TOTAL ENERGY LOSSES DUE TO THE RADIATION IN AN ACOUSTICALLY BASED UNDULATOR: THE UNDULATOR AND THE CHANNELING RADIATION INCLUDED

ANDREI V. KOROL†,‡, ANDREY V. SOLOV’YOV†,§,∥∗, and WALTER GREINER¶

†Department of Physics, St. Petersburg State Maritime Technical University, Lenninski prospect 101, St. Petersburg 198262, Russia
‡A.F.Ioffe Physical-Technical Institute of the Academy of Sciences of Russia, Polytechnicheskaya 26, St. Petersburg 194021, Russia
¶Institut für Theoretische Physik der Johann Wolfgang Goethe-Universität, Robert-Mayer Str. 8-10, 60054 Frankfurt am Main, Germany

Received (received date)
Revised (revised date)

This paper is devoted to the investigation of the radiation energy losses of an ultra-relativistic charged particle channeling along a crystal plane which is periodically bent by a transverse acoustic wave. In such a system there are two essential mechanisms leading to the photon emission. The first one is the ordinary channeling radiation. This radiation is generated as a result of the transverse oscillatory motion of the particle in the channel. The second one is the acoustically induced radiation. This radiation is emitted because of the periodic bending of the particle’s trajectory created by the acoustic wave. The general formalism described in our work is applicable for the calculation of the total radiative losses accounting for the contributions of both radiation mechanisms. We analyze the relative importance of the two mechanisms at various amplitudes and lengths of the acoustic wave and the energy of the projectile particle. We establish the ranges of projectile particle energies, in which total energy loss is small for the LiH, C, Si, Ge, Fe and W crystals. This result is important for the determination of the projectile particle energy region, in which acoustically induced radiation of the undulator type and also the stimulated photon emission can be effectively generated. The latter effects have been described in our previous works.

1. Introduction

This paper is devoted to the investigation of the radiation energy losses of an ultra-relativistic charged particle channeling along a crystal plane which is periodically bent by a transverse acoustic wave. In such a system, there are two essential mechanisms leading to the photon emission. The first one is the ordinary channeling radiation. This radiation is generated as a result of the transverse oscillatory motion of the particle in the channel. This radiation mechanism was suggested in Refs. 1,2 for linear crystals and later studied in numerous theoretical and experimental works (see e.g. Refs. 3,4,5). The second one is the acoustically induced
radiation (AIR)\textsuperscript{6,7}. The AIR is emitted because of the periodic bending of the particle’s trajectory created by the acoustic wave. This mechanism is of particular interest, because the AIR has all the features of the undulator radiation, including the possibility of the stimulated photon emission\textsuperscript{6,7}.

Since the mechanism of AIR generation was suggested only recently, let us describe it here in more details. This mechanism is illustrated in Fig. 1.

Under the action of a transverse acoustic wave propagating along the $z$-direction, which defines the center line of an initially straight channel (not plotted in the figure) the channel becomes periodically bent. Provided certain conditions are fulfilled\textsuperscript{6,7}, the beam of positrons, which enters the crystal at a small incident angle with respect to the curved crystallographic plane, will penetrate through the crystal following the bendings of its channel. It results in the transverse oscillations of the beam particles while travelling along the $z$ axis. These oscillations become an effective source of spontaneous radiation of undulator type due to the constructive interference of the photons emitted from similar parts of the trajectory. As demonstrated in Refs. \textsuperscript{6,7}, the number of oscillations can vary in a wide range from a few up to a few thousands per cm depending on the beam energy, the AW amplitude and wavelength the type of the crystal and the crystallographic plane. In addition to the spontaneous photon emission by the undulator, the scheme presented in Fig. 1 leads to a possibility to generate stimulated emission. This is due to the fact, that photons, emitted at the points of the maximum curvature of the trajectory, travel almost parallel to the beam and, thus, stimulate the photon generation in the vicinity of all successive maxima and minima of the trajectory.

As demonstrated in Refs. \textsuperscript{6,7}, the AIR can be well separated from the ordinary channeling radiation, if certain conditions on the amplitude and the length of the acoustic wave are fulfilled. One of the criteria formulated in Refs. \textsuperscript{6,7} for the stable
work of the AIR type of undulator is the small energy loss of the beam of particles penetrating through the crystal.

In this paper we describe general formalism for the calculation of the total radiative energy loss accounting for the contributions of both radiation mechanisms. We perform such a calculation for the first time. We analyze the relative importance of the two mechanisms at various amplitudes and lengths of the acoustic wave and the energy of the projectile particle. We establish the ranges of the projectile particle energy, in which total radiative energy loss is negligible for the LiH, C, Si, Ge, Fe and W crystals. This result is important for the determination of the projectile particle energy region, in which acoustically induced radiation is of the undulator type and also the stimulated photon emission can be effectively generated.

An adequate approach to the problem of the radiation emission by an ultra-relativistic particle moving in an external field was developed by Baier and Katkov and was called by the authors the “operator quasi-classical method”. The details of that formalism can be found in Ref. 5. We use this formalism to tackle our problem.

For convenience below we enlist the notations used throughout the paper.

- \( \varepsilon, m, \) and \( \gamma = \varepsilon/mc^2 \) are, respectively, the energy, mass, and relativistic factor of a projectile, \( q \) is its charge measured in units of the elementary charge \( e \), \( c \) is the velocity of light.

- \( a_u, \lambda_u \) are the AW amplitude and wave length. The undulator parameter equals to \( p_u = \gamma \xi_u \), where \( \xi_u = 2\pi a_u/\lambda_u \ll 1 \).

- \( \omega \) is the photon frequency, \( \mathbf{n} \) is the unit vector in the direction of the emission.

- \( L \) is the crystal thickness, \( T \approx L/c \) is the time of flight of the projectile through the crystal.

- \( d \) is the interplanar spacing. It is assumed that \( d \) satisfies the condition \( d \ll \lambda_u \).

- \( U = U(\rho) \) is the interplanar potential, \( \rho = [-d/2,+d/2] \) is the distance from the midplane. The quantity \( U_o \) stands for the maximum value of \( U(\rho) \).

- \( C \) stands for the factor \( \varepsilon/(R_{min} qeU'_{max}) \approx \varepsilon d/(R_{min} 2qeU_o) \), where \( R_{min} = (k_u^2 a_u)^{-1} \) is the minimum curvature radius of an acoustically bent channel, and \( k_u = 2\pi/\lambda_u, U'_{max} \approx 2U_o/d \) is the maximum gradient of the interplanar field.

- \( a_c, \lambda_c \) are, respectively, the amplitude and wave length characterizing the channeling motion. The corresponding undulator parameter reads as \( p_c = \gamma \xi_c \). The \( \xi \)-parameter is defined as \( \xi_c = 2\pi a_c/\lambda_c \). More details on these parameters are given in the text.
• $\alpha = e^2/h\,c \approx 1/137$ is the fine structure constant, $r_e = e^2/(m_e c^2) = 2.818 \times 10^{-13}$ cm is the electron classical radius.

2. Quasi-classical formalism for the radiative energy loss

The energy losses, $\Delta E$, due to the emission of photons by a charged projectile moving in an external field are defined as

$$\Delta E = \int_0^{\varepsilon/\hbar} d\omega \int d\Omega_n \frac{dE_\omega(n)}{d\omega d\Omega_n}.$$  \hspace{1cm} (1)

Within the framework of the quasi-classical approach the distribution of the energy radiated by an ultra-relativistic particle (of a spin $s = 1/2$) in given direction $n$ and summed over the polarizations of the photon and the projectile, is given by the following expression, which is written up to the terms $\gamma^{-2}$:

$$\frac{dE_\omega(n)}{d\omega d\Omega_n} = \hbar \alpha \frac{q^2 \omega^2}{4\pi^2} \int_0^T dt_1 \int_0^T dt_2 \, e^{i\omega' \varphi(t_1, t_2)} \, f(t_1, t_2).$$  \hspace{1cm} (2)

The functions $\varphi(t_1, t_2)$ and $f(t_1, t_2)$ equal to

$$\varphi(t_1, t_2) = t_1 - t_2 - \frac{1}{c} \mathbf{n} \cdot (\mathbf{r}_1 - \mathbf{r}_2),$$  \hspace{1cm} (3)

$$f(t_1, t_2) = \frac{1}{2} \left\{ (1 + (1 + u)^2) \left( \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{c^2} - 1 \right) + \frac{u^2}{\gamma^2} \right\}.\hspace{1cm} (4)$$

The notations used are $\mathbf{r}_{1,2} = \mathbf{r}(t_{1,2})$, $\mathbf{v}_{1,2} = \mathbf{v}(t_{1,2})$, with $\mathbf{r}$ and $\mathbf{v}$ standing for projectile’s radius vector and velocity, respectively.

Expression (4) looks almost like the classical formula, although with quantum corrections:

$$\omega \longrightarrow \omega' = \frac{\varepsilon}{\varepsilon - \hbar \omega} \omega, \quad u = \frac{\hbar \omega}{\varepsilon - \hbar \omega},$$  \hspace{1cm} (5)

which take into account the radiative recoil.

Expressing the photon frequency and the quantity $\omega'$ via the dimensionless variable $u$, $\hbar \omega = \varepsilon u/(1 + u)$, $\hbar \omega' = u \varepsilon$, and taking into account the relation $\omega^2 d\omega = (\varepsilon/\hbar)^2 u^2/(1 + u)^4 \, du$, one obtains the following general expression for the relative energy losses:

$$\frac{\Delta E}{\varepsilon} = \alpha \frac{q^2}{4\pi^2} \left( \frac{\varepsilon}{\hbar} \right)^2 \int_0^\infty \frac{u^2 \, du}{(1 + u)^4} \times \int d\Omega_n \int_0^T dt_1 \int_0^T dt_2 \exp \left( i \frac{\varepsilon u}{\hbar} \varphi(t_1, t_2) \right) f(t_1, t_2).$$  \hspace{1cm} (6)

Let us transform the functions $f(t_1, t_2)$ and $\varphi(t_1, t_2)$ retaining the terms of orders up to $\gamma^{-2}$ and omitting the higher-order terms.
To evaluating the factor \( v_1 \cdot v_2/c^2 - 1 \) from (4) one makes use of Eqs. (A.14) and (A.15). Then, neglecting the term \( \Delta \tilde{z}(t_1) \Delta \tilde{z}(t_2) \), one gets

\[
\frac{v_1 \cdot v_2}{c^2} - 1 \approx -\frac{1}{\gamma^2} - \frac{1}{2} \left( \frac{v_y(t_1)}{c} - \frac{v_y(t_2)}{c} \right)^2.
\]

(7)

Substituting this expression into (4) we obtain

\[
f(t_1, t_2) = -\frac{1}{2} \left\{ \frac{1 + (1 + u)^2}{2} \left( \frac{v_y(t_1)}{c} - \frac{v_y(t_2)}{c} \right)^2 + 2(1 + u) \right\}.
\]

(8)

To transform the phase function \( \varphi(t_1, t_2) \) from (3) let us first write the trajectory \( \mathbf{r}(t) \) in the following form:

\[
\mathbf{r}(t) = \mathbf{e}_z \cdot (c t + \Delta z(t)) + \mathbf{e}_y \cdot y(t).
\]

(9)

For an arbitrary interplanar potential the function \( \Delta z(t) \) can be presented in the form (see 1)

\[
\Delta z(t) = -c t \left[ \frac{1}{2 \gamma^2} + \frac{\xi_u^2}{4} + \frac{\xi_c^2}{4} \right] + \Delta z_c(t),
\]

(10)

where the parameters \( \xi_u^2 \) and \( \xi_c^2 \) are related to the mean-square values of the transverse velocities of, respectively, the undulator and the channeling motions

\[
\frac{\xi_u^2}{2} = \left( \frac{1}{c} \frac{dy_u}{dt} \right)^2, \quad \frac{\xi_c^2}{2} = \left( \frac{1}{c} \frac{dy_c}{dt} \right)^2.
\]

(11)

For the undulator motion (i.e. the motion along the centerline of the acoustically bent crystal, \( y_u(t) = a_u \sin(2\pi c t/\lambda_u) \)) the first relation from (11) produces the result \( \xi_u = 2\pi a_u/\lambda_u \). Analogously, the parameter \( \xi_c \) may be written as \( \xi_c = 2\pi a_c/\lambda_c \), where \( a_c \) is the amplitude (mean) of the channeling oscillations \( a_c \leq d/2 \) and \( \lambda_c \) is the period (mean) of the channeling oscillatory motion. To estimate the magnitude of \( \xi_c \) one can do the following: \( \xi_c \sim a_c/(c \tau_c) \) with \( \tau_c \sim \sqrt{m \omega d^2/q U_o} \) standing for the period of the channeling oscillations. Hence \( \xi_c^2 \sim q U_o/\varepsilon \ll 1 \).

The term \( \Delta z(t) \) in the order of magnitude equals to \( \Delta z(t) = O(\xi_u^2, \xi_c^2, \xi_u \xi_c) \). It contains only oscillatory terms which satisfy the condition \( \Delta z(t) = 0 \) if the averaging is carried out over the interval \( \Delta T' > \lambda_u/c, \tau_c \).

Now, to write down the term \( \mathbf{n} \cdot (\mathbf{r}_1 - \mathbf{r}_2) \) from (3) let us notice that for an ultra-relativistic particle the radiation occurs into a narrow cone with the axis along the \( z \)-direction. The width of the cone is defined by three parameters, \( \gamma^{-2}, \xi_c^2 \), and \( \xi_u^2 \) and is equal to

\[
\theta_{max} \sim \max(\gamma^{-2}, \xi_c^2, \xi_u^2) \ll 1.
\]

(12)

The relations established above allow to write down the following expression for \( \varphi(t_1, t_2) \) which explicitly accounts for all the terms of orders \( \gamma^{-2}, \xi_c^2 \), and \( \xi_u^2 \):

\[
\varphi(t_1, t_2) = \varphi_u(t_1, t_2) + \Delta \varphi(t_1, t_2),
\]

(13)

\[
\varphi_u(t_1, t_2) = \kappa^2 \tau - \frac{1}{c} (\Delta z(t_1) - \Delta z(t_2)),
\]

(14)

\[
\Delta \varphi(t_1, t_2) = \frac{\tau}{2} \theta^2 - \theta \frac{y(t_1) - y(t_2)}{c} \cos \phi,
\]

(15)
where the following notations are introduced:

$$\kappa^2 = \frac{1}{2\gamma^2} + \frac{\xi^2}{4} + \frac{\xi_z^2}{4}, \quad \tau = t_1 - t_2. \quad (16)$$

When writing (15) we took into account that $y(t_1) - y(t_2) \sim O(\xi_u, \xi_c) c \tau$ which results in $\sin \theta (y(t_1) - y(t_2)) = \theta (y(t_1) - y(t_2))$.

Using Eqs. (8) and (15) in (6) one gets

$$\frac{\Delta E}{\varepsilon} = \frac{\alpha q^2}{8\pi^2} \left( \frac{\varepsilon}{\hbar} \right)^2 \int_0^T dt_1 \int_0^T dt_2 \int_0^\infty \frac{u^2 du}{(1+u)^4} \exp \left( \frac{i\varepsilon u}{\hbar} \varepsilon_o(t_1, t_2) \right) \times \left\{ \frac{1}{2} \left( \frac{v_y(t_1)}{c} - \frac{v_y(t_2)}{c} \right)^2 + \frac{2(1+u)}{\gamma^2} \right\}$$

$$\times \int d\Omega \exp \left( \frac{i\varepsilon u}{\hbar} \Delta \varphi(t_1, t_2) \right). \quad (17)$$

Due to the relation (12) the main contribution to the integral over $\Omega = (\theta, \phi)$ comes from the region $\theta \ll 1$. Therefore, one may write

$$I = \int d\Omega \exp \left( \frac{i\varepsilon u}{\hbar} \Delta \varphi(t_1, t_2) \right) = \int_0^\infty \theta d\theta \ e^{i\omega' \tau^2/2} \int_0^{2\pi} d\phi \ e^{-i\omega' \Delta y \theta \cos \phi/c}. \quad (18)$$

For short the notations $\omega' = u \varepsilon / \hbar$ (see (5)) and $\Delta y = y(t_1) - y(t_2)$ were used.

The integrals over $\phi$ and $\theta$ are carried out by using the formulae

$$\int_0^{2\pi} d\phi \ e^{i z \cos \phi} = 2\pi J_0(z), \quad \int_0^\infty \exp (iax^2) J_0(bx) x \ dx = \frac{i}{2a} \exp \left( -i \frac{b^2}{4a} \right), \quad (19)$$

where $J_0(z)$ is the Bessel function of order 0.

Applying these integrals to (18) one gets

$$I = \frac{\hbar}{\varepsilon} \frac{2\pi i}{\tau} \exp \left[ -i \frac{\varepsilon u}{\hbar} \tau \left( 1 + \frac{(y(t_1) - y(t_2))^2}{2\gamma^2 \tau^2} \right) \right]. \quad (20)$$

When substituting this result into (17) we first introduce the quantity

$$\zeta = \frac{\varepsilon}{\hbar} \tau \left\{ \kappa^2 - \frac{\Delta z(t_1) - \Delta z(t_2)}{c \tau} - \frac{1}{2} \left( \frac{y(t_1) - y(t_2)}{c \tau} \right)^2 \right\}. \quad (21)$$

Hence

$$\frac{\Delta E}{\varepsilon} = -\frac{\alpha q^2}{4\pi} \left( \frac{\varepsilon}{\hbar} \right)^2 \int_0^T dt_1 \int_0^T dt_2 \int_0^\infty \frac{u^2 du}{(1+u)^4} \exp (i\zeta u) \times \left\{ \frac{1}{2} \left( \frac{v_y(t_1)}{c} - \frac{v_y(t_2)}{c} \right)^2 + \frac{2(1+u)}{\gamma^2} \right\}. \quad (22)$$

Let us demonstrate that the principal contribution to (22) comes from the region $|\zeta| < 1$. To do this we first evaluate the integral over $u$ and then analyze the result.
Making a substitution \( \exp i\zeta u \rightarrow i \sin \zeta u \) in the integrand (which does not affect the three-fold integral on the right-hand side) one is left with two basic integrals

\[
j_1 = \frac{1}{2} \int_0^\infty \frac{u^2 \, du}{(1 + u)^4} \left( 1 + (1 + u)^2 \right) \sin \zeta u ,
\]
\[
j_2 = 2 \int_0^\infty \frac{u \, du}{(1 + u)^3} \sin \zeta u ,
\]
the evaluation of which is elementary but lengthly. The result is

\[
j_1 = \frac{1}{2} \left[ \zeta^3 + \left( 1 - \frac{\zeta^2}{2} \right) f(\zeta) - \left( \zeta - \frac{\zeta^3}{6} \right) g(\zeta) \right] ,
\]
\[
j_2 = -\zeta + \zeta^2 f(\zeta) + 2\zeta g(\zeta) ,
\]
where the functions \( f(\zeta) \) and \( g(\zeta) \) stand for the integrals:

\[
f(\zeta) = \int_0^\infty \frac{\sin \zeta u}{1 + u} \, du , \quad g(\zeta) = \int_0^\infty \frac{\cos \zeta u}{1 + u} \, du .
\]

For \(|\zeta| > 1\) the expansions for \( f(\zeta) \) and \( g(\zeta) \) one finds in Ref. 8. Using them one gets:

\[
j_1 \approx \frac{6}{\zeta^3} , \quad j_2 \approx \frac{12}{\zeta^3} .
\]

Large values of \( \zeta \) correspond to the \( \tau \)-values \(|\tau| \gg \hbar/\varepsilon \kappa^2 \) (see (21)). With the expressions (28) taken into account the integrand in (22) behaves as \( \sim \tau^{-4} \). Thus, the range of large \(|\zeta| \) (or \(|\tau|\)) does not contribute effectively to the integral (22).

The leading contribution to the energy losses (22) comes from the range \(|\zeta| < 1\) which corresponds to

\[
|t_1 - t_2| \equiv |\tau| < \frac{\varepsilon \kappa^2}{\hbar} .
\]

3. Energy losses in the case of harmonic interplanar potential

Instead of evaluating the remaining integrals in (22) in the general case we will consider the harmonic approximation for the interplanar potential. It allows to carry out all the calculations explicitly. The final result can be generalized to the case of an arbitrary interplanar potential.

3.1. Evaluation of the formula for the energy losses.

The harmonic potential, expressed in terms of the relative distance from the midplane, \( \tilde{y} \), can be written in the form

\[
U(\tilde{y}) = 4 U_o \frac{\tilde{y}^2}{d^2} .
\]

The coefficient is chosen to satisfy \( U(\pm d/2) = U_o \), with \( U_o \) standing for the maximum value of the potential.
Assuming the following strong inequality (see 1)

$$\frac{\omega_u}{\omega_a} \gg 1,$$

(31)

which is valid almost in all cases, the dependences $\Delta_2(t)$ and $y(t)$ read as (compare with (A.16) and (A.19))

$$y(t) = a_u \sin \Omega_u t + a_c \sin(\Omega_c t + \phi_0),$$

(32)

$$\Delta_2(t) = -\frac{1}{16\pi} \left\{ \lambda_u \xi_c^2 \sin 2\Omega_u t + \lambda_c \xi_c^2 \sin(2\Omega_c t + 2\phi_0),
+ 4 \lambda_c \xi_c \xi_a \left[ \cos ((\Omega_c + \Omega_u) t + \phi_0) + \cos ((\Omega_c - \Omega_u) t + \phi_0) \right] \right\}.$$ 

(33)

In these formulas the subscripts “u” and “c” indicate that a quantity is related to the undulator motion (the index “u”) or to the channeling motion (the index “c”).

The parameters $a_u$, $\lambda_u$, $\xi_u = 2\pi a_u/\lambda_u$, and $\Omega_u = 2\pi c/\lambda_u$ are explained in .

The quantities characterizing the channeling motion, which are: the frequency of the channeling oscillations $\Omega_c$, the wave length of one oscillation $\lambda_c$, and the parameter $\xi_c = 2\pi a_c/\lambda_c$, are conveniently expressed through the parameter $\mu$ (see Eq. (A.18)):

$$\Omega_c = \frac{2 \mu_c}{d}, \quad \lambda_c = \frac{\pi d}{\mu}, \quad \xi_c = \frac{2 \mu a_c}{d}.$$ 

(34)

It is also worth noting that expressions (32) and (33) embrace both the $a_u \gg d$ and $a_u \ll d$ regions. In the latter case $\xi_u = 0$, and the formulae produce the result for the channeling motion in a linear channel.

To evaluate the integral (22) let us first analyze the functions $((\Delta_2(t_1) - \Delta_2(t_2))/c \tau$, $(y(t_1) - y(t_2))/c \tau$ (see (21)) and $(v_y(t_1) - v_y(t_2))/c \tau$ from the integrand.

It can be easily verified that when substituted into $(\Delta_2(t_1) - \Delta_2(t_2))/c \tau$ the first term from the right-hand side of (33) results in the term of the order $\xi_u^2$, the second one produces the term $\sim \xi_c^2$ and the last one gives the $\sim \xi_c \xi_u$ term. To illustrate this let us consider the contribution of the first term from (33):

$$\frac{\lambda_u \xi_u^2}{16 \pi c} \left| \frac{\sin 2\Omega_u t_1 - \sin 2\Omega_u t_2}{t_1 - t_2} \right| = \frac{\lambda_u \xi_u^2 \Omega_u}{8 \pi c \Omega_u \tau} \left| \frac{\sin \Omega_u \tau}{\Omega_u \tau} \cos \Omega_u (t_1 + t_2) \right| \leq \frac{\xi_u^2}{4}. \quad \text{(35)}$$

Next consideration regarding the difference $((\Delta_2(t_1) - \Delta_2(t_2))/c \tau$ is that for any fixed value of $\tau$ this function is highly oscillatory because of the factors of the type $\cos \Omega_u (t_1 + t_2), \cos \Omega_c (t_1 + t_2)$, and/or $\sin ((\Omega_c + \Omega_u)(t_1 + t_2) + \phi_0)$. It leads to the relation $\Delta_2(t_1) - \Delta_2(t_2) = 0$ for any value of $\tau$. The leading value of the integral (22) will not be changed if in (21) one omits these highly oscillatory terms.

By using similar arguments let substitute the function $((y(t_1) - y(t_2))/c \tau$ from (21) with its non-oscillatory part. Making use of (32) one gets

$$\frac{1}{2} \left( \frac{y(t_1) - y(t_2)}{c \tau} \right)^2 \rightarrow \frac{1}{2} \left( \frac{y(t_1) - y(t_2)}{c \tau} \right)^2 = \frac{\xi_u^2}{4} \left( \frac{\sin \eta_u}{\eta_u} \right)^2 + \frac{\xi_c^2}{4} \left( \frac{\sin \eta_c}{\eta_c} \right)^2. \quad \text{(36)}$$
where \( \eta_{u,c} = \Omega_{u,c} \tau / 2 \).

The main contribution to the integral (22) comes from the region of small \( \tau \) (see (29)), therefore, one may use \((\sin \eta_{u,c}/\eta_{u,c})^2 = 1 - \eta^2_{u,c}/3\).

Thus, the function \( \zeta \) defined in (21) can be substituted with
\[
\zeta \rightarrow x \tau + a \tau^3, \tag{37}
\]
where the following short-hand notations are introduced
\[
x = \frac{\varepsilon}{2\gamma^2 \hbar}, \quad a = \frac{\varepsilon}{2h} \frac{\xi_u^2 \Omega_u^2 + \xi_c^2 \Omega_c^2}{24}. \tag{38}
\]

The last function to be transformed is \((v_y(t_1) - v_y(t_2))^2 / c^2\) from the integrand in (22). The non-oscillatory part of it reads as
\[
\left( \frac{v_y(t_1) - v_y(t_2)}{c} \right)^2 \rightarrow -\frac{\xi_u^2 \Omega_u^2 + \xi_c^2 \Omega_c^2}{2} \tau^2. \tag{39}
\]

Substituting (37)–(39) into (22), and introducing the integration variable \( \tau \) one obtains
\[
\frac{\Delta E}{\varepsilon} = -i \frac{\alpha q^2 \varepsilon}{4\pi^2 \hbar} \int_0^T \int_0^\infty \frac{u^2 du}{(1 + u)^4} \int_{-t}^t \frac{d\tau}{\tau} \exp \left[ i u(x \tau + a \tau^3) \right] \times \left\{ 1 + (1 + u) \frac{\xi_u^2 \Omega_u^2 + \xi_c^2 \Omega_c^2}{2} + \frac{2(1 + u)}{\gamma^2} \right\}. \tag{40}
\]

The limits of the integration over \( \tau \) can be extended to \( \pm \infty \). Then one is left with two integrals. The first one equals to
\[
\int_{-\infty}^{\infty} d\tau \tau \exp \left[ i u(x \tau + a \tau^3) \right] = \frac{2}{i} \frac{\partial}{\partial x} \int_0^\infty d\tau \cos \left[ u(x \tau + a \tau^3) \right] = -\frac{2\pi i}{(3au)^{2/3}} \text{Ai}'(z). \tag{41}
\]

Here \( \text{Ai}'(z) \) is the derivative of the Airy’s function, and the parameter \( z = xu/(3au)^{1/3} \).

The second integral reads as follows
\[
H \equiv \int_{-\infty}^{\infty} d\tau \frac{\exp \left[ i u(x \tau + a \tau^3) \right]}{\tau} = 2i \int_0^\infty \frac{dv}{v} \sin \left( zv + \frac{v^3}{3} \right). \tag{42}
\]

The derivative of \( H \) with respect to \( z \) reduces to the Airy’s function: \( dH/dz = 2\pi i \text{Ai}(z) \). Hence
\[
H = -2\pi i \int_z^\infty dv \text{Ai}(v). \tag{43}
\]

Here the integration constant is chosen to produce \( H = 0 \) for \( z = \infty \).

Substituting (41) and (43) into (40) and changing the variable of integration from \( u \) to \( z \) and, afterwards, integrating by parts the term containing the integral \( \int_z^\infty dv \text{Ai}(v) \) one gets
\[
\frac{\Delta E}{\varepsilon} = -\frac{3}{2} \frac{\alpha q^2}{c} L \frac{\varepsilon}{\hbar \gamma^2} \chi^2 \int_0^\infty \frac{dz}{\beta^2(z)} \left[ \frac{1 + \beta^2(z)}{\beta^2(z)} \text{Ai}'(z) + \frac{z^2}{3} \text{Ai}(z) \right], \tag{44}
\]
where $\beta(z) = 1 + z^{3/2} \chi$.

The parameter $\chi$, the value of which plays the crucial role in defining the magnitude of the energy losses, is defined as

$$
\chi = \frac{\hbar \gamma^3}{\varepsilon} \left[ \frac{\xi_u^2 \Omega_u^2 + \xi_c^2 \Omega_c^2}{2} \right]^{1/2}.
$$

(45)

To clarify the meaning of $\chi$ let us for a moment “switch off” the channeling motion by putting $\xi_c = 0$. Then (omitting the factor $\sqrt{2}$) $\chi \approx \hbar \gamma^3 \xi_u / \varepsilon = \hbar \gamma^2 p_u \Omega_u / \varepsilon$. Here the quantity $\hbar \gamma^2 p_u \Omega_u$ is the frequency of the radiated intensity maximum, $\omega_{\text{max}}$ (in the case $p_u > 1$). For $\omega \gg \omega_{\text{max}}$ the intensity $dE/d\omega$ exponentially decreases. If $\chi < 1$ then the intensity reaches its maximum in the “physical” domain, i.e. $\hbar \omega_{\text{max}} < \varepsilon$. The opposite case $\chi > 1$ (and, consequently $\hbar \omega_{\text{max}} > \varepsilon$) corresponds to the situation when a projectile can emit photons of all frequencies lying within the range $\hbar \omega = [0, \varepsilon]$ so that the spectrum intensity never reaches the maximum. In this case the radiative energy losses are dominated by the radiation of highly energetic photons, $\hbar \omega \sim \varepsilon$.

The analogous arguments can be provided to analyze the case of the channeling radiation only, i.e. $\xi_u = 0$.

Hence, the formula (45) is a generalization of the definition of $\chi$ to the case when both motions, the undulator and the channeling ones, exist. This expression does not contain the cross-term $\propto \xi_u \xi_c$ which could have been originated from the interference of the photons emitted due to two different types of motion. The absence of the cross-term simply reflects the fact that, generally, the frequencies of $\Omega_u$ and $\Omega_c$ (and, correspondingly, the frequencies of all possible harmonics) are incompatible and, thus, the corresponding electromagnetic waves do not interfere.

The expression (44) can be rewritten in the form which is frequently used in the theory of energy losses due to the synchrotron and/or undulator radiation. To do this let us introduce the energy losses calculated in the classical limit. This limit corresponds to $\hbar \omega / \varepsilon \ll 1$, which, in turn, means that $\chi \ll 1$. Neglecting $\chi$ in the integrand in (44) (it results in putting $\beta(z) = 1$) one gets:

$$
\left( \frac{\Delta E}{\varepsilon} \right)_{cl} = -\frac{3}{2} \frac{\alpha q^2}{c} L \frac{\varepsilon}{\hbar \gamma^2} \chi^2 \int_0^\infty zdz \left[ 2 - \frac{z^3}{12} \right] \text{Ai}'(z).
$$

(46)

Here the second term in the integrand was obtained by integrating by parts the term proportional to $\text{Ai}(z)$ in (44).

The integrals are evaluated by using the formula:

$$
\int_0^\infty dz z^\nu \text{Ai}'(z) = -\frac{3 (4 \nu - 1)^{3/4}}{2 \pi} \Gamma \left( \frac{\nu}{3} + 1 \right) \Gamma \left( \frac{\nu}{3} + \frac{1}{3} \right),
$$

(47)

yielding

$$
\left( \frac{\Delta E}{\varepsilon} \right)_{cl} = \frac{2}{3} \frac{\alpha q^2}{c} L \frac{\varepsilon}{\hbar \gamma^2} \chi^2.
$$

(48)
The functional dependence presented by (48) coincides with the well-known expression\textsuperscript{10}.

For a projectile positron (which is, actually, of a prime interest) it is convenient to denote the coefficients in (48) in another form by making use of the relations \(\varepsilon/(\hbar c \gamma) = m_e c^2/(\hbar c) = \alpha/r_e\). Hence

\[
\left( \frac{\Delta E}{\varepsilon} \right)_{cl} = \frac{2}{3} \frac{\alpha^2}{r_e} L \frac{\chi^2}{\gamma}. \tag{49}
\]

Inserting (48) into (44) we obtain the formula for the energy losses within the framework of the quasi-classical approach:

\[
\frac{\Delta E}{\varepsilon} = \left( \frac{\Delta E}{\varepsilon} \right)_{cl} \Phi(\chi), \tag{50}
\]

where \(\Phi(\chi)\) is given by

\[
\Phi(\chi) = -\frac{9}{4} \int_0^\infty \frac{dz}{\beta^2(z)} \left[ 1 + \frac{\beta^2(z)}{\beta^2(z)} \text{Ai}'(z) + \frac{z^2}{3} \text{Ai}(z) \right]. \tag{51}
\]

The functions \(\Phi(\chi)\) and \(\chi^2 \Phi(\chi)\) are presented in Fig. 2.
3.2. **Comparison of the contributions of undulator and channeling mechanisms to the energy radiative losses.**

From (48) (or (49)) and (50)–(51) it is clear that the magnitude of the energy losses for a given projectile energy and a crystal length depends solely on the value of the quantity $\chi$, see (45). Thus, to compare the relative contributions of the two types of radiation it is sufficient to analyze the ratio

$$
\eta \equiv \frac{\xi_u \Omega_u}{\xi_c \Omega_c},
$$

(52)

By making use of (34) and recalling the definition $C = \varepsilon / (R_{min} q e U'_{max})$ one gets

$$
\eta = C \frac{d}{|a_c|},
$$

(53)

In a linear channel ($\lambda_u = \infty$, $a_u = 0$, $R_{min} = \infty$) the range of the $a_c$ values corresponding to the channeling motion is $a_c \in [-d/2, d/2]$.

For a channel periodically bent by an acoustic wave the range of the $a_c$ values, for which the stable channeling motion can occur, is narrower $|a_c| \leq a_c^{(max)} < d/2$. Let us first establish the magnitude of $a_c^{(max)}$ for given projectile energy and AW wave length and amplitude.

Inside the channel the motion of the channeled particle is determined by the effective potential

$$
q e U_{eff}(\tilde{y}) = q e U(\tilde{y}) - \frac{\varepsilon}{R} \tilde{y},
$$

(54)

where $R^{-1} = R_{min}^{-1} \sin(2\pi z/\lambda_u) \in [-R_{min}^{-1}, R_{min}^{-1}]$ is the local curvature radius of the channel. Written in terms of the dimensionless variable $Y = \tilde{y}/d \in [-0.5, 0.5]$ and in the case of the harmonic interplanar potential the quantity $U_{eff}$ reads as

$$
U_{eff}(Y) = 4 q e U_o \left( Y^2 - C Y \sin \frac{2\pi z}{\lambda_u} \right).
$$

(55)

Schematically, the dependences $U_{eff}(Y)$ calculated in the points of the AW maximum ($\sin(2\pi z/\lambda_u) = +1$) and minimum ($\sin(2\pi z/\lambda_u) = -1$) are presented in Fig. 3. The particle will be trapped into the channeling mode only if its energy, $\varepsilon_y$, associated with the transverse motion is less than $4 q e U_o \left( |Y|^2 - C |Y| \right)_{|Y|=0.5}$ i.e. the minimum of two values $4 q e U_o \left( |Y|^2 \pm C |Y| \right)_{|Y|=0.5}$ which are the heights of the asymmetric wells corresponding to the effective potential (55) in the vicinity of the AW minima/maxima. Hence, in an acoustically bent channel the range $|a_c| \leq a_c^{(max)}$ is determined by following two inequalities (see Fig. 3):

$$
Y^2 - C Y \leq \frac{1}{4} (1 - C)^2, \quad Y^2 + C Y \leq \frac{1}{4} (1 - C)^2,
$$

(56)

which result in

$$
|a_c| \leq a_c^{(max)} = \frac{d}{2} (1 - C).
$$

(57)
Total energy losses due to the radiation in an acoustically based undulator

\[ U_{\text{eff}}(Y) = 4 q e U_o (Y^2 - CY), \text{ and minimum, } U_{\text{eff}}(-Y) = 4 q e U_o (Y^2 + CY), \text{ (see (55)) versus the dimensionless distance from the centerline, } Y = 2 \tilde{y}/d \in [-0.5, 0.5]. \]

\[ \varepsilon_y^{\text{max}} = U_{\text{eff}}(-0.5) = U_{\text{eff}}(0.5) \]

\[ \text{is the maximum transverse energy for which channeling in an acoustically bent crystal can occur.} \]

The quantity \( Y_{\text{max}} \) defines the maximum value of the parameter \( \alpha_c \), \( Y_{\text{max}} = 2 \alpha_c \max /d \). Further explanations are given in the text.

Let us assume that a uniform beam is ideally collimated along the centerline when entering the crystal, so that the particles of the beam differ only in the value of the initial coordinate \( \tilde{y}_0 \). Then, the parameter \( 2 \alpha_c \max /d = 1 - C \) defines the relative part of the beam particles which are trapped into the channeling mode in the acoustically bent crystal.

Equations (53) and (57) allow to define the ranges of the parameter \( \alpha_c \) for which either the undulator or the channeling type of radiation dominate in the total energy losses. Namely, for \( \eta > 1 \), and, consequently, \( |\alpha_c| < dC \), the undulator radiation contributes more to the total energy loss. If \( \eta > 1 \Rightarrow |\alpha_c| > dC \) then the channeling radiation plays the dominant role. If \( |\alpha_c| = C \) then both mechanisms contribute equally to the radiative losses. The value \( \alpha_c = C \leq \alpha_c \max /d \) can be reached only if \( C \leq 1/3 \). In the opposite case, \( C > 1/3 \), the parameter \( \eta \) is greater than 1 for all \( \alpha_c \) values consistent with (57), and, hence, the losses due to the undulator radiation are higher than those due to the channeling one.

3.3. **Estimation of the magnitude of the parameter \( \chi \)**

With (53) taken into account the parameter \( \chi \) from (45) becomes:

\[ \chi = \frac{1}{\sqrt{2}} \chi_c^{\text{max}} \sqrt{4C^2 + 4 \frac{a^2_c}{d^2}}. \]
where (see (33) and (A.18))

$$\chi_c^{(max)} = \frac{\hbar \gamma^3}{\varepsilon} \left[ \xi_c \Omega_c \right]_{a_c=d/2} = \frac{\hbar \gamma^3\, 2\mu^2 c}{\varepsilon \, d} \tag{59}$$

is the maximum value of $\chi$ due to the channeling radiation in the case of a linear channel, i.e. when $a_c^{(max)} = d/2$.

The factor $2 \sqrt{C^2 + a_c^2/d^2}$ in (58) reaches its maximum value of 2 for $C = 1$ (to get this one substitutes $a_c$ with its maximum value given by (57) and, afterwards, finds the maximum value with respect to $C$ in the range $C = [0, 1]$ within which the channeling in an acoustically bent crystal can occur). Hence

$$\chi_{max} = \sqrt{2} \chi_c^{(max)} = \sqrt{2} \frac{\hbar \gamma^3\, 2\mu^2 c}{\varepsilon \, d}. \tag{60}$$

Let us estimate the magnitude of the right-hand side of (60) for a positron, $\varepsilon = 0.511 \cdot 10^6$ eV. By taking into account the definition (A.18) one gets after some simple algebra

$$\chi_{max} = 4.3 \times 10^{-8} \gamma \frac{v_o}{d_\AA}. \tag{61}$$

Here $v_o$ is the magnitude of $q e U_o$ measured in eV and $d_\AA$ is the interplanar spacing measured in Å. In table 1 the values of $v_o$ and $d_\AA$ are presented for (110) channels in various crystals as indicated. The data for C, Si, Fe, Ge and W were taken from Ref. 5, the $d_\AA$ and $v_o$ values for LiH were adopted from Ref. 12. The last column of the table corresponds to the values of a positron relativistic factor which produce $\chi_{max} = 1$. Fig. 4 presents the dependences $\chi_{max}(\gamma)$ obtained for the (110) channels of LiH, C, Si, Fe, Ge and W crystals.

Table 1. The values of $d_\AA$, $v_o$ and $\gamma_{min}$ calculated for a positron channeling in (110) planar channel in C, Si, Ge, F and W crystals.

<table>
<thead>
<tr>
<th>Crystal</th>
<th>$d_\AA$ ($\AA$)</th>
<th>$v_o$ (eV)</th>
<th>$\gamma$ for $\chi_{max} = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1.26</td>
<td>23</td>
<td>$1.27 \times 10^6$</td>
</tr>
<tr>
<td>Si</td>
<td>1.92</td>
<td>23</td>
<td>$1.94 \times 10^6$</td>
</tr>
<tr>
<td>Ge</td>
<td>2.00</td>
<td>40</td>
<td>$1.16 \times 10^6$</td>
</tr>
<tr>
<td>F</td>
<td>1.02</td>
<td>70</td>
<td>$3.39 \times 10^5$</td>
</tr>
<tr>
<td>W</td>
<td>1.12</td>
<td>130</td>
<td>$2.00 \times 10^5$</td>
</tr>
</tbody>
</table>

One may conclude, based on the values of $\gamma_{\chi_{max} = 1}$, and on the curves in Fig. 4, that in the range $\gamma < 10^5$ the parameter $\chi$ can be chosen to satisfy the condition $\chi < \chi_{max} < 1$.

3.4. 

**Realistic calculation of the total energy losses by a positron bunch channeling in an acoustically bent crystal.**

Formulae (49)–(51) allow to calculate the energy loss for the particular trajectory (which is specified by the parameter $a_c$) and for arbitrary crystal length $L$. A more
realistic approach must take into account, firstly, the effect of the decrease in the beam volume density with the penetration distance, i.e. the dechanneling effect, and, secondly, the distribution of the beam particles in $a_c$.

3.4.1. Account for the dechanneling effect.

Random scattering of the channeling particle by the electrons and nuclei of the crystal leads to a gradual increase of the particle energy associated with the transverse oscillations in the channel. As a result, the transverse energy at some distance from the entrance point exceeds the depth of the interplanar potential well, and the particle leaves the channel. This distance is called the dechanneling length $L_d(\gamma, R)$. For a given crystal and channel it depends on a positron energy (relativistic factor) and on the curvature radius $R$.

Therefore, for realistic estimations it is sufficient to assume that the crystal length $L$ does not exceed $L_d(\gamma, R)$ and one may calculate the relative energy loss for $L = L_d(\gamma, R)$.

It can be demonstrated\textsuperscript{11} that for a crystal bent with a constant curvature radius $R$ the dechanneling length $L_d(\gamma, R)$ satisfies the relation

$$L_d(\gamma, R) = \left(1 - \frac{R_c}{R}\right)^2 L_d(\gamma, \infty),$$

Fig. 4. Parameter $\chi_{\text{max}}$ from (61) versus the relativistic factor of a positron channeled in (110) channels of various crystals as indicated.
where \( L_d(\gamma, \infty) \) is the dechanneling length of a positron of the same energy in a straight channel \((R = \infty)\) and

\[
R_c = \frac{\varepsilon}{e q U'_{\text{max}}}
\]

is the critical (minimal) radius consistent with the channeling condition in a bent crystal, “the centrifugal force < the interplanar force”\(^{13}\).

In an acoustically bent crystal the curvature \( R^{-1}(z) = R^{-1}_{\text{min}} \sin(2\pi z/\lambda_u) \) is not constant. Therefore, it is natural to consider the mean curvature \( 1/R \) which is obtained by averaging \( 1/|R(z)| \) over the undulator period

\[
\frac{1}{R} = \frac{1}{\lambda_u} \int_0^{\lambda_u} k_u^2 a_u |\sin k_u z| dz = \frac{2}{\pi} \frac{1}{R_{\text{min}}}.
\]

Then the dechanneling length in an acoustically bent channel is estimated as follows:

\[
L_d(\gamma, R) = (1 - \frac{2}{\pi} C)^2 \gamma \alpha_d(\gamma),
\]

where we introduced the reduced dechanneling length \( \alpha(\gamma) \equiv L_d(\gamma, \infty)/\gamma \). For a given crystal and crystallographic plane this quantity depends weakly on \( \gamma \). Its explicit expression, calculated by using the Lindhard approximation for the potential of a planar channel, reads\(^{11}\)

\[
\alpha(\gamma) = \frac{256}{9\pi^2} \frac{a_{\text{TF}}}{r_{\text{cl}}} \frac{d}{\ln(2\varepsilon/I) - 1}.
\]

Here \( a_{\text{TF}} = 0.8853 Z_c^{1/3} a_0 \) and \( I = 16Z_c^{0.9} \) eV are the Thomas-Fermi atomic radius and ionization potential, respectively. \( Z_c \) is the atomic number of the crystal atoms, and \( a_0 \) is the Bohr radius. The dependences \( \alpha_d(\gamma) \) for various planar channels can be found in Ref. \(^7\).

Taking into account the quantities introduced above one obtains the following expression for the relative energy loss due to the electromagnetic radiation emitted in a crystal of the length \( L = L_d(\gamma, R)\):

\[
\frac{\Delta E}{\varepsilon} = 2.3 \times 10^{-7} \left( \frac{\gamma v_o}{d_A} \right)^2 \alpha_d(\gamma) (1 - \frac{2}{\pi} C)^2 \left( C^2 + \frac{a_c^2}{d^2} \right) \Phi(\chi).
\]

Here the factor \( 2.3 \times 10^{-7} \left( \gamma v_o/d_A \right)^2 \) originates from \( 2\alpha^2 \chi^2/3r_c \) (see (49)) with the parameter \( \chi \) taken in the form given by (58) and, in turn, \( \chi_{\text{max}} = \chi_{\text{max}}/\sqrt{2} \) is calculated from (61).

3.4.2. **Energy losses averaged over the parameter \( a_c \).**

When a bunch of positrons enters the crystal the particles have various values of the initial coordinate \( y_0 \) and of the incidence angle \( \theta_0 \) between the momentum of the
incident particle and the tangent to the crystal centerline $a_u \sin(k_u z)$. Let us assume that the bunch is ideally collimated so that all the particles have $\theta_0 = 0$ at the entrance. Furthermore, let us assume that the particles are uniformly distributed in the $y_0$ space. Then, when considering the energy losses by those particles of the bunch which are trapped into the channeling mode of motion, it is meaningful to carry out the averaging of $\Delta E/\varepsilon$ over the $a_c$ values satisfying (57). Thus, the average energy losses are defined as follows:

$$\overline{\Delta E/\varepsilon} = 2.3 \times 10^{-7} \left( \frac{\gamma v_o}{d_A} \right)^2 \alpha_d(\gamma) \left( 1 - \frac{2}{\pi} C \right)^2 G\left( C, \chi_c^{(\max)} \right),$$

(68)

where the function $G(C, \chi_c^{(\max)})$ is given by

$$G\left( C, \chi_c^{(\max)} \right) = \frac{2}{x_0} \int_0^{x_0} dx \left( C^2 + x^2 \right) \Phi(\chi).$$

(69)

Here $x = a_c/d$, $\Phi(\chi)$ is defined in (51) and $\chi = \sqrt{2} \chi_c^{(\max)} \sqrt{C^2 + x^2}$ according to (58). The upper limit of integration equals $x_0 = a_c^{(\max)}/d = (1 - C)/2$.

Although the integration over $x$ can be carried out explicitly, the final result is rather cumbersome to be reproduced here. It is simplified in the case $\chi \ll 1$ when

$$\Phi(\chi) = 1 - \frac{55 \sqrt{3}}{16} \chi + 48 \chi^2 \ldots$$

(70)

Then

$$G\left( C, \chi_c^{(\max)} \right) = \left( C^2 + \frac{x_0^2}{3} \right) - \frac{55 \sqrt{3}}{16} \chi_c^{(\max)} A_1 + 48 \left( \chi_c^{(\max)} \right)^2 A_2,$$

(71)

where

$$A_1 = \frac{1}{8} \left\{ \left( C^2 + x_0^2 \right)^{1/2} \left[ 5C^2 + 2x_0^2 \right] + \frac{3C^4}{x_0} \ln \frac{x_0 + \sqrt{C^2 + x_0^2}}{C} \right\},$$

(72)

$$A_2 = C^4 + \frac{2}{3} C^2 x_0^2 + \frac{1}{5} x_0^4.$$

(73)

3.4.3. Results of the numerical calculations of the averaged energy losses.

As it was mentioned in Sec. there are two types of radiation accompanying the channeling process of an ultra-relativistic particle in an acoustically bent channel: the ordinary channeling radiation and the AIR. Both of them belong to an undulator type of radiation. It is known from general theory of a planar undulator radiation (see e.g. Ref. 4) that its frequency-angular distribution $dE_\omega(n)/d\omega d\Omega_n$ is represented by the sets of characteristic frequencies (harmonics) $\omega^{(K)}_n$ ($K = 1, 2, 3 \ldots$) each of the width $\Gamma^{(K)} = (2/N) (\omega^{(K)}/K)$, where $N$ is the number of the undulator periods. The frequencies of harmonics are defined from
the relation $\hbar \omega^{(K)} = 4\gamma^2 \Omega K/(2 + p^2)$ with $\Omega$ and $p$ standing, respectively, for the undulator frequency and parameter.

In the present paper we analyze mainly the case $\Omega_c/\Omega_u \gg 1$ (see Eq. (31)) which can be achieved by choosing $a_u \gg d$ (see Eq. (A.21)). Hence, the frequencies of harmonics characterizing the AIR and the channeling radiation are well separated satisfying the condition $\omega^{(K)}_c/\omega^{(K)}_u \gg 1$ (Refs. 7,14). It is the AIR mechanism which brings the novelty into the problem. Therefore, let us analyze the stability of the undulator AIR radiation towards the decrease in a projectile positron energy due to the radiative losses.

The frequencies of harmonics of the AIR radiation one calculates from (see Ref. 7)

$$
\bar{h} \omega^{(K)}_u = \frac{4\gamma^2 \Omega_u K}{2 + 2\theta^2 \gamma^2 + p^2_u},
$$

(74)

where $p_u = \gamma \xi_u$, and $\theta$ is the emission angle with respect to the undulator axis (which is the centerline of the initially linear channel, see Fig. 1). The magnitude of $\theta$ satisfies to $\theta \leq \theta_{\text{max}} = \max\{\gamma^{-1}, \xi_u\}$ (Ref. 7).

From (74) one gets the following relation between the total energy losses $\Delta E/\varepsilon$ and the quantity $\Delta \omega^{(K)}_u$ which is the shift of the $K$th harmonic frequency from its unperturbed value $\omega^{(K)}_u$

$$
\Delta \omega^{(K)}_u = \omega^{(K)}_u - \frac{2 \Delta E}{\varepsilon} < \omega^{(K)}_u - \frac{2 \Delta E}{\varepsilon}
$$

(75)

The spontaneous AIR radiation formed during the passage of a positron through an acoustically bent crystal of total length $L_d(\gamma, R)$ is stable towards the energy loss of the positron provided the shift $\Delta \omega^{(K)}_u$ is smaller than the natural line half-width $\Gamma_u^{(K)}/2$. The latter is given by $\Gamma_u^{(K)}/2 = (1/N_u)(\omega^{(K)}_u/K)$, where $N_u = L_d(\gamma, R)/\lambda_u$. Therefore, from (74) one deduces

$$
\frac{\Delta E}{\varepsilon} \leq \frac{1}{2KN_u}
$$

(76)

It was estimated in Ref. 7 and analyzed in more detail in Ref. 14 that the realistic range of $N_u$ in an acoustically based undulator is $N_u = 10\ldots25$ and the corresponding number of the harmonics emitted via the AIR mechanism is $K \sim 1$. Thus, the stability of the AIR radiation will occur if $\Delta E/\varepsilon < 0.01$.

Figures 5 represent the dependences $\Delta E/\varepsilon$ versus $\gamma$ calculated, according to (68)–(69), for a positron channeling in (110) channels of LiH, C, Si, Fe, Ge and W crystals and for several values of the parameter $C < 1$ which characterizes the bending of the channel.

The chosen crystals are commonly used in experiments devoted to the investigation of the channeling phenomena and, in addition, this set includes crystals composed of light (LiH, C, Si), intermediate (Fe, Ge) and heavy (W) atoms.

Figures 5 allow to estimate the range of validity of the condition (76) for $N_u = 10\ldots25$. It is seen from the figures that the inequality (76) is well-fulfilled for
Total energy losses due to the radiation in an acoustically based undulator... 19

\( \gamma < 10^5 \) in the case of LiH crystal, \( \gamma < 5 \times 10^3 \) for C, Si, and Ge, \( \gamma < 2 \times 10^3 \) for Fe and W.

For higher values of \( \gamma \) the increase of \( \frac{\Delta E}{\varepsilon} \) will result in the shift \( \Delta \omega^{(K)}_u \) greater than the half-width \( \Gamma^{(K)}_u / 2 \) thus smearing the line over the wider range of frequencies. In this case it is meaningful to consider not the harmonic acoustic wave but rather the one with varying amplitude and period analogously to how it was proposed when considering undulator radiation formed in the tampered magnetic wigglers\(^{16}\).


In this work we have described the general formalism for the calculation of the total radiative energy loss accounting for the contributions of both radiation mechanisms, i.e. the acoustically induced radiation\(^6\),\(^7\) and the ordinary channeling radiation. Our formalism is based on the quasi-classical approach (see e.g. Ref.\(^5\)). We have analyzed the relative importance of ordinary channeling radiation and the AIR to the total radiation energy loss at various amplitudes and lengths of the acoustic wave and as a function of the energy of the projectile particle. We established the ranges of the projectile particle energy, in which the total radiative energy loss is negligible for the LiH, C, Si, Ge, Fe and W crystals. This result is important for the determination of the projectile particle energy region, in which acoustically induced radiation of the undulator type and also the stimulated photon emission can be effectively generated.

We consider our present research as a milestone for the advanced theoretical description of the AIR phenomenon. The goal of further investigation in this field is to achieve an accurate quantitative description of the undulator radiation and of the corresponding laser effect. In our recent work\(^7\) we have outlined the phenomena, which must be thoroughly considered.

Here we mention only the problem, which is closely connected to the present research. Using the formalism very similar to the one described here, it is interesting to calculate the total frequency and angular distribution of of photons emitted due to the mechanisms of the ordinary channeling radiation and the AIR. This work is in progress at the moment and will become the subject of another publication in the near future.

Acknowledgements

The authors acknowledge support from the DFG, GSI, BMBF and the Alexander von Humboldt Foundation.
Fig. 5. Averaged relative energy losses (68) for the crystal length \( L = L_d(\gamma, R) \) versus the relativistic factor of a positron channeling in (110) channels of various crystals: (a) LiH, (b) C (diamond), (c) Si, (d) Fe, (e) Ge, (f) W. The curves corresponds to different values of the parameter \( C = \epsilon/(R_{\min} q \ell U_{\max}') \) as indicated.
Total energy losses due to the radiation in an acoustically based undulator...

Fig. 5 (Continued)

(c) Si, (110)

(d) Fe, (110)
Fig. 5 (Continued)
Appendix A

5. Particle’s motion in an acoustically bent channel

5.1. Approximations

In this section we outline the approximations which have been used when considering both the motion and the radiation of an ultra-relativistic charged particle undergoing planar channeling in a crystal bent by means of a transverse, harmonic, plane acoustic wave transmitted along the z-direction which coincides with a crystallographic direction in the initially linear crystal. It is assumed that the crystallographic planes are equally spaced and being parallel to the (xy)-plane.

The ultra-relativistic particle enters the crystal at \( z = 0 \) having only the \( y \)- and \( z \)-velocity components, \( v_{yo}, v_{zo} \). It is assumed that \( v_{yo} \ll v_{zo} \) and \( v_{zo} \approx c \).

If one neglects random scattering of the particle by the electrons and nuclei of the crystal, then the particle’s trajectory lies in the (xy)-plane and is subject to the joint action of the interplanar force \( U'(\rho) \) and of the centrifugal force due to the crystal bending.

The necessary (but not sufficient) condition for a projectile to be trapped into the channeling mode of motion in a bent crystal is \( \Theta < \Theta_c \), where \( \Theta \) is the entrance angle between the particle’s velocity and the channel centerline, and \( \Theta_c \) is some critical angle (the estimates of \( \Theta_c \) in the case of bent channel can be found in Ref. 11). In a linear crystal \( \Theta_c \) coincides with the Lindhard’s angle 15.

The strong inequality \( \Theta \ll \Theta_c \) allows to introduce the continuum approximation for the interaction potential \( U \) between the charged projectile and lattice atoms arranged in atomic planes. In our paper this approximation is used to describe the equations of motion for the particle in a bent channel.

5.1.1. Approximations related to the crystal bending

The shape of a channel centerline in an acoustically bent crystal is described by

\[
y(z) = a_u \sin k_u z, \tag{A.1}\]

with \( k_u = 2\pi/\lambda_u \).

It was demonstrated in Ref. 7 that the realistic ranges for the AW amplitude and wavelength are

\[
a_u = 10^{-8} \ldots 10^{-6} \text{ cm}, \quad \lambda_u = 10^{-3} \ldots 10^{-1} \text{ cm}. \tag{A.2}\]

Therefore, we introduce an approximation by assuming that the following strong inequality is fulfilled:

\[
\xi_u = 2\pi a_u / \lambda_u \ll 1. \tag{A.3}\]

All the final formulae written below take into account the contributions of the terms proportional to \( \xi_u^0 \) and \( \xi_u^1 \), while the terms the order \( \xi_u^2 \) and higher are omitted (except for the cases when the \( \xi_u^2 \) terms are the leading ones).
It is easily verified that the length of a centerline and the interplanar spacing in linear \((L, d)\) and in acoustically bent \((L', d')\) channels are related as follows

\[
L' = L \left(1 + O(\xi_u^2)\right), \quad d' = d \left(1 + O(\xi_u^2)\right).
\] (A.4)

The analogous relationship one finds for the distance \(\rho\) between some inner point \((y, z)\) of the bent channel and its centerline (see Fig. A.1):

\[
\rho^2 = (y-y_0)^2 + (z-z_0)^2 = (y-y_0)^2 \left(1 + \frac{(z-z_0)^2}{(y-y_0)^2}\right) = (y-y_0)^2 (1 + O(\xi_u^2)).
\] (A.5)

Therefore, when neglecting the terms of the order \(\xi_u^2\) and higher one may put \(L' = L\), \(d' = d\), and disregard the difference between \(\rho\) and \(y - a_u \sin(k_u z)\).

Eqs. (A.4)–(A.5) allow us to introduce the constant field approximation for the interplanar potential which assumes that: (1) within any bent channel the potential \(U\) depends only on the variable \(\rho\) which is the distance of the \((y, z)\) point from the channel centerline, (2) the explicit dependence of \(U\) on \(\rho\) in the acoustically bent channel is identical to the \(U(\rho)\) dependence in the linear channel.

In a linear channel, where the coordinates \(y\) and \(\rho\) are basically the same, the potential depends only on the transverse coordinate \(y\) and is a periodic function with the period \(d\): \(U \equiv U(y) = U(y + N d)\), where the index \(N = \ldots, -1, 0, 1, \ldots\) enumerates the channels.

When the channel is bent by the AW then the \(y\) coordinate becomes \(y = a_u \sin(k_u z)\). The displacement along the \(z\) axis is proportional to \(\xi_u^2\) and, thus,
may be disregarded. Hence, the magnitude of \( U \) in the linear channel in the point \((y, z)\) corresponds to the the magnitude of the potential in the bent channel calculated at the point \((y + a_u \sin(k_u z), z)\). The dependence of \( U \) on \( y \) and \( z \) in an acoustically bent channel is given by

\[
U(y, z) = U(\tilde{y}), \quad \tilde{y} = y - a_u \sin(k_u z)
\]

The constant field approximation means that the explicit dependence \( U(\tilde{y}) \) in the bent channel is equivalent to the dependence \( U(y) \) in a linear one. The coordinate \( \rho \), in the bent channel up to the terms \( \sim \xi_u \), is equivalent to \( \tilde{y} \).

### 5.1.2. Approximations related to the energy of a projectile

We assume that for an ultra-relativistic particle, which enters the crystal having the energy \( \varepsilon_0 = mc^2 \gamma_0 \) (here the index "0" indicates that the quantity is measured at the entrance) the following condition is valid while it channels in the crystal

\[
\frac{qeU_o}{\varepsilon_0} \ll 1.
\]

Here the quantity \( U_o \) stands for the depth of the interplanar potential well. For a positron \((q = 1)\) typical values for \( qeU_o \) are \( \sim 10 \ldots 100 \) eV, so that (A.7) is well fulfilled in the ultra-relativistic case.

The strong inequality (A.7) justifies the classical description of the particle’s motion in both linear and bent crystals.

### 5.2. The equations of motion

The Hamiltonian function of a relativistic particle moving in a scalar potential \( U \) is given by

\[
H = \sqrt{(c p)^2 + m^2 c^4} + qe U(\tilde{y}),
\]

where \( p = m\gamma v \) is the momentum, and the coordinate \( \tilde{y} \) is defined in (A.6).

This Hamiltonian does not depend on time. Therefore, the total energy of the particle, \( m\gamma c^2 + qeU \) is conserved. Hence, the relativistic factor satisfies the condition \( \gamma = \gamma_0 (1 - qeU/\varepsilon_0) \). If one neglects the term \( qeU/\varepsilon_0 \) then the relativistic factor

\[
\gamma^{-1} = \sqrt{1 - \frac{\tilde{y}^2}{c^2} - \frac{\tilde{z}^2}{c^2}}
\]

is the integral of motion \( \gamma = \gamma_0 = \text{const} \).

By introducing the variable \( \hat{y} \) in the equation of motion \( \hat{p} = -\partial H/\partial r \) one derives

\[
\dot{\hat{y}} = qe \frac{dU}{d\hat{y}} + k_u \xi_u c^2 \sin(k_u ct), \quad \dot{\hat{z}} = 0.
\]

This system was obtained by omitting the terms \( \propto qeU/\varepsilon_0 \) and \( \propto \xi_u^n, n \geq 2 \). The first term on the right-hand side of (A.10) represents the acceleration due to the
action of the interplanar force. The second one is due to the channel bending and can be written in the form \( \gamma u^2 / R \) which explicitly indicates the centrifugal acceleration (here, \( v \approx c \) for an ultra-relativistic projectile, and \( R^{-1} = k_u \xi_u \sin(k_u z) \) with \( z \approx c t \) is the curvature of the centerline (A.1)). This term vanishes in the case of a linear channel when \( a_u = 0 \) and/or \( \lambda_u \rightarrow \infty \).

The equation (A.11) is readily integrated yielding \( z = c t \). The correction to this dependence one finds by using the relation \( \gamma = \gamma_0 = \text{const} \) in (A.9):

\[
\frac{v_y^2(t)}{c^2} = 1 - \left( \frac{1}{\gamma^2} + \frac{v_y^2(t)}{c^2} \right). \tag{A.12}
\]

Let us now estimate the ratio \( v_y^2(t)/c^2 \). There are two typical scales for the velocity in the \( y \) direction. The first one, \( v_y^{(1)} \), is related to the motion of the projectile along the centerline of the channel. The period of this motion equals \( \approx \lambda_u / c \), hence, \( v_y^{(1)} \sim 2a_u / (\lambda_u / c) \). The second characteristic velocity, \( v_y^{(2)} \), is connected with the particle oscillations inside the channel due to the action of the interplanar field \( U \).

The period of this oscillations is estimated as \( \tau_c \sim 2\pi \sqrt{m \gamma / qe U''} \sim \pi d \sqrt{m \gamma / qe U_0} \), so that \( v_y^{(2)} \sim 2d / \tau_c \). Hence

\[
\left( \frac{v_y^{(1)}}{c} \right)^2 \sim \xi_u^2 \ll 1, \quad \left( \frac{v_y^{(2)}}{c} \right)^2 \sim \frac{qe U_0}{\varepsilon_0} \ll 1. \tag{A.13}
\]

These estimates, combined with the relation \( v_y(t) = \dot{y} = \dot{\tilde{y}} + \xi_u c \cos(kc t) \) (see (A.1) and (A.6)) produce

\[
z(t) = c t + \Delta z(t), \tag{A.14}
\]

with \( \Delta z(t) \) satisfying the equation

\[
\frac{d \Delta z}{dt} = -\frac{c}{2} \left[ \frac{1}{\gamma^2} + \frac{(\dot{\tilde{y}}(t) + \xi_u c \cos(kc t))^2}{c^2} \right]. \tag{A.15}
\]

For an arbitrary function \( \tilde{y}(\tilde{y}) \) the system (A.10)–(A.15) can be easily integrated numerically by setting the initial conditions \( \tilde{y}(0), \dot{\tilde{y}}(0), \Delta z(0) \).

The function \( \tilde{y}(t) \) describes the motion of the particle with respect to the centerline (A.1) of the acoustically bent channel. According to (A.6), the total \( y(t) \) dependence is obtained by combining \( \tilde{y}(t) \) and the term \( a_u \sin(ku z t) \).

**5.3. As a specific case: harmonic approximation for the interplanar potential**

The case of the harmonic interplanar potential is of a particular interest because it allows an analytical solution of the equations of motion. Substituting the function (30) into the right-hand side of (A.10) one gets the well-known equation for a driven pendulum. Its solution \( \tilde{y}(t) \) reads

\[
\tilde{y}(t) = a_c \sin(\Omega_c t + \phi_0) + \frac{a_u}{\sigma^2 - 1} \sin \Omega_u t, \tag{A.16}
\]
where the quantities $\Omega_u = 2\pi c / \lambda_u$ and $\Omega_c = (q e U'' / d^2 m \gamma)^{1/2}$ are the frequencies of, respectively, the undulator motion, i.e. the motion along the centerline of the acoustically bent channel, and the channeling motion due to the action of the interplanar potential. The amplitude of the channeling oscillations $a_c$ and the parameter $\phi_0$ are defined by the initial conditions of the particle entering the crystal. As short-hand notation, $\sigma$ stands for the ratio of the frequencies

$$\sigma = \frac{\Omega_c}{\Omega_u} = \frac{\lambda_u \mu}{\pi d} \quad (A.17)$$

with

$$\mu^2 = \frac{2q e U_o}{\varepsilon} \ll 1. \quad (A.18)$$

We remind that the dependence $\tilde{y}(t)$ defines the deviation of the trajectory from the channel’s centerline. The total $y(t)$ dependence, according to (A.6), is obtained by combining (A.16) with the term $a_u \sin(k_u ct)$.

Substituting (A.16) into (A.15) and integrating the resulting equation one obtains the $z(t)$ dependence

$$z(t) = ct \left[ 1 - \frac{1}{2\gamma^2} - \frac{\xi_u^2}{4} \frac{\sigma^4}{(\sigma^2 - 1)^2} - \frac{\mu^2 a_c^2}{d^2} \right]$$

$$- \frac{\pi}{4} \frac{\sigma}{\lambda_u} a^2_c \sin(2\Omega_c t + 2\phi_0) - \frac{\xi_u^2 \lambda_u}{16\pi} \frac{\sigma^4}{(\sigma^2 - 1)^2} \sin 2\Omega_o t$$

$$- \frac{\xi_u a_c}{2} \frac{\sigma^3}{\sigma^2 - 1} \left[ \cos \left( (\Omega_c + \Omega_u) t + \phi_0 \right) \right] + \cos \left( (\Omega_c - \Omega_u) t + \phi_0 \right) / \sigma - 1 \right]. \quad (A.19)$$

The analytic solution (A.16) allows to establish several quantitative statements about the conditions which must be satisfied to consider the stable channeling motion, i.e. $\tilde{y}(t) \in [-0.5d, 0.5d]$, as well as to make estimates of the relative magnitudes of the frequencies $\Omega_u$ and $\Omega_c$.

The following conditions must be fulfilled.

1. $|a_c| < d/2$. This condition means that the amplitude of the channeling oscillations due to the action of the interplanar potential must not exceed the half-width of the channel.

   It is equivalent to the Lindhard’s condition which established the maximal entrance angle of the particle with respect to the midplane in a linear channel, $\Theta < \Theta_L = (2 q e U_o / \varepsilon)^{1/2}$, where $\Theta_L$ is the Lindhard’s critical angle$^{15}$.

   This relation reflects the fact that in a linear channel the energy $\varepsilon_y$ associated with the transverse motion is less than the potential barrier $U_o$. It is easy to verify that in the case of a harmonic potential this condition is equivalent to $|a_c| < d/2$.

2. The condition for the channeling in a bent channel. Driven oscillations (the second term on the right-hand side of (A.16)) must not result in the particle’s
leaving the channel. Hence, the relation $a_u/(\sigma^2 - 1) < d/2$ must be fulfilled. This inequality can be written in the form which clearly exhibits the physical condition for the possibility of the channeling process in an acoustically bent crystal. Recalling the definition (A.18) one gets

$$\frac{\varepsilon}{q e U'_{\text{max}} R_{\min}} < \frac{1}{1 + d/2a_u} < 1.$$  \hspace{1cm} (A.20)

The left-hand side of this relation is the ratio of the maximum centrifugal force $\varepsilon/R_{\min} = \varepsilon k_u^2 a_u$ and the maximum interplanar force $q e U'_{\text{max}} = 4q e U_o/d$. Channeling in an acoustically bent crystal can occur only if this ration is less than 1. For a channel bent with a constant radius such a condition was formulated by Tsyganov\textsuperscript{13}.

(3) \textit{The relationship between the undulator and the channeling frequencies.} From (A.17) combined with (A.20) one obtains the ratio of the frequencies $\Omega_c$ and $\Omega_u$

$$\sigma^2 > 1 + \frac{2a_u}{d} > 1,$$  \hspace{1cm} (A.21)

which demonstrates that the frequency of the channeling motion is always larger than that of the undulator motion.

References

Total energy losses due to the radiation in an acoustically based undulator...