We discuss the asymptotic dynamical evolution of spatially homogeneous brane-world cosmological models close to the initial singularity. We find that generically the cosmological singularity is isotropic in Bianchi type IX brane-world models and consequently these models do not exhibit Mixmaster or chaotic-like behaviour close to the initial singularity. We argue that this is typical of more general cosmological models in the brane-world scenario. In particular, we show that an isotropic singularity is a past-attractor in all orthogonal Bianchi models and is a local past-attractor in a class of inhomogeneous brane-world models.

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I. INTRODUCTION

Higher-dimensional gravity theories inspired by string theory, in which the matter fields are confined to a 3-dimensional ‘brane-world’ embedded in 1 + 3 + d dimensions while the gravitational field can also propagate in the d extra dimensions (i.e., in the ‘bulk’) [1], are currently of great interest. The d extra dimensions need not be small or even compact in these theories, which effectively become 4-dimensional at lower energies. In recent work Randall and Sundrum [2] have shown that for d = 1, gravity can be localized on a single 3-brane even when the fifth dimension is infinite. In this paradigm, Einstein’s theory of general relativity (GR) must be modified at high energies (i.e., at early times). An elegant geometric formulation and generalization of the Randall-Sundrum scenario has been given [3,4]. Much effort has already been devoted to understand the cosmology of such a Randall-Sundrum-type brane-world scenario [5,6].

The dynamical equations on the 3-brane differ from the GR equations by terms that carry the effects of imbedding and of the free gravitational field in the five-dimensional bulk [3,4]. The local (quadratic) energy-momentum corrections are significant only at very high energies. In addition to the matter fields corrections, there are nonlocal effects from the free gravitational field in the bulk, transmitted via the projection $E_{\mu\nu}$ of the bulk Weyl tensor, that contribute further corrections to the Einstein equations (e.g., the Friedmann equation) on the brane. Due to its symmetry properties, $E_{\mu\nu}$ can be irreducibly decomposed (with respect to a timelike congruence $u^\mu$) into

$$E_{\mu\nu} = -\left(\frac{\kappa}{\kappa}\right)^4 \left[ U (u_\mu u_\nu + \frac{1}{3}h_{\mu\nu}) + P_{\mu\nu} + 2Q_{(\mu}u_{\nu)}\right], \quad (1)$$

in terms of an effective nonlocal energy density on the brane, $U$, arising from the free gravitational field in the bulk, an effective nonlocal anisotropic stress on the brane, $P_{\mu\nu}$, and an effective nonlocal energy flux on the brane, $Q_\mu$ [4].

In general, the conservation equations do not determine all of the independent components of $E_{\mu\nu}$ on the brane. In particular, there is no evolution equation for $P_{\mu\nu}$ and hence, in general, the projection of the 5-dimensional field equations onto the brane does not lead to a closed system. However, in the cosmological context, in which the background metric is spatially homogeneous and isotropic, we have that

$$D_\mu U = Q_\mu = P_{\mu\nu} = 0, \quad (2)$$

where $D_\mu$ is the totally projected part of the brane covariant derivative. Since $P_{\mu\nu} = 0$, in this case the evolution of $E_{\mu\nu}$ is fully determined [2]. In general $U = U(t) \neq 0$ (and can be negative) in the Friedmann background [5,6]. For a spatially homogeneous and isotropic model on the brane equation (2) follows, and similar conditions apply self-consistently in other Bianchi models [4]. The Friedmann brane-world models have been extensively investigated [5,6].

There are many reasons to consider the classical dynamical evolution in more general spatially homogeneous Randall-Sundrum-type brane-world cosmological models, particularly in the early Universe in which the dynamical behaviour can be completely different to that of GR. In this article we shall consider the local dynamical behaviour of the general Bianchi type IX models close to the singularity.

Due to the existence of monotone functions, it is known that there are no periodic or recurrent orbits in orthogonal spatially homogeneous Bianchi type IX models in GR. In particular, there are no sources or sinks and generically...
Bianchi type IX models have an oscillatory behaviour with chaotic-like characteristics, with the matter density becoming dynamically negligible as one follows the evolution into the past towards the initial singularity. Using qualitative techniques, Ma and Wainwright (see [7]) have shown that the orbits of the associated cosmological dynamical system are negatively asymptotic to a lower two-dimensional attractor. This is the union of three ellipsoids in $\mathbb{R}^3$ consisting of the Kasner ring joined by Taub separatrices; the orbits spend most of the time near the self-similar Kasner vacuum equilibrium points. More rigorous global results are possible. Ringström has proven that a curvature invariant is unbounded in the incomplete directions of inextendible null geodesics for generic vacuum Bianchi models, and has rigorously shown that the Mixmaster attractor is the past attractor of Bianchi type IX models with an orthogonal perfect fluid [8].

All spatially homogeneous models in GR expand indefinitely except for the Bianchi type IX models. Bianchi type IX models obey the “closed universe recollapse” conjecture [9], whereby initially expanding models enter a contracting phase and recollapse to a future “Big Crunch”. All orbits in the Bianchi IX invariant sets are positively departing; in order to analyse the future asymptotic states of such models it is necessary to compactify phase-space. The description of these models in terms of conventional Hubble- or expansion-normalized variables is only valid up to the point of maximum expansion (where $H = 0$). An appropriate set of alternative normalised variables, which leads to the compactification of Bianchi IX state space, were suggested in [7] (see section 8.5.2) and have been utilized in the qualitative study of locally rotationally symmetric perfect fluid models [10], of scalar field models with an exponential potential [11], and scalar field models with a perfect fluid [12]. In particular, spatially homogeneous Bianchi models containing a scalar field $\phi$ with an exponential potential of the form $V(\phi) = \Lambda e^{k\phi}$ were studied in [13]. For Bianchi type IX models, if the parameter $k < \sqrt{2}$ then there exists a set of ever-expanding anisotropic Bianchi type IX models that isotropize and inflate towards an expanding power-law inflationary solution, except for the subset that recollapse [14], and if $k > \sqrt{2}$ then initially expanding Bianchi type IX models do not isotropize towards an ever-expanding isotropic model [11].

II. GOVERNING EQUATIONS

The field equations induced on the brane are derived via an elegant geometric approach by Shiromizu et al. [3,4], using the Gauss-Codazzi equations, matching conditions and $Z_2$ symmetry. The result is a modification of the standard Einstein equations, with the new terms carrying bulk effects onto the brane:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \kappa^4 S_{\mu\nu} - \mathcal{E}_{\mu\nu},$$

where

$$\kappa^2 = \frac{8\pi}{M_p^2}, \quad \lambda = \frac{\kappa^2}{\kappa^4}, \quad \Lambda = \frac{2\pi}{M_p^2} \left[ \bar{\Lambda} + \left( \frac{4\pi}{3M_p^3} \right) \lambda^2 \right].$$

The brane energy-momentum tensor for a perfect fluid or a minimally-coupled scalar field is given by

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu},$$

where $u^\mu$ is the 4-velocity, $\rho$ and $p$ are the energy density and isotropic pressure, and the projection tensor $h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$ projects orthogonal to $u^\mu$. We shall assume that the matter content is equivalent to that of a non-tilting perfect fluid with a linear barotropic equation of state for the fluid, i.e., $p = (\gamma - 1)\rho$, where the energy conditions impose the restriction $\rho \geq 0$, and the constant $\gamma$ satisfies $\gamma \in [0, 2]$ from causality requirements. For a minimally coupled scalar field the energy density and pressure are, respectively,

$$\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2} \dot{\phi}^2 - V(\phi),$$

and the conservation law is equivalent to the Klein-Gordon equation. A dynamical analysis of scalar field models indicates that at early times the scalar field is effectively massless [12]. A massless scalar field is equivalent to a perfect fluid with a stiff equation of state parameter $\gamma = 2$. In the early Universe, and close to the singularity, we expect that $\gamma \geq 4/3$.

The bulk corrections to the Einstein equations on the brane are of two forms: firstly, the matter fields contribute local quadratic energy-momentum corrections via the tensor $S_{\mu\nu}$, and secondly, there are nonlocal effects from the free gravitational field in the bulk, transmitted via the projection $\mathcal{E}_{\mu\nu}$ of the bulk Weyl tensor. The matter corrections are given by
\[ S_{\mu\nu} = \frac{1}{12} T_{\alpha\beta} \Gamma_{\mu\nu}^{\alpha\beta} - \frac{1}{4} T_{\mu\alpha} T_{\nu}^{\alpha} + \frac{1}{24} g_{\mu\nu} \left[ 3 T_{\alpha\beta} T_{\alpha\beta} - (T_{\alpha})^2 \right], \]  

which is equivalent to

\[ S_{\mu\nu} = \frac{1}{12} \rho \mu_\mu + \frac{1}{4} \rho (p + 2 p) \eta_{\mu\nu}, \]

for a perfect fluid or minimally-coupled scalar field. The quadratic energy-momentum corrections to standard GR will be significant for \( \tilde{\kappa}^2 \rho^2 \gtrsim \kappa^2 \rho \) in the high-energy regime.

All of the bulk corrections may be consolidated into an effective total energy density, pressure, anisotropic stress and energy flux, as follows. The modified Einstein equations take the standard Einstein form with a redefined energy-momentum tensor:

\[ G_{\mu\nu} = \kappa^2 T_{\mu\nu}^{\text{tot}}, \]

where

\[ T_{\mu\nu}^{\text{tot}} = T_{\mu\nu} + \tilde{\kappa}^4 S_{\mu\nu} - \frac{1}{\kappa^2} \mathcal{E}_{\mu\nu} - \frac{\Lambda}{\kappa^2} g_{\mu\nu} \]  

is the redefined perfect fluid (or minimally coupled scalar field) energy-momentum tensor with

\[ \rho^{\text{tot}} = \rho + \frac{\tilde{\kappa}^4}{\kappa^6} \left[ \kappa^4 \frac{1}{12} \rho^2 + \mathcal{U} \right] + \frac{\Lambda}{\kappa^2} \]

and

\[ p^{\text{tot}} = p + \frac{\tilde{\kappa}^4}{\kappa^6} \left[ \kappa^4 \frac{1}{12} (\rho^2 + 2 \rho p) + \frac{1}{3} \mathcal{U} \right] - \frac{\Lambda}{\kappa^2} \]

where we recall that \( D_\mu U = Q_\mu = P_{\mu\nu} = 0 \) in the cosmological case of interest here.

As a consequence of the form of the bulk energy-momentum tensor and of \( Z_2 \) symmetry, it follows [3] that the brane energy-momentum tensor separately satisfies the conservation equations, i.e.,

\[ \nabla^\nu T_{\mu\nu} = 0, \]

whence the Bianchi identities on the brane imply that the projected Weyl tensor obeys the constraint

\[ \nabla^\mu \mathcal{E}_{\mu\nu} = \tilde{\kappa}^4 \nabla^\mu S_{\mu\nu}. \]

In the spatially homogeneous case, the conservation equations for a non-tilting perfect fluid with a linear barotropic equation of state (or massless scalar field) reduce to

\[ \dot{\rho} + 3 H (\rho + p) = 0, \]

where the Hubble parameter \( H \equiv \dot{a}/a \) gives the volume expansion rate, and the nonlocal conservation equations reduce to an evolution equation for \( \mathcal{U} \):

\[ \dot{\mathcal{U}} + 4 H \mathcal{U} = 0, \]

In the Friedmann case Eq. (15) yields the ‘dark radiation’ solution \( \mathcal{U} = \mathcal{U}_0 (a_0/a)^4 \).

The field equations are given by Eqns (8) and (9). The generalized Friedmann equation, which determines the expansion of the universe or the Hubble function, in the case of spatially homogeneous cosmological models is

\[ H^2 = \frac{1}{3} \kappa^2 \rho \left( 1 + \frac{\rho}{2 \Lambda} \right) - \frac{1}{6} 3 R + \frac{1}{3} \sigma^2 + \frac{1}{3} \Lambda + \frac{2 \mathcal{U}}{\lambda \kappa^2}, \]

where \( 3 R \) is the scalar curvature of the hypersurfaces orthogonal to the fluid flow, which we associate with the cosmological fluid, and \( 2 \sigma^2 = \sigma_{ab} \sigma_{ab} \) is the shear scalar in terms of the shear \( \sigma_{\mu\nu} \).

For a perfect fluid or minimally-coupled scalar field, the generalized Raychaudhuri equation on the brane becomes [4]:

\[ \dot{H} + H^2 + \frac{2}{3} \sigma^2 + \frac{1}{6} \kappa^2 (\rho + 3 p) - \frac{1}{3} \Lambda = - \frac{1}{36} \left( \frac{\kappa}{\kappa} \right)^4 \left[ \kappa^4 \rho (2 \rho + 3 p) + 12 \mathcal{U} \right], \]

where we have assumed (2), and in particular \( P_{\mu\nu} = 0 \), self-consistently on the brane [4].
Following [7,16] we define Hubble-normalized (and dimensionless) shear variables $\Sigma_+, \Sigma_-$, curvature variables $N_1, N_2, N_3$ and matter variables $\Omega_i, P_i$ (relative to a group-invariant orthonormal frame), and a logarithmic (dimensionless) time variable, $\tau$, defined by $d\tau = Hdt$. These variables do not lead to a global compact phase space, but they are bounded close to the singularity [7].

The governing evolution equations for these quantities are then

$$
\Sigma'_+ = (q - 2)\Sigma_+ - S_+
$$

(18)

$$
\Sigma'_- = (q - 2)\Sigma_- - S_-
$$

(19)

$$
N'_1 = (q - 4\Sigma_+)N_1
$$

(20)

$$
N'_2 = (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2
$$

(21)

$$
N'_3 = (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3,
$$

(22)

where a prime denotes differentiation with respect to $\tau$, and $K, S_+ and S_-$ are curvature terms that are defined as follows:

$$
K \equiv \frac{1}{12} \left( (N_1^2 + N_2^2 + N_3^2) - 2(N_1N_2 + N_2N_3 + N_1N_3) \right)
$$

(23a)

$$
S_+ \equiv \frac{1}{6} (N_2 - N_3)^2 - \frac{1}{6} N_1 \left( 2N_1 - N_2 - N_3 \right)
$$

(23b)

$$
S_- \equiv \frac{1}{6} \sqrt{3} (N_2 - N_3) \left( -N_1 + N_2 + N_3 \right)
$$

(23c)

The quantity $q$ is the deceleration parameter given by

$$
q \equiv 2\Sigma^2_+ + 2\Sigma^2_- + \frac{1}{2} \sum \Omega_i + \frac{3}{2} \sum P_i.
$$

(24)

The decoupled Raychaudhuri equation becomes

$$
H' = -(1 + q)H.
$$

(25)

In addition, the generalized Friedmann equation reduces to the constraint

$$
1 = \Sigma^2_+ + \Sigma^2_- + \sum \Omega_i + K.
$$

(26)

Due to the symmetries in the dynamical system, we can restrict ourselves to the set $N_1 \geq 0, N_2 \geq 0, and N_3 \geq 0$, without loss of generality. (Note, for Bianchi IX models all of the $N$’s must be of the same sign.) We again note that these new normalized variables, for which the evolution equation for $H$ has decoupled from the remaining equations, are bounded close to the initial singularity.

In the above

$$
\sum \Omega_i = \frac{\kappa^2 \rho_{tot}}{3H^2}, \quad \sum P_i = \frac{\kappa^2 P_{tot}}{3H^2},
$$

(27)

where the $\Omega_i \ (i = 1 - 4)$ are given by

$$
\Omega_1 = \Omega = \frac{\kappa^2 \rho}{3H^2}; \quad P = (\gamma - 1)\Omega
$$

(28a)

$$
\Omega_2 = \Omega_b = \frac{\kappa^2 \rho^2}{6\lambda H^2}; \quad P_b = (2\gamma - 1)\Omega_b
$$

(28b)

$$
\Omega_3 = \Omega_d = \frac{2\dot{U}}{\kappa^2 \lambda H^2}; \quad P_d = \frac{1}{3} \Omega_d
$$

(28c)

$$
\Omega_4 = \Omega_{\Lambda} = \frac{\Lambda}{3\kappa^2 H^2}; \quad P_{\Lambda} = -\Omega_{\Lambda},
$$

(28d)

which satisfy the equations

$$
\Omega'_i = [2(1 + q) - 3\Gamma_i] \Omega_i,
$$

(29)
where

\[ \Gamma_1 = \gamma, \Gamma_2 = \Gamma_3 = \Gamma_4 = \frac{4}{3}, \Gamma_5 = \Gamma_\Lambda = 0. \]  \hspace{1cm} (30)

A minimally coupled homogeneous scalar field \( \phi = \phi(t) \) with an exponential potential can also be included, where the energy-momentum tensor is given by

\[ T_{ab}^{sf} = \phi \partial_a \phi \partial_b - g_{ab} \left( \frac{1}{2} \phi^2 \phi^i \phi^j + V(\phi) \right). \]  \hspace{1cm} (31)

The usual (linear) terms can be included via [12]

\[ \Omega_\phi = \Psi^2 + \Phi, \quad P_\phi = \Psi^2 - \Phi, \]  \hspace{1cm} (32)

where

\[ \Phi = \sqrt{3V/(3H)}, \quad \Psi = \sqrt{3 \phi / 2H}. \]  \hspace{1cm} (33)

The evolution of these variables is then given by the conservation or Klein-Gordon equation, which can be written as

\[ \Psi' = (q - 2)\Psi - \frac{\sqrt{6}}{2} k \Phi^2 \]  \hspace{1cm} (34)

\[ \Phi' = \Phi \left( (1 + q) + \frac{\sqrt{6}}{2} k \Psi \right) \]  \hspace{1cm} (35)

The local quadratic corrections \( \Omega_{b\phi} \) and \( P_{b\phi} \) can be included via \( \Phi_b \sim \frac{V}{\eta} \) and \( \Psi_b \sim \frac{\dot{\phi}}{\eta} \) and a generalized (quadratic) Klein-Gordon equation. Since an extensive analysis of scalar field models has shown that close to the initial singularity the scalar field must be massless [12,13], it is plausible that scalar field models can be approximated by a stiff perfect fluid close to the initial singularity (particularly regarding questions of stability).

A. Initial Singularity

If \( \rho^{tot} > 0 \) and \( \rho^{tot} + 3p^{tot} > 0 \) (which follows from the various energy conditions) for all \( t < t_0 \), then from the generalized Raychaudhuri equation (17) (or (25)) and using the generalized Friedmann equation (16) (or (26)) and the conservation equations, it follows that for \( \dot{a}_0 > 0 \) (where \( a_0 \equiv a(t_0) \)) there exists a time \( t_b \) with \( t_b < t_0 \) such that \( a(t_b) = 0 \), and there exists a singularity at \( t_b \), where we can rescale time so that \( t_b = 0 \) and the singularity occurs at the origin.

From Eqn (17) we can find the precise constraints on \( \Lambda \) and \( \mathcal{U}_b \) in terms of \( a_0 \) in order for these conditions to be satisfied at \( t = t_0 \). It then follows from the generalized Raychaudhuri equation, the generalized Friedmann equation and the conservation equations that if these conditions are satisfied at \( t = t_0 \) they are satisfied for all \( 0 < t < t_0 \), and a singularity necessarily results. These conditions are indeed satisfied for regular matter undergoing thermal collapse in which the the local energy density and pressure satisfy \( \rho(2\rho + 3p) > 0 \) (and is certainly satisfied for perfect fluid matter satisfying the weak energy condition \( \rho \geq 0 \) and a linear barotropic equation of state with \( \gamma \geq 1 \)). On the other hand, it is known that a large positive cosmological constant \( \Lambda \) or a significant negative nonlocal term \( \mathcal{U} \) counteracts gravitational collapse and can lead to the singularity being avoided in exceptional circumstances.

Since the variables are bounded close to the singularity, it follows from Eqn (24) that \( 0 < q < 2 \). Hence, from Eqn (25), \( H \) diverges as the initial singularity is approached. At an equilibrium point \( q = q^\star \), where \( q^\star \) is a constant with \( 0 < q^\star < 2 \), so that from Eqn (25) we have that \( H \rightarrow (1 + q^\star)^{-1/2} t^{-1} \) as \( t \rightarrow 0^+ \) (\( \tau \rightarrow -\infty \)).

From Eqns (24), (26) and the conservation laws it then follows that \( \rho \rightarrow \infty \) as \( t \rightarrow 0^+ \). It then follows directly from the conservation laws (29) that \( \Omega_b \) dominates as \( t \rightarrow 0^+ \) and that all of the other \( \Omega_i \) are negligible dynamically as the singularity is approached. The fact that the effective equation of state at high densities become ultra stiff, so that the matter can dominate the shear dynamically, is a unique feature of brane cosmology.

Note that the function defined in Bianchi IX models by

\[ Z \equiv (N_1 N_2 N_3)^2, \]  \hspace{1cm} (36)
and is consequently monotone close to the singularity [7].

In summary, generically the Bianchi type IX models have a cosmological initial singularity in which \( \rho \to \infty \), and consequently \( \Omega_b \) dominates, as \( t \to 0^+ \). This can be proven by more rigorous methods [17,8]. It can also be shown by qualitative methods that the spatial 3-curvature is negligible at the initial singularity and from a comprehensive analysis that at later times \( \rho \) decreases and the low density regime in which GR is valid ensues.

\[ Z' = 6qZ, \tag{37} \]

B. The Isotropic Equilibrium Point

Hence we have shown that close to the singularity \( \rho^{\text{tot}} = \rho_b \), so that \( \sum \Omega_i = \Omega_b \) and \( \sum P_i = P_b \), where \( P_b = (2\gamma - 1)\Omega_b \). Consequently, Eqn (26) can be written as

\[ \Omega_b = 1 - \Sigma_+^2 - \Sigma_-^2 - K, \tag{38} \]

which can be used to eliminate \( \Omega_b \) from the governing equations. In particular, Eqn (24) becomes

\[ q = 3(1 - \gamma)(\Sigma_+^2 + \Sigma_-^2) + (3\gamma - 1)(1 - K), \tag{39} \]

and the governing equations are given by the dynamical system (18-22), where \( q \) now given by the above expression.

There is an equilibrium point of the dynamical system, denoted by \( \mathcal{F}_b \), given by \( \Sigma_+ = \Sigma_- = 0 \), and \( N_1 = N_2 = N_3 = 0 \), which corresponds to a spatially homogeneous and isotropic non-general-relativistic brane-world (without brane tension; \( \mathcal{U} = 0 \)) model, first discussed by Binétruy, Deffayet and Langlois [6], in which \( a(t) \sim t^{\frac{1}{3\gamma}} \), which is valid at very high energies \( (\rho \gg \lambda) \) as the initial singularity is approached; \( t \to 0 \). Note that these solutions are self-similar, and are referred to as Friedmann brane-worlds [5], Binétruy, Deffayet and Langlois solutions [6] or Brane-Robertson-Walker models [15].

For the equilibrium point \( \mathcal{F}_b \), the 5 eigenvalues are:

\[ 3(\gamma - 1), 3(\gamma - 1), (3\gamma - 1), (3\gamma - 1), (3\gamma - 1) \tag{40} \]

Hence, for all physically relevant values of \( \gamma (\gamma \geq 1) \), \( \mathcal{F}_b \) is a source (or past-attractor) in the brane-world scenario and the singularity is isotropic. This contrasts to the situation in GR in which anisotropy dominates for \( \gamma < 2 \). This is also consistent with previous analyses of Bianchi type I and V models where \( \mathcal{F}_b \) is always a source for \( \gamma \geq 1 \) (in the FRW models, \( \mathcal{F}_b \) is a source when \( \gamma \geq 1/3 \) when \( \mathcal{U} = 0 \) and \( \gamma \geq 2/3 \) when \( \mathcal{U} \neq 0 \) [16].

C. Discussion

There are no other equilibrium points of the dynamical system that correspond to sources. However, it is instructive to consider the Kasner equilibrium points.

The one-parameter set (circle) of (Bianchi I, shearing) Kasner vacuum (\( \Omega_i = 0 \)) equilibrium points \( \mathcal{K} \) are given by \( \Sigma_+^2 + \Sigma_-^2 = 1 \), and \( N_1 = N_2 = N_3 = 0 \). The eigenvalues are

\[ 2(1 - 2\Sigma_+), 2(1 + \Sigma_+ + \sqrt{3}\Sigma_-), 2(1 + \Sigma_+ - \sqrt{3}\Sigma_-), -6(\gamma - 1) \tag{41} \]

and a fifth eigenvalue which is zero due to the fact that \( \mathcal{K} \) is a one-parameter set of equilibrium points. All of these equilibrium points are saddles; in particular, they can never be sources. The fourth eigenvalue is negative, whereas in GR it is positive, so that the structure of the stable and unstable manifolds close to \( \mathcal{K} \) is altered. In GR it was argued [7] that \( \Omega \) and \( \Delta \), defined by

\[ \Delta \equiv (N_1N_2)^2 + (N_2N_3)^2 + (N_3N_1)^2, \tag{42} \]

satisfy \( \Omega \to 0 \), \( \Delta \to 0 \) as \( \tau \to -\infty \), and hence in general all orbits approach \( \mathcal{K} \) (which is characterized by \( \Omega = 0 \), \( \Delta = 0 \)) into the past. A qualitative and numerical investigation indicates that \( \Omega \) does not tend to zero as \( \tau \to -\infty \), and hence generically \( \mathcal{K} \) is not, nor does it constitute part of, a (past-) attracting set in the models under investigation.

Close to the singularity we expect that \( \gamma \geq 4/3 \). However, we note that there is a bifurcation at \( \gamma = 1 \), and \( \mathcal{F}_b \) and \( \mathcal{K} \) coalesce to form a two-parameter set of equilibrium points \( \mathcal{J} \) which are analogues of the Jacobs stiff fluid solution in GR in which \( \Omega_b \neq 0 \). A subset of these equilibrium points are sources. In general \( \Sigma_+^2 + \Sigma_-^2 \neq 0 \) (the equilibrium points with zero shear are the special points in \( \mathcal{J} \) equivalent to \( \mathcal{F}_b \)); however, there are sources in \( \mathcal{J} \) in which \( \Sigma_+^2 + \Sigma_-^2 \) (and hence the shear) is arbitrarily small.
Therefore, generically the Bianchi IX brane-world models do not have space-like and oscillatory singularities in the past, and consequently brane-world cosmological models do not exhibit Mixmaster and chaotic-like behaviour close to the initial singularity.

We expect this to be a generic feature of more general cosmological models in the brane-world scenario. In particular, we anticipate that \( \mathcal{F}_b \) is a source in all spatially homogeneous models. Following arguments similar to those of section III.A, it follows that there exists a singularity of a similar nature in all orthogonal Bianchi brane-world models. In general, \( \Omega_h \) will again dominate as the initial singularity is approached into the past and the qualitative results of section III.B will follow. In particular, \( \mathcal{F}_b \) will be a local source and in general the initial singularity will be isotropic.

### A. Inhomogeneous Brane-World Models

We consider the dynamics of a class of spatially inhomogeneous cosmological models with one spatial degree of freedom in the brane-world scenario. The \( G_2 \) cosmological models admit a 2-parameter Abelian isometry group acting transitively on spacelike 2-surfaces. These models admit one degree of freedom as regards spatial inhomogeneity, and the resulting governing system of evolution equations constitute a system of autonomous partial differential equations in two independent variables. We follow the formalism of [18] which utilizes area expansion normalized scale-invariant dependent variables, and we use the timelike area gauge to discuss the asymptotic evolution of the class of orthogonally transitive \( G_2 \) cosmologies near the cosmological initial singularity. The initial singularity can be shown to be characterized by \( E_1^1 \to 0 \) as \( \tau \to -\infty \), where \( E_1^1 \) is a normalized frame variable [18].

The ‘equilibrium point’ \( \mathcal{F}_b \) is characterized by zero shear \( (\Sigma_\gamma = \Sigma_x = 0) \) and zero curvature \( (N_\gamma = N_x = 0) \) (where we note that the area expansion normalized variables utilized in [18] are equivalent to H-normalized variables close to \( \mathcal{F}_b \)). Linearizing the evolution equations about \( \mathcal{F}_b \), using the same spatial reparametrisation as in [18] (so that \( E_1^1 = \exp((3\gamma - 1)\tau) \)), we obtain the following general solution of the linearized equations in which the shear variables satisfy

\[
\Sigma_\gamma = a_1(x) \exp (3(\gamma - 1)\tau), \quad \Sigma_x = a_2(x) \exp (3(\gamma - 1)\tau),
\]

the curvature variables satisfy

\[
N_\gamma = a_3(x) \exp ((3\gamma - 1)\tau), \quad N_x = a_4(x) \exp ((3\gamma - 1)\tau),
\]

and

\[
\Omega_b = 1 + a_5(x) \exp (3(\gamma - 1)\tau),
\]

where the \( a_i(x) \) are arbitrary functions of the space coordinate. As in [18], it can be shown that the tilt or peculiar velocity \( v \) between the timelike area gauge and the fluid 4-velocity obeys \( v = a_6(x) \exp ((3\gamma - 1)\tau) \) and hence tends to zero as \( \tau \to -\infty \) and, in addition, it follows from the conservation laws that \( \Omega \to 0, \Omega_d \to 0 \) and \( \Omega_A \to 0 \) as the initial singularity is approached.

The above linearized solution represents a general solution in the neighbourhood of the initial singularity. Hence \( \mathcal{F}_b \) is a local source or past-attractor in this family of spatially inhomogeneous cosmological models for \( \gamma > 1 \). In particular, we see that the shear and curvature asymptote to zero as \( \tau \to -\infty \), and hence the singularity is isotropic. We also note that, unlike the analysis of the perfect fluid GR models in [18], the Kasner equilibrium set \( \mathcal{K} \) are found to be saddles in the class of \( G_2 \) brane-world cosmological models.

The most detailed proposal for the structure of space–time singularities in GR are the conjectures of Belinskii, Khalatnikov and Lifshitz (BKL) [19], one aspect of which is that each spatial point evolves towards the singularity as if it were a spatially homogeneous (Bianchi) cosmology. That is, generic space–times have the property that spatial points decouple near the singularity, the EFE effectively reduce to ordinary differential equations (i.e., the spatial derivatives have a negligible effect on the dynamics) and the local dynamical behavior is asymptotically like Bianchi models locally near the singularity. Support for this conjecture comes from recent analyses that show that the presence of the inhomogeneity ceases to govern the dynamics asymptotically toward the singularity in particular classes of inhomogeneous models. In a special class of Abelian \( G_2 \) spatially inhomogeneous models, the so-called “velocity-dominated” spacetimes, the evolution at different spatial points will approach that of different Kasner solutions [20]. A numerical investigation of a class of vacuum Gowdy \( G_2 \) cosmological spacetimes that represent an inhomogeneous generalization of Bianchi type VI\(_0\) models with a magnetic field, has shown evidence that at a generic
point in space the evolution toward the initial singularity is asymptotically that of a spatially homogeneous spacetime with Mixmaster behaviour [21]. More rigorous results are given in [18,22].

The dynamics of the class of inhomogeneous brane-world models considered above, together with the BKL conjectures, indicate that a wide class of inhomogeneous brane-world models will have an isotropic initial singularity.

**B. Discussion**

Therefore, it is plausible that generically the brane-world cosmological models have an *isotropic initial singularity*, whose generic evolution near the cosmological initial singularity is approximated by a spatially homogeneous and isotropic model in a rigorously defined mathematical sense [24] (in contrast to the situation in GR). Therefore, it is possible that brane cosmology provides for a quiescent [25] initial cosmological period in which the Universe is smooth and highly symmetric, perhaps explained by entropy arguments and the second law of thermodynamics [26], thereby naturally providing the precise conditions for inflation to subsequently take place and consequently avoiding the problem of initial conditions in inflation [15].

The results in this paper are incomplete in that a description of the gravitational field in the bulk is not provided. Unfortunately, the evolution of the anisotropic stress part is not determined on the brane. These nonlocal terms also enter into crucial dynamical equations, such as the Raychaudhuri equation and the shear propagation equation, and can lead to important changes from the GR case. The correction terms must be consistently derived from the higher-dimensional equations. Additional modifications can also occur for higher-dimensions (than 5), for more general (than static) higher-dimensional (bulk) geometry, for higher (non-GR) curvature corrections, for higher-dimensional matter fields (e.g., scalar fields) in the bulk and for motion of the brane [23]. Further work is therefore necessary. However, it is plausible that the main results of this paper will persist when these additional effects are included.

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