Comment on ’Another form of the Klein-Gordon equation’

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Abstract

It is shown that the alternative Klein-Gordon equation with positive definite probability density proposed in a letter by M.D. Kostin does not meet the requirements of relativistic (quantum) field theory and therefore does not allow for a meaningful physical interpretation.

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The alternative formulation of the Klein-Gordon equation [1, 2] proposed by M.D. Kostin reads [3]

\[ i\hbar \partial_\phi \frac{\partial \phi}{\partial t} = +mc^2 \phi + c(\tilde{\vec{p}}^* \vec{\psi}) \quad (1) \]
\[ i\hbar \partial_\tilde{\vec{\psi}} \frac{\partial \tilde{\vec{\psi}}}{\partial t} = -mc^2 \tilde{\vec{\psi}} + c\hat{\tilde{\vec{p}}} \phi \quad , \]

where \( \phi(\vec{r}, t) \) and \( \tilde{\vec{\psi}}(\vec{r}, t) \) are ‘scalar’ and ‘vector’ probability amplitudes, respectively, and \( \hat{\tilde{\vec{p}}} = -i\hbar \vec{\nabla} \). Defining the probability density

\[ P = \phi^* \phi + (\tilde{\vec{\psi}}^* \vec{\psi}) \quad (3) \]

and the probability current density

\[ \vec{S} = c(\phi^* \tilde{\vec{\psi}} + \phi \tilde{\vec{\psi}}^*) \quad , \]

one readily derives the probability conservation equation

\[ \frac{\partial P}{\partial t} + \vec{\nabla} \vec{S} = 0 \quad . \]

(5)

It is a nice feature of the probability density \( P(\vec{r}, t) \) to be positive definite, although is is clear that the non-existence of a positive definite probability density for the Klein-Gordon equation is no more a problem in quantum field theory.

Multiplying (1) with \((i\hbar \frac{\partial}{\partial t} + mc^2)\) and (2) by \(c\hat{\tilde{\vec{p}}}\) and combining the results, one obtains

\[ \hbar^2 \frac{\partial^2 \phi}{\partial t^2} - c^2 \hbar^2 \vec{\nabla}^2 \phi + m^2 c^4 \phi = 0 \quad , \]

i.e. \( \phi \) satisfies the Klein-Gordon equation, but in a similar way one immediately sees that the components of \( \tilde{\vec{\psi}} \) fulfil the (non-covariant) equation

\[ \hbar^2 \frac{\partial^2 \tilde{\vec{\psi}}}{\partial t^2} - c^2 \hbar^2 (\vec{\nabla}^2 \tilde{\vec{\psi}}) + m^2 c^4 \tilde{\vec{\psi}} = 0 \quad . \]

(7)

Although the problematic nature of equations (1)-(5) can be uncovered easily, their tempting form sometimes leads to confusion and the equations have even found their way into literature [4]. Furthermore, when the scalar particle described by \((\phi, \tilde{\vec{\psi}})\) is coupled to an electromagnetic potential, different results are obtained as in the case of the Klein-Gordon equation. One must therefore ask if the proposed equations should be treated on an equal footing with the usual Klein-Gordon equation.

We give simple arguments in the following which show that the alternative form of the Klein-Gordon equation is hard to interpret in a meaningful way. Obviously, (1) and (2) can be cast in a Dirac-like form

\[ i\hbar \frac{\partial}{\partial t} \Psi = mc^2 \beta \Psi + c(\vec{\alpha} \tilde{\vec{p}}) \Psi \quad , \]

(8)

with appropriate matrices \( \beta \) and \( \vec{\alpha} \) and the four-component wave function

\[ \Psi = \left( \begin{array}{c} \phi \\ \tilde{\vec{\psi}} \end{array} \right) \quad , \]

(9)

or, using a more compact notation in the following where \( \hbar = c = 1 \)

\[ \{i\gamma^\mu \partial_\mu - m\} \Psi(x) = \{\gamma^\mu \hat{\partial}_\mu - m\} \Psi(x) = 0 \quad . \]

(10)
Then it is easy to show by straightforward calculation that matrices $S(\Lambda)$ which relate the wave functions in different coordinates $x, x'$

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \quad x^\nu = (ct, \vec{r}) \quad \Lambda^\mu_\nu \gamma^\rho = S^{-1}(\Lambda) \gamma^\mu S(\Lambda),$$

according to

$$\Psi'(x') = S(\Lambda) \Psi(x) = S(\Lambda) \Psi(\Lambda^{-1} x') \quad \{\gamma^\mu \hat{P}_\mu - m\} \Psi(x) = \{\gamma^\mu \hat{P}_\mu - m\} \Psi'(x') = 0$$

exist trivially for rotations, but not for general Lorentz transformations [5].

A severe problem arises when one considers the propagators for the proposed theory. The Dirac equation can be written in an explicit form as follows

$$\begin{pmatrix}
\hat{p}_0 - m & 0 & \hat{p}_3 & \hat{p}_1 - i\hat{p}_2 \\
0 & \hat{p}_0 - m & \hat{p}_1 + i\hat{p}_2 & -\hat{p}_3 \\
-\hat{p}_3 & -\hat{p}_1 + i\hat{p}_2 & -\hat{p}_0 - m & 0 \\
-\hat{p}_1 - i\hat{p}_2 & \hat{p}_3 & 0 & -\hat{p}_0 - m
\end{pmatrix}
\begin{pmatrix}
\Psi_1 \\
\Psi_2 \\
\Psi_3 \\
\Psi_4
\end{pmatrix} = 0,$$

and by formal inversion of the matrix in (14) the retarded (advanced) propagator can be constructed in momentum space

$$\tilde{S}_{R,A}(p) \sim \frac{\gamma^\mu p_\mu + m}{p^2 - m^2 \pm ip_0},$$

which has causal support in real space

$$\text{supp}(S_{R,A}(x)) \subseteq V^\pm, \quad V^+ = \{x \in \mathbb{R}^4 | x^2 \geq 0, x^0 \geq 0\} \quad V^- = \{x \in \mathbb{R}^4 | x^2 \geq 0, x^0 \leq 0\},$$

a fact which expresses, roughly speaking, the causal structure of the theory [6]. The support property (16) of the tempered distributions $S_{R,A} \in S'(\mathbb{R}^4)$ means that the product $<S_{R,A}|f>$ vanishes for all rapidly decreasing test functions in Schwartz space $f \in S(\mathbb{R}^4)$ which have their support outside the forward (backward) light cone. But in the present case, inversion of the differential operator

$$\begin{pmatrix}
\hat{p}_0 - m & -\hat{p}_1 & -\hat{p}_2 & -\hat{p}_3 \\
\hat{p}_1 & -\hat{p}_0 - m & 0 & 0 \\
\hat{p}_2 & 0 & -\hat{p}_0 - m & 0 \\
\hat{p}_3 & 0 & 0 & -\hat{p}_0 - m
\end{pmatrix},$$

leads to a result

$$\sim \frac{1}{p^2 - m^2}
\begin{pmatrix}
p_0 - m & -p_1 & -p_2 & -p_3 \\
p_1 & -p_0^2 + p_1^2 + p_2^2 + m^2 & -p_1 p_2 & -p_1 p_3 \\
p_2 & -p_1 p_2 & -p_0^2 + p_1^2 + p_2^2 + m^2 & -p_2 p_3 \\
p_3 & -p_1 p_3 & -p_2 p_3 & -p_0^2 + p_1^2 + p_2^2 + m^2
\end{pmatrix},$$

which is in conflict with the requirements of the local structure of quantum field theory due to the non-local operator $\sim (\hat{p}_0 + m)^{-1}$ in the propagator. The description of a scalar particle in the Duffin-Kemmer-Petiau formalism [7, 8, 9] by a five-component wave function is equivalent (at least on the classical level) to the usual Klein-Gordon equation and causes no problems of that kind [10].
References