\pm}^M\} \) specify perturbed hypersurfaces \( \Sigma_{\pm} \). As pointed out in ref. [24] there are two kinds of gauge transformations, one in the bulk and another on the branes. The former is called 5-gauge transformation and is of the form
\[
x^M \rightarrow x^M + \xi^M(x).
\]
The latter is called 4-gauge transformation and is of the form
\[
y_{\pm}^\mu \rightarrow y_{\pm}^\mu + c_{\pm}^\mu(y_{\pm}).
\]

Following ref. [24], it is straightforward to calculate perturbations of the induced metric \( q_{\pm\mu\nu} \) and the extrinsic curvature \( K_{\pm\mu\nu} \) from \( \delta g_{MN} \) and \( Z_{\pm}^M \). The result for a general background is
\[
q_{\pm\mu\nu} = q_{\pm\mu\nu}^{(0)} + \delta q_{\pm\mu\nu},
\]
\[
K_{\pm\mu\nu} = K_{\pm\mu\nu}^{(0)} + \delta K_{\pm\mu\nu},
\]
where
\[
\delta q_{\pm\mu\nu} = e_{\pm\mu}^{(0)} e_{\pm\nu}^{(0)N} (\delta g_{MN} + \mathcal{L}_{\delta Z_{\pm}} g_{MN}^{(0)}),
\]
\[
\delta K_{\pm\mu\nu} = \frac{1}{2} h_{\pm}^{(0)M} n_{\pm}^{(0)N} (\delta g_{MN} + 2 \delta Z_{\pm M;N}) K_{\pm\mu\nu}^{(0)}
\]
\[
- \frac{1}{2} n_{\pm}^{(0)L} e_{\pm\mu}^{(0)M} e_{\pm\nu}^{(0)N} \left[ 2 \mathcal{L}_{\delta L} + \delta Z_{\pm L;MN} + \delta Z_{\pm L;NM} + (R_\perp^{(0)LMN} + R_\perp^{(0)LNM}) \delta Z_{\pm}^{L} \right].
\]

Here, \( \mathcal{L} \) denotes the Lie derivative defined in the five-dimensional spacetime, the semicolon denotes the covariant derivative compatible with the background metric \( g_{MN}^{(0)} \), \( R_\perp^{(0)LMN} \) is the Riemann tensor of \( g_{MN}^{(0)} \) and \( \delta \Gamma_{LMN} = (1/2)(\delta g_{LM,N} + \delta g_{LN,M} - \delta g_{MN,L}) \). The right hand sides of the above expressions for perturbations \( \delta q_{\pm\mu\nu} \) and \( \delta K_{\pm\mu\nu} \) can be evaluated on the unperturbed hypersurface \( \Sigma_{\pm}^{(0)} \) at \( x = Z_{\pm}^{(0)}(y_{\pm}) \). It was shown in ref. [24] that \( \delta q_{\pm\mu\nu} \) and \( \delta K_{\pm\mu\nu} \) are invariant under the 5-gauge transformation and transform under the 4-gauge transformation as
\[
\delta q_{\pm\mu\nu} \rightarrow \delta q_{\pm\mu\nu} - \mathcal{L}_{\xi_{\pm}} q_{\pm\mu\nu}^{(0)},
\]
\[
\delta K_{\pm\mu\nu} \rightarrow \delta K_{\pm\mu\nu} - \mathcal{L}_{\xi_{\pm}} K_{\pm\mu\nu}^{(0)},
\]
where \( \mathcal{L} \) denotes the Lie derivative defined in the 4-dimensional manifold \( \Sigma_{\pm}^{(0)} \). Trivially, \( \delta S_{\pm\mu\nu} \) is also invariant under the 5-gauge transformation and transforms under the 4-gauge transformation as
\[
\delta \tilde{S}_{\pm\mu\nu} \rightarrow \delta \tilde{S}_{\pm\mu\nu} - \mathcal{L}_{\xi_{\pm}} \tilde{S}_{\pm\mu\nu}^{(0)}.
\]

Applying these formulas to our background, we obtain the following expressions for \( \delta q_{\pm\mu\nu} \) and \( \delta K_{\pm\mu\nu} \),
\[
\delta q_{\pm\mu\nu} = \delta g_{\mu\nu} + \partial_{\mu} \delta Z_{\pm\nu} + \partial_{\nu} \delta Z_{\pm\mu} - 2 A \eta_{\mu\nu} \delta Z_{\pm w}
\]
\[
\delta K_{\pm\mu\nu} = \mp \frac{1}{2} e^A \left[ \delta g_{\mu\nu} - \partial_{\mu} \delta g_{\nu w} - \partial_{\nu} \delta g_{\mu w} + \dot{A}_{\mu\nu} \delta g_{w w} - 2 \dot{A} \left( \partial_{\mu} \delta Z_{\pm\nu} + \partial_{\nu} \delta Z_{\pm\mu} \right) - 2 \partial_{\mu} \partial_{\nu} \delta Z_{\pm w} + 2 e^{A} (e^{-A}) \cdot \eta_{\mu\nu} \delta Z_{\pm w} \right],
\]
where \( \delta Z_{\pm}^{M} = g_{MN}^{(0)} \delta Z_{\pm}^{N} \).

Similarly, we obtain the following expressions for perturbations of \( \Psi_{\pm} \) and \( \partial_{\perp} \Psi_{\pm} \),
\[
\Psi_{\pm} = \Psi_{\pm}^{(0)} + \delta \Psi_{\pm},
\]
\[
\partial_{\perp} \Psi_{\pm} = \partial_{\perp} \Psi_{\pm}^{(0)} + \delta \partial_{\perp} \Psi_{\pm},
\]
\[
(38)
\]
\[
\delta \Psi \pm = \delta \Psi + e^{2A} \dot{\Psi}^{(0)} \delta Z_{\pm w}, \\
\delta \partial_\pm \Psi \pm = \mp \left[ 3A \delta g_{\mu\nu} \dot{\Psi}^{(0)} - 2e^{2A} \delta g_{\mu w} \dot{\Psi}^{(0)} + 2e^A \delta \dot{\Psi} + 2e^A (e^{A} \dot{\Psi}^{(0)} - \delta Z_{\pm w}) \right]. 
\]

(39)

Again, the right hand side can be evaluated on the unperturbed hypersurface \( \Sigma^{(0)} \) at \( x = Z^{(0)}(y) \). It is easy to calculate perturbations of the physical metric \( \bar{\mu}_{\pm \mu} \) on \( \Sigma_{\pm \mu} \) by using the following relation

\[
\delta \bar{q}_{\pm \mu \nu} = e^{-\alpha_{\pm}^{(0)}} \delta q_{\pm \mu \nu} - \alpha_{\pm}^{(0)} e^{-\alpha_{\pm}^{(0)}} e^{-2A_{\pm} \eta_{\pm \mu} \bar{q}_{\pm \mu \nu}} \delta \Psi_{\pm}, 
\]

where \( \alpha_{\pm}, \alpha_{\pm}^{(0)} \) and \( \alpha_{\pm}^{(')}(\Psi_{\pm}^{(0)}) \) represent \( \alpha_{\pm}(x_{\pm}) \), \( \alpha_{\pm}(\Psi_{\pm}^{(0)}) \) and \( \alpha_{\pm}^{(')}(\Psi_{\pm}^{(0)}) \), respectively. We have the following 4-gauge transformation.

\[
\delta \tilde{S}_{\pm \mu \nu} = \delta \bar{S}_{\pm \mu \nu} - \frac{1}{2}(K_{\pm \mu}^{(0)} \delta q_{\pm \mu \nu} + K_{\pm \nu}^{(0)} \delta q_{\pm \nu}) \delta \Psi_{\pm}, 
\]

(42)

and \( K_{\pm \mu}^{(0)} = K_{\pm \mu}^{(0)} \xi_{\pm}^{(0)} \). To see the 5-gauge invariance of \( \delta \tilde{S}_{\pm \mu \nu} \) note that we have redefined \( V_{\pm} \) and \( \mathcal{L}_{\pm} \) so that \( \bar{S}_{\pm}^{(0)} \) vanishes (see eq.(25) and the footnote after that.). The perturbed junction condition can be written in terms of these variables as

\[
2\delta \tilde{K}_{\pm \mu \nu} = -\kappa^2 \left( \delta \tilde{S}_{\pm \mu \nu} - \frac{1}{3} \delta \bar{S}_{\pm \mu \nu} \right), \\
\delta \tilde{S}_{\pm \mu \nu} = -e^{-2A_{\pm}} V_{\pm}^{(0)} \delta \Psi_{\pm} \eta_{\mu \nu} + e^{-\alpha_{\pm}} \delta \tilde{S}_{\pm \mu \nu}, 
\]

(43)

and \( \delta \tilde{S}_{\pm} = q_{\pm}^{(0)} \mu \nu \tilde{S}_{\pm \mu \nu} \), where \( V_{\pm}^{(0)} \) represents \( V_{\pm}^{(0)}(\Psi_{\pm}^{(0)}) \). The scalar field matching condition is written as

\[
\mp \delta \partial_\pm \Psi_{\pm} = V_{\pm}^{(0)} \delta \Psi_{\pm} + \frac{1}{2} \alpha_{\pm}^{(')} \delta \tilde{S}_{\pm}, 
\]

(44)

where \( V_{\pm}^{(0)} \) and \( \delta \tilde{S}_{\pm} \) represent \( V_{\pm}^{(0)}(\Psi_{\pm}^{(0)}) \) and \( q_{\pm}^{(0)} \mu \nu \delta \tilde{S}_{\pm \mu \nu} \), respectively.

On the other hand, the 5-gauge transformation is

\[
\delta g_{\mu \nu} \rightarrow \delta g_{\mu \nu} - \partial_\mu \xi_{\nu} - \partial_\nu \xi_{\mu} - 2A_{\mu} \eta_{\mu \nu} \xi_{w}, \\
\delta g_{\mu w} \rightarrow \delta g_{\mu w} - e^{-2A} (e^{2A} \xi_{w}) - \partial_\mu \xi_{w}, \\
\delta g_{w w} \rightarrow \delta g_{w w} - 2e^{-A} (e^{2A} \xi_{w}) \xi_{w}, \\
\delta Z_{\pm M} \rightarrow \delta Z_{\pm M} + \xi_{M}, \\
\delta \Psi \rightarrow \delta \Psi - e^{2A} \xi_{w} \dot{\Psi}^{(0)}, 
\]

(45)

where \( \xi_{M} = g_{MN}^{(0)} \xi^{N} \).

Since the background has 4-dimensional Poincare symmetry it is convenient to expand the perturbations in terms of scalar, vector and tensor harmonics in 4-dimensional Minkowski spacetime:

\[
\delta g_{MN} dx^M dx^N = \left( h_{(T)(T)} \gamma_{\mu \nu} + h_{(LT)(TL)} \gamma_{\mu \nu} + h_{(LL)(TT)} \gamma_{\mu \nu} + h_{(TV)(TV)} \gamma_{\nu} \right) dx^\mu dy^\nu, \\
+ 2 \left( h_{(TV)} \gamma_{\nu} w_{(TV)} \gamma_{\mu} \right) dx^\mu dw + h_{ww} Y dw^2, \\
\delta \Psi = w Y, \\
\delta Z_{\pm M} dx^M = \left( z_{\pm (T)} \gamma_{\mu} + z_{\pm (L)} \gamma_{\mu} \right) dx^\mu + z_{\pm w} Y dw, 
\]

(46)

and
where $T_{(T,LT,LL,Y)\mu\nu}$, $V_{(T,L)\mu}$ and $Y$ are harmonics in 4-dimensional Minkowski spacetime (see appendix A for their definition), the $k$-dependent Fourier coefficients $h_{(T,LT,LL,Y)}$, $h_{(T,L)w}$, $h_{ww}$, and $\psi$ are functions of $w$ only, and the other coefficients $z_{\pm}(T,L)$, $z_{\pm}w$, and $\tau_{k(T,LT,LL,Y)}$ are constants. We omitted the integration with respect to $k$. Let us expand the gauge parameters $\xi_{M}$ and $\zeta_{\mu}$, too.

$$\delta S_{\pm \mu \nu} = \tilde{\tau}_{\pm}(T)T_{(T)\mu \nu} + \tilde{\tau}_{\pm}(LT)T_{(LT)\mu \nu} + \tilde{\tau}_{\pm}(LL)T_{(LL)\mu \nu} + \tilde{\tau}_{\pm}(Y)T_{(Y)\mu \nu},$$

(47) where $\tilde{\tau}_{\pm}(T)$ and $\tilde{\tau}_{\pm}(LT)$ are functions of $w$ only and $\tilde{\tau}_{\pm}(T,L)$ are constants.

Using the harmonic expansions (46), we can obtain the corresponding harmonic expansion of $\delta q_{\pm \mu \nu}$, $\tilde{K}_{\pm \mu \nu}$, $\delta \Psi_{\pm}$ and $\delta \partial_{\pm} \Psi_{\pm}$. The result is

$$\delta q_{\pm \mu \nu} = \tilde{s}_{\pm}(T)T_{(T)\mu \nu} + \tilde{s}_{\pm}(LT)T_{(LT)\mu \nu} + \tilde{s}_{\pm}(LL)T_{(LL)\mu \nu} + \tilde{s}_{\pm}(Y)T_{(Y)\mu \nu},$$

$$\delta \tilde{K}_{\pm \mu \nu} = k_{\pm}(T)T_{(T)\mu \nu} + k_{\pm}(LT)T_{(LT)\mu \nu} + k_{\pm}(LL)T_{(LL)\mu \nu} + k_{\pm}(Y)T_{(Y)\mu \nu},$$

$$\delta \Psi_{\pm} = \psi_{\pm} Y,$$

$$\delta \partial_{\pm} \Psi_{\pm} = \psi_{\pm} Y,$$

(49) where $k$-dependent Fourier coefficients are

$$\tilde{s}_{\pm}(T) = e^{-A(0)} F_{(T)},$$

$$\tilde{s}_{\pm}(LT) = e^{-A(0)} \phi_{\pm}(T),$$

$$\tilde{s}_{\pm}(LL) = e^{-A(0)} \phi_{\pm}(L),$$

$$\tilde{s}_{\pm}(Y) = e^{-A(0)} \left[ F - 2 \tilde{A} \phi_{\pm} w - \frac{1}{2} \eta^{\mu \nu} k_{\mu} k_{\nu} \phi_{\pm}(L) - \alpha_{\pm}^{(0)} (e^{-2A} \varphi + \phi_{\pm} w \tilde{\psi}(0)) \right],$$

$$k_{\pm}(T) = \frac{1}{2} e^{-A\left(e^{2A} F(T)\right)},$$

$$k_{\pm}(LT) = \frac{1}{2} e^{-A} F_{w},$$

$$k_{\pm}(LL) = \frac{1}{2} e^{A} \phi_{\pm},$$

$$k_{\pm}(Y) = \frac{1}{2} \left( e^{-A} (e^{2A} F') + (e^{A}) F_{ww} + \left( \frac{1}{2} e^{A} \eta^{\mu \nu} k_{\mu} k_{\nu} - 2 (e^{A}) \right) \phi_{\pm} \right),$$

$$\psi_{\pm} = \bar{\psi} + e^{A} \tilde{\psi}(0) \phi_{\pm},$$

$$\psi_{\pm} = \frac{1}{2} \left[ -e^{-A} \psi(0) F_{ww} + 2 e^{A} \bar{\psi} + 2 e^{A} (e^{A} \tilde{\psi}(0)) \phi_{\pm} w \right].$$

(50) Here, the right hand sides of (50) are evaluated at $w = w_{\pm}$ respectively and have been written in terms of 5-gauge invariant variables defined by

$$F_{(T)} = h_{(T)},$$

$$F_{w} = h_{(T)w} - e^{-2A} (e^{2A} h_{(LT)})',$n

$$F = h_{(Y)} + 2 \tilde{A} X_{w} + \frac{1}{2} \eta^{\mu \nu} k_{\mu} k_{\nu} h_{(LL)},$$

$$F_{ww} = h_{ww} - 2 e^{-A} (e^{A} X_{w})',$$

$$\varphi = \psi - e^{2A} \tilde{\psi}(0) X_{w},$$

$$\phi_{\pm}(T) = z_{\pm}(T) + h_{(LT)} \big|_{w = w_{\pm}},$$

$$\phi_{\pm}(L) = z_{\pm}(L) + h_{(LL)} \big|_{w = w_{\pm}},$$

$$\phi_{\pm} = z_{\pm} w + X_{w} \big|_{w = w_{\pm}},$$

(51) where $X_{w} = h_{(L)w} - e^{-2A} (e^{2A} h_{(LL)})'$. The reason why the coefficients of $\delta q_{\pm \mu \nu}$, $\delta \tilde{K}_{\pm \mu \nu}$, $\delta \Psi_{\pm}$ and $\delta \partial_{\pm} \Psi_{\pm}$ were written by 5-gauge-invariant variables only is that they themselves are 5-gauge-invariant as shown in ref. [24] for $\delta q_{\pm \mu \nu}$ and
we can construct the following doubly-gauge-invariant variables.

\[ f_{\pm}(T) = \bar{\sigma}_{\pm}(T) = e^{-\alpha_{\pm}^{(0)}} F(T), \]

\[ f_{\pm} = \bar{\sigma}_{\pm}(Y) + \frac{1}{2} \eta^{\mu\nu} k_\mu k_\nu \bar{\sigma}_{\pm(LL)} e^{-\alpha_{\pm}^{(0)}} \left[ F - 2\tilde{A} \phi_{\pm w} - \alpha_{\pm}^{(0)} \left( e^{-2A} \varphi + \phi_{\pm w} \tilde{\Psi}^{(0)} \right) \right]. \]  

Note that the coefficients \( \bar{\tau}_{(T,L,T,LL,Y)} \) of \( \delta S_{\pm \mu \nu} \) are doubly-gauge invariant variables by themselves.

Since there are many coefficients in the above harmonic expansions, let us divide these into three classes. The first class is the scalar perturbations and consists of coefficients of \( Y, V(L)_\mu, T(LL)_\mu \) and \( T(Y)_\mu \). The second is the vector perturbations and consists of coefficients of \( V(T)_\mu \) and \( T(TT)_\mu \). The last is the tensor perturbations and consists of coefficients of \( T(T)_\mu \). It is not difficult to show that perturbations in different classes are decoupled from each other.

Before analyzing each class in detail, here we summarize some differences between our formalism and one in the literature [18,20]. What we have to do is exactly the same in both formalisms: we have to solve perturbed Einstein and scalar equations in the 5-dimensional bulk with boundary conditions given by Israel’s junction condition and the scalar field matching condition on the branes. Of course, this system can be considered as a set of inhomogeneous differential equations with \( \mathbb{Z}_2 \) symmetry since the non-trivial boundary conditions can be interpreted as source terms with the delta-function form [18,20,26]. Hence, mathematically there are two approaches to this problem: one is to solve the differential equations with the boundary conditions (or the inhomogeneous equations) directly; another is to use the Green’s function method to solve the inhomogeneous equations by using the complete set of homogeneous solutions. The formalism in the literature [18,20] is based on the second approach. In this approach we need the complete set of homogeneous solutions in order to construct Green’s function. The set of homogeneous solutions consists of the so called zero modes and Kaluza-Klein modes. These homogeneous solutions satisfy the boundary conditions corresponding to vanishing matter stress energy on the branes. Accordingly, the homogeneous solutions have discrete spectra. On the other hand, in our formalism we treat the differential equations with the boundary conditions (or the inhomogeneous equations) directly. By doing so, we can Fourier transform all equations with respect to the 4-dimensional coordinates to reduce the problem into a set of purely 1-dimensional problems. We also classify the Fourier components into irreducible representations (scalar, vector and tensor modes) of the little group of 4-dimensional Poincare symmetry. Note that \( \eta^{\mu\nu} k_\mu k_\nu \) is continuous in our formalism.

Although \( \eta^{\mu\nu} k_\mu k_\nu \) is continuous and there is no discreteness in it, the scalar and tensor Kaluza-Klein masses may still have important roles in our formalism. Here, a scalar (or tensor) Kaluza-Klein mass \( m_{KK} \) is, as in the literature, such a non-zero value of \( \sqrt{-\eta^{\mu\nu} k_\mu k_\nu} \) that the scalar (or tensor, respectively) perturbation equation have a non-trivial solution with vanishing matter stress energy on the branes (homogeneous solution). It is expected that \( \eta^{\mu\nu} k_\mu k_\nu = -m_{KK}^2 \) corresponds to a resonance point in the continuous spectrum of \( \eta^{\mu\nu} k_\mu k_\nu \). Hence, there may be interesting phenomena near the point corresponding to the Kaluza-Klein mass, \( \eta^{\mu\nu} k_\mu k_\nu \approx -m_{KK}^2 \). However, we shall not see this in this paper since we shall adopt the expansion by a parameter \( \mu = l^{\eta^{\mu\nu} k_\mu k_\nu} \), where \( l \) is a length scale of the model. Further analysis near the resonance may be an interesting future work.

### A. Scalar perturbations

For scalar perturbations, we can use the the background equation to reduce the perturbed junction condition to

\[ 2\bar{\tau}_{(Y)} = 3\eta^{\mu\nu} k_\mu k_\nu \bar{\tau}_{(LL)}, \]

\[ \phi_{\pm w} = \mp k^2 e^{-A - \alpha_{\pm}^{(0)}} \bar{\tau}_{(LL)}. \]  

Here, we have assumed that \( k^\mu \neq 0 \) since a mode with \( k^\mu = 0 \) preserves 4-dimensional Poincare symmetry and can be included in the background. The first equation is a gauge-invariant expression of the perturbed conservation equation. The second equation gives the gauge-invariant perturbation \( \phi_{\pm w} \) of the position of each brane in terms of matter perturbations on the brane. The perturbed matching condition of the scalar field at \( w = w_\pm \), respectively, is

\[ \mp \left[-e^{3A} \tilde{\Psi}^{(0)} F_{ww} + 2e^A \varphi + 2e^{2A} (e^A \tilde{\Psi}^{(0)}) \phi_{\pm w} \right] = Y_\pm^{(0)} (\varphi + e^{2A} \tilde{\Psi}^{(0)} \phi_{\pm w} + 2\alpha_\pm^{(0)} e^{2A} \bar{\tau}_{(Y)}), \]  

\[ (54) \]
thus is easy to check, for instance, that ˜coefficients

\[ - \frac{3\kappa_5^2}{2\kappa_5^2 (\Psi)'(0)} \hat{F}, \]

and

\[ \hat{F} + \left[ 2\hat{A} - \frac{(e^A)''}{(e^A)'} \right] \hat{F} - \left[ 2e^{-A}(e^A)' + \eta^{\mu\nu} k_\mu k_\nu \right] F = 0. \] (56)

In the following arguments, we shall consider modes with \( \eta^{\mu\nu} k_\mu k_\nu = 0 \). These modes represent massless fields on branes and are called zero modes. As we shall see in the next section, for scalar perturbations it is actually sufficient to analyze zero modes in order to compare the low energy behavior of gravity in the brane world scenario with 4-dimensional Einstein gravity.

Setting \( \eta^{\mu\nu} k_\mu k_\nu = 0 \), it is easy to find a zero-mode solution \( F = F^{(1)} \equiv (e^A)' \) of (56). Another independent solution \( F = F^{(2)} \) can be easily found by using the fact that the Wronskian \( \Delta \equiv \hat{F}^{(1)} F^{(2)} - F^{(1)} \hat{F}^{(2)} \) obeys the equation \( \Delta = -B_1 \Delta \), where \( B_1 \) is the coefficient of \( \hat{F} \) in (56). The solution to the Wronskian equation is \( \Delta \propto e^{-2A(e^A)'}, \) and thus

\[ F^{(2)} \propto (e^A) \int dw \frac{\ddot{A} + \dot{A}^2}{A^2} e^{-3A}. \]

Hence, a general solution is

\[ F(w) = C_1 (e^A(w)) + C_2 (e^A(w)) \int_0^w dw' \frac{\ddot{A}(w') + \dot{A}(w')^2}{A(w')^2} e^{-3A(w')} \] (57)

For the solution (57), the perturbed matching condition (54) of the scalar field is

\[ \left( C_1 + C_2 B(w_+) \pm 2\kappa_5^2 e^{-2A_\pm -\alpha_\pm(0)} \bar{\tau}_{\pm(LL)} \right) \hat{B}_\pm = 0, \] (58)

where

\[ B(w) = \int_0^w dw' \frac{\ddot{A}(w') + \dot{A}(w')^2}{A(w')^2} e^{-3A(w')} + \frac{e^{-3A(w)}}{A(w)}, \]

\[ \hat{B}_\pm = (\ddot{A} + \dot{A}^2) \left( \Psi(0) + \hat{\Psi} \pm \frac{1}{2} e^{-A} V_\pm(0) \right) \bigg|_{w=w_\pm} = \frac{\kappa_5^2}{12} e^{-4AV_\pm'(0)^2} \left( U_\pm'(0) + \frac{\kappa_5^2}{3} V_\pm(0) V_\pm'(0) - \frac{1}{4} V_\pm'(0) V_\pm(0) \right), \] (59)

and \( A_\pm, \alpha_\pm(0) \) and \( U_\pm'(0) \) are abbreviations of \( A(w_\pm), \alpha_\pm(\Psi_\pm)' \) and \( U'(\Psi_\pm)' \), respectively. Here we have used the perturbed junction condition (53) and the background equations to simplify the equation. Hence, if \( \hat{B}_\pm \neq 0 \) then the coefficients \( C_1 \) and \( C_2 \) are uniquely determined as

\[ C_1 = \frac{2\kappa_5^2}{B(w_-) - B(w_+)} \left[ B(w_-) e^{-2A_- - \alpha_- (0)} \bar{\tau}_{+LL} + B(w_+ e^{-2A_+ - \alpha_+ (0)} \bar{\tau}_{-LL} \right], \]

\[ C_2 = \frac{-2\kappa_5^2}{B(w_+) - B(w_-)} \left[ e^{-2A_- - \alpha_- (0)} \bar{\tau}_{+LL} + e^{-2A_+ - \alpha_+ (0)} \bar{\tau}_{-LL} \right]. \] (60)

Note that there are no further conditions and that this solution represents a set of gauge invariant perturbations.

For further analysis it is rather important that \( \hat{B}_\pm \neq 0 \). The rare case of \( \hat{B}_\pm = 0 \) would correspond to the situation when scalar brane matter perturbations are not coupled with the bulk perturbations. It takes place, for instance for the RS models with constant \( U \) and \( V_\pm \). In general, the forms of the potentials not necessary lead to \( \hat{B}_\pm = 0 \). It is easy to check, for instance, that \( \hat{B}_\pm \neq 0 \) in the 5d reduction of HW theory [4] with \( U(\Psi) = \beta_5 \exp(-2\sqrt{2}\kappa_5 \Psi) \).
and $V_\pm = \pm \beta \sqrt{2} \exp (-\sqrt{2} \kappa_5 \psi)$, $\beta$ is a constant. For these potentials the radion is not stabilized in the sense that the inter-brane distance is arbitrary. On the other hand, we also may expect that there is non-vanishing coupling $\hat{B}_\pm$ between brane matter perturbations and bulk perturbations even in the case when the background inter-brane distance is arbitrary. A concrete example is the 5d reduction of HW theory as shown above. We conclude that the condition $\hat{B}_\pm \neq 0$ is insensitive to the issue of the radion stabilization.

Therefore, by using the formula (52), the doubly gauge invariant variable $\tilde{f}$ representing perturbations of the physical metric is written as

$$
\tilde{f}_\pm = -\frac{\kappa_5^2 e^{-2A_+ - \alpha_0^0}}{f_{w-}^\mu dw^\nu e^{-3A(w^-)}} \left[ e^{-2A_+ - \alpha_0^0} \tilde{\tau}_+(LL) + e^{-2A_- - \alpha_0^0} \tilde{\tau}_-(LL) \right].
$$

(61)

Here, we have used the fact that $\hat{B} = -2e^{-3A}$ to simplify $B(w_+) - B(w_-)$, and we have also used the background equation.

Although we worked in a sector with $\eta^{\mu\nu}k_\mu k_\nu = 0$, the result (61) can be considered as the lowest order term in the expansion of $\tilde{f}_\pm$ by the parameter $\mu \equiv l^2 \eta^{\mu\nu}k_\mu k_\nu$, where $l$ is a characteristic length scale of the model. (See (75) for the expansion of tensor perturbations.) Hence, the above result for $\mu = 0$ is enough to give the following expression of the lowest order term in the $\mu$ expansion for a small non-zero $\mu$.

$$
\tilde{f}_\pm = -\frac{\kappa_5^2 e^{-2A_+ - \alpha_0^0}}{f_{w-}^\mu dw^\nu e^{-3A(w^-)}} \left[ e^{-2A_+ - \alpha_0^0} \tilde{\tau}_+(LL) + e^{-2A_- - \alpha_0^0} \tilde{\tau}_-(LL) \right] + O(\mu).
$$

(62)

\textbf{B. Vector perturbations}

For vector perturbations, the perturbed junction condition at $w = w_\pm$, respectively, is

$$
F_w = \mp \kappa_5^2 e^{-A_\pm - \alpha_0^0} \tilde{\tau}_\pm(LL),
$$

(63)

and there is no matching condition for the scalar field.

The perturbed Einstein equation in the bulk for vector perturbations is

$$
\eta^{\mu\nu}k_\mu k_\nu F_w = 0, \quad (e^{-A} F_w') = 0.
$$

(64)

Hence, the solution is

$$
F_w = \begin{cases} 
C e^A, & \text{for } \eta^{\mu\nu}k_\mu k_\nu = 0 \\
0, & \text{for } \eta^{\mu\nu}k_\mu k_\nu \neq 0
\end{cases}
$$

(65)

where $C$ is a constant. For $\eta^{\mu\nu}k_\mu k_\nu = 0$, the junction condition overdetermines the coefficient $C$:

$$
C = -\kappa_5^2 e^{-2A_+ - \alpha_0^0} \tilde{\tau}_+(LL) = \kappa_5^2 e^{-2A_- - \alpha_0^0} \tilde{\tau}_-(LL).
$$

(66)

Hence, for $\eta^{\mu\nu}k_\mu k_\nu = 0$ we have the following relation between $\tilde{\tau}_+(LL)$ and $\tilde{\tau}_-(LL)$.

$$
e^{-2A_+ - \alpha_0^0} \tilde{\tau}_+(LL) + e^{-2A_- - \alpha_0^0} \tilde{\tau}_-(LL) = 0.
$$

(67)

As we shall see in the next section, this relation is important for restoring 4-dimensional Einstein gravity at low energies. For $\eta^{\mu\nu}k_\mu k_\nu \neq 0$, the junction condition becomes

$$
\tilde{\tau}_+(LL) = \tilde{\tau}_-(LL) = 0.
$$

(68)

Note that for vector perturbations there is no doubly gauge invariant perturbation of physical metric on the branes.
For tensor perturbations, the perturbed junction condition at \( w = w_{\pm} \), respectively, is reduced to

\[
(e^{2A} F_{(T)})' = \pm \kappa_5^2 e^{A_{\pm} - \alpha_{\pm}^{(0)}} \tilde{\tau}_{\pm(T)}
\]

and there is no matching condition for the scalar field.

The perturbed Einstein equation for tensor perturbations in the bulk is

\[
e^A \left[ e^{-3A} (e^{2A} F_{(T)})' \right] - \eta^\mu\nu k_\mu k_\nu F_{(T)} = 0.
\]

In the following arguments, we shall first consider zero modes, or modes with \( \eta^{\mu\nu} k_\mu k_\nu = 0 \). Contrary to the scalar perturbations, we will see in the next section that for tensor perturbations it is not sufficient to analyze zero modes in order to compare the low energy behavior of gravity in the brane world scenario with 4-dimensional Einstein gravity. Hence, after analyzing zero modes, we shall investigate non-zero modes, or so called Kaluza-Klein modes by perturbing with respect to \( \eta^{\mu\nu} k_\mu k_\nu \) around the zero modes.

For \( \eta^{\mu\nu} k_\mu k_\nu = 0 \), the solution of the bulk Einstein equation (70) is

\[
F_{(T)}(w) = D_1 e^{-2A(w)} \int_0^w dw' e^{3A(w')} + D_2 e^{-2A(w)},
\]

where \( D_1 \) and \( D_2 \) are constants. By using the formula (52), the doubly gauge invariant variable \( \tilde{f}_{\pm(T)} \) representing perturbations of the physical metric can be written as

\[
\tilde{f}_{\pm(T)} = D_1 e^{-2A_{\pm} - \alpha_{\pm}^{(0)}} \int_0^w dw' e^{3A(w')} + D_2 e^{-2A_{\pm} - \alpha_{\pm}^{(0)}}.
\]

The junction condition overdetermines the coefficient \( D_1 \):

\[
D_1 = \kappa_5^2 e^{-2A_{\pm} - \alpha_{\pm}^{(0)}} \tilde{\tau}_{\pm(T)} = -\kappa_5^2 e^{-2A_{\pm} - \alpha_{\pm}^{(0)}} \tilde{\tau}_{\mp(T)}.
\]

Hence, for \( \eta^{\mu\nu} k_\mu k_\nu = 0 \) we have the following relation between \( \tilde{\tau}_{\mp(T)} \) and \( \tilde{\tau}_{\pm(T)} \).

\[
e^{-2A_{\pm} - \alpha_{\pm}^{(0)}} \tilde{\tau}_{\pm(T)} + e^{-2A_{\pm} - \alpha_{\pm}^{(0)}} \tilde{\tau}_{\mp(T)} = 0.
\]

As we shall see in the next section, this relation is important for restoring 4-dimensional Einstein gravity at low energies.

For \( \eta^{\mu\nu} k_\mu k_\nu \neq 0 \) it is not trivial to solve (70) without specifying the model potentials. Hence, we shall perform perturbation with respect to \( \mu = l^2 \eta^{\mu\nu} k_\mu k_\nu \) around the zero mode solution, where \( l \) is a characteristic length scale of the model.

By expanding \( F_{(T)} \) and \( \tilde{\tau}_{(T)} \) as

\[
F_{(T)} = F^{[0]}_{(T)} + \mu F^{[1]}_{(T)} + O(\mu^2),
\]

\[
\tilde{\tau}_{(T)} = \bar{\tau}_{[0]}_{(T)} + \mu \bar{\tau}_{[1]}_{(T)} + O(\mu^2),
\]

we shall solve the Einstein equation (70) in the bulk and the junction condition (69) order by order. The zeroth order analysis is exactly the same as the previous analysis of the zero modes. Hence, the zeroth order Einstein equation in the bulk has the solution

\[
F^{[0]}_{(T)}(w) = \tilde{D}_1 e^{-2A(w)} \int_0^w dw' e^{3A(w')} + \tilde{D}_2 e^{-2A(w)},
\]

where \( \tilde{D}_1 \) and \( \tilde{D}_2 \) are constants, and the zeroth order junction condition overdetermines the coefficient \( \tilde{D}_1 \):

---

\[3\] The characteristic length \( l \) is determined by the background solution [22] and determines the energy scale below which the low energy description is valid.
\[
\tilde{D}_1 = \kappa_5^2 e^{-2A_+ - \alpha_+^{(0)}} \tau_+^{(T)} = -\kappa_5^2 e^{-2A_- - \alpha_-^{(0)}} \tau_-^{(T)},
\]

and thus we have the following relation between \(\tilde{\tau}^{[0]}_+\) and \(\tilde{\tau}^{[0]}_-\):

\[
e^{-2A_+ - \alpha_+^{(0)}} \tau_+^{[0]} + e^{-2A_- - \alpha_-^{(0)}} \tau_-^{[0]} = 0.
\]

Note that the zeroth order analysis leaves the coefficient \(\tilde{D}_2\) undetermined.

To first order in \(\mu\), the Einstein equation (70) becomes

\[
l^2 e^A \left[ e^{-3A} (e^{2A} F_1^{(1)}(T)) \right] - F_1^{(0)} = 0.
\]

Hence, we obtain

\[
l^2 e^{-3A} (e^{2A} F_1^{(1)}(T)) = \tilde{D}_1 B_1(w) + \tilde{D}_2 B_2(w) + \tilde{D}_3,
\]

where \(\tilde{D}_3\) is a constant and

\[
B_1(w) = \int_0^w dw' e^{-3A(w')} \int_0^{w'} dw'' e^{3A(w'')},
\]

\[
B_2(w) = \int_0^w dw' e^{-3A(w')}.
\]

By integrating (80) once more we obtain a solution of the bulk Einstein equation to this order. For the solution, the junction condition (69) becomes

\[
\tilde{D}_1 B_1(w_\pm) + \tilde{D}_2 B_2(w_\pm) + \tilde{D}_3 = \pm \kappa_5^2 e^{-2A_+ - \alpha_+^{(0)}} \tilde{\tau}_\pm^{[1]}(T).
\]

Hence, the constants \(\tilde{D}_2\) and \(\tilde{D}_3\) are uniquely determined by \(\tilde{\tau}_\pm^{[0,1]}(T)\). In particular,

\[
\tilde{D}_2 = \frac{\kappa_5^2}{B_2(w_+) - B_2(w_-)} \left[ e^{-2A_+ - \alpha_+^{(0)}} \tilde{\tau}_+^{[1]}(T) + e^{-2A_- - \alpha_-^{(0)}} \tilde{\tau}_-^{[1]}(T) \right] - \frac{B_1(w_+) - B_1(w_-)}{B_2(w_+) - B_2(w_-)} \tilde{D}_1.
\]

Therefore, for small \(|\mu| = |l^2 \eta^{\mu\nu} k_{\mu} k_{\nu}|\), by using the formula (52), the doubly gauge invariant variable \(\tilde{f}_\pm(T)\) representing perturbations of the physical metric can be given by

\[
e^{2A_+ + \alpha_+^{(0)}} \tilde{f}_\pm(T) = -\tilde{D}_1 \frac{\int_{w_-}^{w_+} \int_{w_-}^{w_+} dw' e^{-3A(w')} \int_{w_-}^{w_+} dw'' e^{3A(w'')}}{\int_{w_-}^{w_+} dw' e^{-3A(w')}}
\]

\[
+ \frac{\kappa_5^2}{B_2(w_+) - B_2(w_-)} \left[ e^{-2A_+ - \alpha_+^{(0)}} \tilde{\tau}_+^{[1]}(T) + e^{-2A_- - \alpha_-^{(0)}} \tilde{\tau}_-^{[1]}(T) \right] + O(\mu),
\]

where \(\tilde{D}_1\) is given by (77) and \(\tilde{\tau}_\pm^{[0]}(T)\) must satisfy the relation (78).

V. COMPARISON WITH FOUR-DIMENSIONAL EINSTEIN GRAVITY

In this section we compare gravity in the brane world scenario with 4-dimensional Einstein gravity. We show that 4-dimensional Einstein gravity is restored at low energies on our brane, provided that matter fields on the hidden brane are not excited. This qualitative result does not depend on which brane we are living. However, quantitatively, the effective 4-dimensional gravitational constants on the two branes are different.
For the purpose of comparison, we review linearized Einstein gravity in 4-dimensions in terms of gauge-invariant variables. We consider general perturbations around the Minkowski spacetime. Namely, we consider the metric
\[ \bar{q}^{(0)}_{\mu\nu} = \Omega^2 \eta_{\mu\nu}, \] (85)
where \( \Omega \) is a non-zero constant conformal factor, and
\[ \delta \bar{q} = \bar{\sigma}(T)T_{(T)\mu\nu} + \bar{\sigma}(LT)T_{(LT)\mu\nu} + \bar{\sigma}(LL)L_{(LL)\mu\nu} + \bar{\sigma}(Y)Y_{(Y)\mu\nu}. \] (87)
Here, the coefficients \( \bar{\sigma}(T), \bar{\sigma}(LT), \bar{\sigma}(LL), \) and \( \bar{\sigma}(Y) \) are constants. Following the previous section, we can construct gauge-invariant variables \( \bar{f}(T) \) and \( \bar{f} \).
\[ \bar{f}(T) = \bar{\sigma}(T), \]
\[ \bar{f} = \bar{\sigma}(Y) + \frac{1}{2} \eta^{\mu\nu}k_\mu k_\nu \bar{\sigma}(LL). \] (88)
As for the stress energy tensor \( \bar{S}_{\mu\nu} \), we consider it as a first order quantity and expand it as follows.
\[ \bar{S}_{\mu\nu} = \bar{\tau}(T)T_{(T)\mu\nu} + \bar{\tau}(LT)T_{(LT)\mu\nu} + \bar{\tau}(LL)L_{(LL)\mu\nu} + \bar{\tau}(Y)Y_{(Y)\mu\nu}, \] (89)
where the coefficients \( \bar{\tau}(T,LT,LL,Y) \) are constants. In the Minkowski background these coefficients are gauge-invariant by themselves.
We can expand the four-dimensional Einstein equation \( \bar{G}_{\mu\nu} = 8\pi G_N \bar{S}_{\mu\nu} \) up to first order in the perturbations, where \( \bar{G}_{\mu\nu} \) is the Einstein tensor constructed from the metric \( \bar{q}_{\mu\nu} \), and express it in terms of the above gauge-invariant variables. The Einstein equation is
\[ 2\bar{f} = 3\bar{\tau}(Y), \]
\[ \bar{f} = -16\pi G_N \Omega^2 \bar{\tau}(LL) \] (90)
for scalar perturbations,
\[ \bar{\tau}(LT) = 0 \] (91)
for vector perturbations, and
\[ \bar{q}^{(0)}_{\mu\nu}k_\mu k_\nu \bar{f}(T) = 16\pi G_N \bar{\tau}(T) \] (92)
for tensor perturbations.
Note that in such a gauge \(^4\) that \( \partial^\nu(\delta \bar{q}_{\mu\nu} - \bar{q}^{(0)}_{\mu\nu}\delta \bar{q}^{(0)}_{\rho\sigma}/2) = 0 \), the variable \( \bar{f} \) is given by \( \bar{f} = (2/3)\bar{\sigma}(Y) \). Hence, in this gauge the second equation of (90) becomes
\[ \bar{q}^{(0)}_{\mu\nu}k_\mu k_\nu \sigma(Y) = -16\pi G_N \bar{\tau}(Y) \] (93)
It is easy to confirm by noting \( \square = -\bar{q}^{(0)}_{\mu\nu}k_\mu k_\nu \) that equations (92) and (93) are equivalent to (8) with \( B = 0 \). Of course it is also easy to derive Newton’s law from equation (3) with \( \Delta K = B = 0 \) [27].

\(^4\)In (3) and (4) we adopted this gauge choice.
Here we shall apply the results of Sec. IV to the case that our 4-dimensional universe is one of the two branes \( \Sigma_\pm \) and that there is no matter excitation on the other brane.

First, let us suppose that our 4-dimensional universe is \( \Sigma_+ \) and that there is no matter excitation on the other brane \( \Sigma_- \). For scalar perturbations, by setting \( \tau_- (Y) = \bar{\tau}-(LL) = 0 \), we obtain

\[
\bar{f}_+ = - \frac{\kappa^2 e^{-2A_+ - \alpha_+ (w)}}{\int_{w_-}^{w_+} dw' e^{-3A(w')} e^{-2A_+ - \alpha_+ (w')} \bar{\tau}_+(LL)}
\]

(94)

for zero modes. This equation can be considered the lowest order equation in an expansion in terms of \( \mu = l^2 \eta^{\mu \nu} k_\mu k_\nu \), where \( l \) is a characteristic length scale of the model \(^5\). Moreover, we have the conservation equation

\[
2\bar{\tau}_+(Y) = 3\eta^{\mu \nu} k_\mu k_\nu \bar{\tau}_+(LL)
\]

(95)

for all modes.

For vector perturbations, by setting \( \bar{\tau}_-(LT) = 0 \) we obtain

\[
\bar{\tau}_+(LT) = 0
\]

(96)

for general \( k_\mu \). Here, we have used the relation (67) for zero modes, or modes with \( \eta^{\mu \nu} k_\mu k_\nu = 0 \). For tensor perturbations, by setting \( \bar{\tau}_-(T) = \bar{\tau}_+(T) = 0 \), we obtain

\[
\bar{q}^{(0) \mu \nu} k_\mu k_\nu \bar{f}_+(T) = \frac{\kappa^2 e^{-2A_+ - \alpha_+ (w)}}{\int_{w_-}^{w_+} dw' e^{-3A(w')} e^{-2A_+ - \alpha_+ (w')} \bar{\tau}_+(T)}
\]

(97)

for small but non-zero \( |\mu| = |l^2 \eta^{\mu \nu} k_\mu k_\nu| \). Here, we have used the relation (78) to show that \( \bar{\tau}_+(T) = 0 \). Actually, eq. (97) remains correct even for \( \eta^{\mu \nu} k_\mu k_\nu \to 0 \) since \( D_2 \) in (72) is arbitrary and the relation (74) implies \( \bar{\tau}_+(T) = 0 \).

Next, let us suppose that our 4-dimensional universe is \( \Sigma_- \) and that there is no matter excitation on the other brane \( \Sigma_+ \). Repeating the above analysis, we can obtain the same result with \( \pm \) replaced by \( \mp \). For scalar perturbations, we have

\[
2\bar{\tau}_-(Y) = 3\eta^{\mu \nu} k_\mu k_\nu \bar{\tau}_-(LL),
\]

\[
\bar{f}_- = - \frac{\kappa^2 e^{-2A_- - \alpha_- (w)}}{\int_{w_-}^{w_+} dw' e^{-3A(w')} e^{-2A_- - \alpha_- (w')} \bar{\tau}_-(LL)}
\]

(98)

for zero modes.

For vector perturbations, we have

\[
\bar{\tau}_-(LT) = 0
\]

(99)

for general \( k_\mu \). For tensor perturbations, we have

\[
\bar{q}^{(0) \mu \nu} k_\mu k_\nu \bar{f}_-(T) = \frac{\kappa^2 e^{-2A_- - \alpha_- (w)}}{\int_{w_-}^{w_+} dw' e^{-3A(w')} e^{-2A_- - \alpha_- (w')} \bar{\tau}_-(T)}
\]

(100)

for \( |\mu| \) zero or sufficiently small.

Therefore, by setting \( \Omega^2 = e^{-2A_\pm - \alpha_\pm (w)} \), 4-dimensional Einstein gravity is restored at low energies, provided that there are no tachyonic modes, or positive \( \mu \) modes, with \( \bar{\tau}_\pm (T, LT, LL, Y) = 0 \). The condition \( \bar{\tau}_\pm (T, LL, Y) = 0 \) for scalar and tensor perturbations corresponds to the boundary condition

\[^5\text{See eq. (75) for tensor perturbations and the footnote before it.}\]
Suppose energy-momentum tensor at the visible brane is from the hidden brane, which enters the final results in a symmetric manner, see the last equations in Sec. IVA and (41). It is instructive to write down the most generic weak gravity equations taking into account the contribution from a bulk scalar field and investigated gauge-invariant perturbations around a background with two parallel Minkowski branes. By using the doubly covariant formalism of perturbation of Israel’s junction conditions [23] developed in Sec. II, use linearized metric perturbations $h_{\mu\nu}$ on both branes. These doubly gauge invariant perturbations include scalar, vector, and tensor perturbations. After that, we have shown that 4-dimensional Einstein gravity is restored on both branes at low energies, provided that $w_-$ is positive on both branes. It is worth while mentioning that the following naive expectation holds.

$$G_{N+} = e^{-2A_+-a_+^{(0)}}$$

$$G_{N-} = e^{-2A_--a_-^{(0)}}$$

where $q_{\pm \mu \nu}$ are the unperturbed physical metrics on $\Sigma^{(0)}_{\pm}$, respectively.

**VI. SUMMARY**

We have investigated gravity in brane worlds. In particular, we considered a general model on a $Z_2$-orbifold with a bulk scalar field and investigated gauge-invariant perturbations around a background with two parallel Minkowski branes. By using the doubly covariant formalism of perturbation of Israel’s junction conditions [23] developed in ref. [24], we have obtained equations governing the low energy dynamics of doubly gauge invariant perturbations on the branes. These doubly gauge invariant perturbations include scalar, vector, and tensor perturbations. After that, we have shown that 4-dimensional Einstein gravity is restored on both branes at low energies, provided that combinations of the bulk and the brane potentials $B_\pm$ given by (59) are not zero and that there are no tachyonic modes for scalar and tensor perturbations with the boundary condition (101). The effective 4-dimensional Newton’s constants $G_{N\pm}$ on $\Sigma_{\pm}$ are given by

$$16\pi G_{N\pm} = \frac{\kappa_5^2 e^{-2A_\pm-a_\pm^{(0)}}}{\int_{w_-} e^{-3A(w')}}$$

Since $w_- < w_+$, $G_{N\pm}$ is positive on both branes. It is worth while mentioning that the following naive expectation holds.

$$\frac{G_{N+}}{G_{N-}} = \frac{e^{-2A_+-a_+^{(0)}}}{e^{-2A_--a_-^{(0)}}} = \frac{\tilde{q}_{\pm \mu \nu}^{(0)}}{\tilde{q}_{- \mu \nu}^{(0)}}$$

Therefore, in principle, we need four gravitational couplings, two for tensor modes and two for scalar modes, to describe the linearized gravity at the visible brane. We also learned that each of them are constructed from the fundamental 5d gravitational constant, $\kappa_5$, and combinations of the warp factor and “effective” extra dimensional distance $l \sim \int_{w_-}^{w_+} dw' e^{-3A(w')}$. It turns out that the BD parameter, which characterizes the gravitational coupling for the scalar mode, depends on the bulk scalar sector. In terms of coefficient $B$ of Sec. II, $B^\pm$ is none-zero for the case without bulk scalar, $B^\pm=0$ for the case of a single scalar field. We conjecture, based on our derivation with a single

\[6\] If we had choose the convention $w_- > w_+$ then the sign in (28) would be reversed and $G_{\pm N}$ would still be positive, as should be.
scalar, that in general case of more than one bulk scalar, $B^\pm$ again will be non-zero. It will be interesting to calculate it explicitly for next simplest case of two bulk scalars.

Equation (104) may have interesting insights from point of view of superstring (Horava-Witten) phenomenology. Indeed, it says that there are, in general, four different Planckian masses! It also teaches us that derivation of 4d effective theory by means of $\int dw$ integration will wash out the subtleties of the actual gravitational dynamics. Finally, from point of view of gravity and its application to astronomy, it gives us rather unusual theory with four gravitational couplings.

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APPENDIX A: HARMONICS IN MINKOWSKI SPACETIME

In this appendix we give definitions of scalar, vector and tensor harmonics in an $n$-dimensional Minkowski spacetime. Throughout this appendix, $n$-dimensional coordinates are $x^\mu (\mu = 0, 1, \cdots, n-1)$, $\eta_{\mu\nu}$ is the Minkowski metric, and all indices are raised and lowered by the Minkowski metric and its inverse $\eta^{\mu\nu}$.

1. Scalar harmonics

The scalar harmonics are given by

$$Y = \exp(-ik_{\rho}x^{\rho}), \quad (A1)$$

by which any function $f$ can be expanded as

$$f = \int dk \: cY, \quad (A2)$$

where $c$ is a constant depending on $k$. Hereafter, $k$ and $dk$ are abbreviations of $\{k^\mu\} (\mu = 0, 1, \cdots, n-1)$ and $\prod_{\mu=0}^{n-1} dk^\mu$, respectively. We omit $k$ in most cases.

2. Vector harmonics

In general, any vector field $v_{\mu}$ can be decomposed as

$$v_{\mu} = v_{(T)\mu} + \partial_{\mu} f, \quad (A3)$$

where $f$ is a function and $v_{(T)\mu}$ is a transverse vector field:

$$\partial^\mu v_{(T)\mu} = 0. \quad (A4)$$

Thus, the vector field $v_{\mu}$ can be expanded by using the scalar harmonics $Y$ and transverse vector harmonics $V_{(T)\mu}$ as

$$v_{\mu} = \int dk \: [c_{(T)} V_{(T)\mu} + c_{(L)} \partial_{\mu} Y]. \quad (A5)$$

Here, $c_{(T)}$ and $c_{(L)}$ are constants depending on $k$, and the transverse vector harmonics $V_{(T)\mu}$ are given by

$$V_{(T)\mu} = u_{\mu} \exp(-ik_\rho x^\rho), \quad (A6)$$

where the constant vector $u_{\mu}$ satisfies the following condition.
and transverse traceless tensor harmonics $T_{\mu \nu}$ that these numbers are equal to the numbers of physical degrees of freedom for massive and massless spin-1 fields in $n$-dimensions, respectively.

Because of the expansion (A5), it is convenient to define longitudinal vector harmonics $V_{(L)\mu}$ by

$$V_{(L)\mu} \equiv \partial_\mu Y = -i k_\mu Y.$$  

(A9)

3. Tensor harmonics

In general, a symmetric second-rank tensor field $t_{\mu \nu}$ can be decomposed as

$$t_{\mu \nu} = t_{(T)\mu \nu} + \partial_\mu v_\nu + \partial_\nu v_\mu + f \eta_{\mu \nu},$$  

(A10)

where $f$ is a function, $v_\mu$ is a vector field and $t_{(T)\mu \nu}$ is a transverse traceless symmetric tensor field:

$$t_{(T)\mu} = 0, \quad \partial_{\mu} t_{(T)\nu} = 0.$$  

(A11)

Thus, the tensor field $t_{\mu \nu}$ can be expanded by using the scalar harmonics $Y$, the vector harmonics $V_{(T)}$ and $V_{(L)}$, and transverse traceless tensor harmonics $T_{(T)}$ as

$$t_{\mu \nu} = \int dk \left[ c_{(T)} T_{(T)\mu \nu} + c_{(LT)} (\partial_\mu V_{(T)\nu} + \partial_\nu V_{(T)\mu}) 
+ c_{(LL)} (\partial_\mu V_{(L)\nu} + \partial_\nu V_{(L)\mu}) + \tilde{c}_{(Y)} Y \eta_{\mu \nu} \right].$$  

(A12)

Here, $c_{(T)}$, $c_{(LT)}$, $c_{(LL)}$, and $\tilde{c}_{(Y)}$ are constants depending on $k$, and the transverse traceless tensor harmonics $T_{(T)}$ are given by

$$T_{(T)\mu \nu} = s_{\mu \nu} \exp(-i k_\rho x^\rho),$$  

(A13)

where the constant symmetric second-rank tensor $s_{\mu \nu}$ satisfies the following condition.

$$k^\mu s_{\mu \nu} = 0, \quad \partial_\mu s_\nu = 0.$$  

(A14)

for $k^\mu k_\mu \neq 0$, and

$$k^\mu s_{\mu \nu} = 0, \quad s_\mu^\nu = 0, \quad \tau^\nu s_{\mu \nu} = 0.$$  

(A15)

for non-vanishing $k_\mu$ satisfying $k^\mu k_\mu = 0$, where $\tau^\mu$ is an arbitrary constant timelike vector. For $k_\mu = 0$, the constant tensor $s_{\mu \nu}$ does not need to satisfy any of the above conditions. For the special case $k^\mu k_\mu = 0$, the last condition in (A15) can be imposed by redefinition of $c_{(LT)}$, $c_{(LL)}$, and $\tilde{c}_{(Y)}$. Actually this condition is necessary to eliminate redundancy. Note that the number of independent symmetric second-rank tensors satisfying the above conditions is $(n + 1)(n - 2)/2$ for $k^\mu k_\mu \neq 0$ and $n(n - 3)/2$ for $k^\mu k_\mu = 0$ and that these numbers are equal to numbers of physical degrees of freedom for massive and massless spin-2 fields in $n$-dimensions, respectively.

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Because of the expansion (A12), it is convenient to define tensor harmonics $T_{(LT)}$, $T_{(LL)}$, and $T_{(Y)}$ by

\[ T_{(LT)}^{\mu\nu} \equiv \partial_\mu V(T)_{\nu} + \partial_\nu V(T)_{\mu}, \]
\[ = -i(u_\mu k_\nu + u_\nu k_\mu)Y, \]
\[ T_{(LL)}^{\mu\nu} \equiv \partial_\mu V(L)_{\nu} + \partial_\nu V(L)_{\mu} - \frac{2}{n}\eta^{\rho}_{\mu\nu}\partial_\rho V(L)_{\rho}, \]
\[ = \left(-2k_\mu k_\nu + \frac{2}{n}k^\rho k_\rho \eta_{\mu\nu}\right)Y, \]
\[ T_{(Y)}^{\mu\nu} \equiv \eta_{\mu\nu} Y. \]  

(A16)