0. Introduction

In an important contribution to the burgeoning study of integrable systems in non-commutative geometry (see [3], [6], [8]), Kapustin, Kuznetsov and Orlov have generalised the Penrose-Ward twistor transform to that setting, [4]. The geometric parts of the construction translate fairly naturally using the standard techniques of non-commutative geometry ([2],[5]), but the form (or even existence) of the non-commutative twistor algebra is less obvious. In [4] such an algebra is presented using braiding properties defined by an appropriate \( R \)-matrix. In this note we consider the extent to which this is determined by the deformation of the original space-time algebra using a dual quantum group description. The ordinary twistor space and the six-dimensional conformal space in which compactified Minkowski space-time is identified with the rays of a cone are both flag manifolds generated by the conformal group action on highest weight vectors, so we consider non-commutative twistor space as a quantum flag manifold for the quantised function algebra of a deformation of the conformal group [1]. We shall also give a coordinate-free expression for the \( R \)-matrix.

1. The non-commutative space-time algebra

We recall that the non-commutative algebra for (four-dimensional) space-time \( M \) is the twisted group algebra defined by a symplectic form \( \theta \) on \( M \). Its Lie algebra is generated by elements \( X(u) \) for \( u \in M \) which satisfy \( [X(u), X(v)] = i\hbar \theta(u,v)I \). where \( I \) is a positive central element. Using the non-degenerate symmetric bilinear form \( G \) on \( M \) we may write \( i\hbar \theta(u,v) = G(u, \Theta v) \) for a suitable skew-symmetric operator \( \Theta \), (which may be regarded as an element of the Lie algebra of the orthogonal group of \( G \)), so that

\[
[X(u), X(v)] = G(u, \Theta v)I.
\]

We can also define \( H = \frac{1}{2}G^{ij}X_iX_j \) (where \( X(u) = X_ju^j \) in terms of coordinates and using the summation convention), and calculate that

\[
[H, X(u)] = X(\Theta u)I.
\]

In order to generalise the conformal theory it is useful to introduce a central element \( T \) such that \( I = T^2 \), and to define \( D = T^{-1}H \), so that the commutation relations take the homogeneous form

\[
[X(u), X(v)] = G(u, \Theta v)T^2, \quad [D, X(u)] = X(\Theta u)T,
\]

defining a quadratic algebra. The identity \( 2TD = 2H = G^{ij}X_iX_j \), gives the non-commutative version of the quadric cone of the conformal theory.

In the commutative case one studies connections with self-dual curvature. This curvature vanishes on isotropic planes and these define twistors (projective spinors for the conformal group) \( T \), [7]. (For an isotropic plane through the origin, we note that the Clifford algebra of \( M \) acts on the space of spinors. Writing \( \gamma(u) \) for the action of \( u \in M \) we may consider the “vacuum” vector \( \Psi_\xi \in T \) with the property that \( \gamma(u)\Psi_\xi = 0 \) for all \( u \) in the plane defined by \( \xi \in \wedge^2 M \), that is for \( u \wedge \xi = 0 \). Since \( \Psi \) is determined only up to multiples
this gives a ray, and defines a point of the projective space of \( T \). Isotropic planes through other points can be obtained using the action of the conformal group.)

The space \( T \) decomposes into odd and even spinors \( T_+ \oplus T_- \). Each of the spaces \( T_{\pm} \) is two-dimensional so that \( \wedge^2 T_{\pm} \) is spanned by a single vector \( \epsilon_{\pm} \). (We normalise these so that \( \| \epsilon_{\pm} \|^2 = 2 \), so that when we choose orthonormal bases \( e_1, e_2 \) for \( T_+ \) and \( e_3, e_4 \) for \( T_- \) we may take \( \epsilon_+ = e_1 \wedge e_2 = e_1 \otimes e_2 - e_2 \otimes e_1 \) and \( \epsilon_- = e_3 \wedge e_4 = e_3 \otimes e_4 - e_4 \otimes e_3 \).)

The six-dimensional space \( \wedge^2 T \) has a natural symmetric bilinear form \( B \) defined by

\[
B(\xi, \eta)\epsilon_+ \wedge \epsilon_- = \xi \wedge \eta.
\]

The action of the conformal group on \( \wedge^2 T \) leaves invariant the null cone \( B(u, u) = 0 \), and the rays of this cone form a compact complex version of space-time. (The finite points of space time can be identified with the rays of the cone for which \( B(\epsilon_, u) \neq 0 \), or equivalently \( \mathcal{M} \) can be identified with \( T_+ \cup T_- \). For brevity we shall call \( \wedge^2 T \) conformal space-time.)

By construction the orthogonal group of \( G \) has its spin representation \( \Gamma \) on \( T \), which leaves each of \( T_{\pm} \) invariant, and so the forms \( \epsilon_{\pm} \) are also invariant under the spin action. We may choose the orthonormal bases \( e_1, e_2 \) for \( T_+ \) and \( e_3, e_4 \) for \( T_- \) so that the matrix \( \Gamma(\Theta) \) is diagonal:

\[
\frac{\hbar}{2} \begin{pmatrix}
(b-a)\sigma_3 & 0 \\
0 & (a+b)\sigma_3
\end{pmatrix}
\]

where \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \).

We now complete \( \epsilon_{\pm} \) to a basis for \( \wedge^2 T \) by introducing \( \epsilon_1 + i \epsilon_4 = e_4 \wedge e_1, \epsilon_1 - i \epsilon_4 = e_2 \wedge e_3, \epsilon_2 + i \epsilon_3 = e_1 \wedge e_4, \) and \( \epsilon_2 - i \epsilon_3 = e_2 \wedge e_4 \). With this notation \( \{ \epsilon_\mu \} \) forms a \( G \)-orthonormal basis of \( \mathcal{M} \sim \wedge^2 T \) when we identify \( G \) with \(-2B\). The matrix of \( \Theta \) with respect to this basis is given by

\[
i\hbar \begin{pmatrix}
0 & 0 & 0 & a \\
0 & 0 & b & 0 \\
0 & -b & 0 & 0 \\
-a & 0 & 0 & 0
\end{pmatrix}.
\]

Unfortunately, the commutation relations of the non-commutative theory destroy the full conformal symmetry. For example, the transformations of the space \( \mathcal{M} \), affecting only the operators \( X(u) \), must not only be orthogonal with respect to \( G \) but also be symplectic with respect to \( \theta \), or equivalently commute with \( \Theta \). In general this forces them to lie in a Cartan subgroup, though, for particular \( \theta \), (when \( a \) vanishes) they may lie in a larger group. We must therefore deform the conformal group, or more precisely we shall quantise its function algebra by deforming the \( R \) matrix associated with the commutation relations. To make explicit the connection between commutation relations and an \( R \) matrix, we consider for any vector space \( U \) an operator \( R : U \otimes U \to U \otimes U \) satisfying the Yang-Baxter equation, and write \( R = \Phi + R' \) where \( \Phi \) denotes the usual flip operator \( x \otimes y \mapsto y \otimes x \) on \( U \otimes U \), and \( R' \) is the deformation. The function algebra defined by \( R \) is given by operators \( A \) on \( U \) which satisfy \( R(A \otimes A) = (A \otimes A)R \). The following is then a straightforward consequence of our definitions.

**PROPOSITION 1.** The relations \( R(A \otimes A) = (A \otimes A)R \) of the function algebra defined by \( R = \Phi + R' \) are equivalent to the commutation relations

\[
[\langle e|Au\rangle, \langle f|Av\rangle] = \langle f \otimes e|(A^2)R' - R'A^2\rangle(u \otimes v),
\]

where \( A^{(2)} = A \otimes A, e, f \in U^*, u, v \in U \), and \( \langle e|u \rangle \) denotes the pairing between the dual spaces \( U^* \) and \( U \).

**Proof:** The relations \( R(A \otimes A) = (A \otimes A)R \) can be rewritten as

\[
\Phi(A \otimes A) - (A \otimes A)\Phi = (A \otimes A)R' - R'(A \otimes A).
\]

Writing \( A^{(2)} = A \otimes A \), and choosing \( e, f \in U^*, u, v \in U \) this gives

\[
\langle f \otimes e|(\Phi A^{(2)} - A^{(2)}\Phi)(u \otimes v) = \langle f \otimes e|(A^{(2)}R' - R'A^{(2)})(u \otimes v).
\]
The left-hand side can be written as
\[ \langle e \otimes f | A^{(2)} u \otimes v \rangle - \langle f \otimes e | A^{(2)} v \otimes u \rangle = \langle e|Au \rangle \langle f|Av \rangle - \langle f|Av \rangle \langle e|Au \rangle = [\langle e|Au \rangle, \langle f|Av \rangle], \]
so that we have the stated result. 

Let us take \( U = \wedge^2 T \) and write the vectors as \( v = v_0 + v_0 + v_0 + v_0 \), with \( v_0 \in T_+ \wedge T_+ \sim U \). To avoid confusion with the twistor examples we write \( \tilde{R} \) and \( \tilde{A} \) instead of \( \tilde{R} \) and \( \tilde{A} \). We are less interested in the whole function algebra than in the quantum flag manifold generated by the highest weight vector \( e^*_+ \) of the dual basis of \( U^* \), and want to identify \( \langle e^*_+ | \tilde{A}v \rangle \) with \( v_0 D + X(v_0) + v_0 T \). In this notation the commutation relations reduce to
\[ [u_0 D + X(u_0) + u_0 T, v_0 D + X(v_0) + v_0 T] = u_0 TX(\Theta v_0) - v_0 X(\Theta u_0)T + G(u_0, \Theta v_0)T^2. \]

One simple way of obtaining these is to consider the deformation given by \( \tilde{R}' \)
\[ \tilde{R}'(u \otimes v) = u_0 \epsilon_+ \otimes \Theta v_0 - \Theta u_0 \otimes v_0 \epsilon_+ + G(u_0, \Theta v_0)\epsilon_- \otimes \epsilon_. \]
Since this has no terms in \( \epsilon_+ \), we have \( \langle \epsilon^*_+ \otimes \epsilon^*_+ | \tilde{R}' \rangle = 0 \) and one may readily check that \( \tilde{R}' \) gives the correct commutation relation. It is useful to note that
\[ \tilde{R}'(u \wedge v) = u_0 \epsilon_+ \otimes S \Theta v_0 - \Theta u_0 \otimes S v_0 \epsilon_+ + G(u_0, \Theta v_0)\epsilon_- \otimes S \epsilon_, \]
and
\[ \tilde{R}'(u \otimes_S v) = u_0 \epsilon_- \otimes \Theta v_0 - \Theta u_0 \otimes v_0 \epsilon_- , \]
so that \( \tilde{R}' \) interchanges symmetric and antisymmetric tensors. It is also easy to check that the square of \( \tilde{R}' \) vanishes. These are in accord with the following simple result.

**Proposition 2.** A family of deformations \( R = \Phi + R' \) satisfies \( R^2 = 1 \), if and only if \( R' \) maps the symmetric tensors to antisymmetric tensors and vice versa, and \( R'^2 = 0 \).

**Proof:** This is a matter of comparing terms of first and second order in \( R' \) in the identity
\[ 1 = R^2 = \Phi^2 + \Phi R' + R' \Phi + R'^2 = 1 + \Phi R' + R' \Phi + R'^2, \]
which gives \( \Phi R' + R' \Phi = 0 = R'^2 \). The first identity can be rewritten in the form
\[ R'(1 + \Phi) = (1 + \Phi)R', \]
which, recalling that \( \frac{1}{2}(1 + \Phi) \) is the projection onto the symmetric tensor product \( \otimes_2 U \) and \( \frac{1}{2}(1 - \Phi) \) projects onto the antisymmetric tensor product \( \wedge^2 U \), shows that \( R' \) maps \( \otimes_2 U \) to \( \wedge^2 U \) and \( \wedge^2 U \) to \( \otimes_2 U \). (For similar reasons \( R' \) also interchanges \( \otimes_2 U = (1 + R)(U \otimes U) \) and \( \wedge^2 U = (1 - R)(U \otimes U) \).)

2. **Yang-Baxter operators for the twistors**

We now want to find a Yang-Baxter operator \( R \) for twistor space which is consistent with that on conformal space-time \( \wedge^2 T \). More precisely, a Yang-Baxter operator \( \tilde{R} \) on \( \wedge^2 T^* \) can be extended to a Yang-Baxter operator \( \tilde{R} : \otimes^4 T \rightarrow \otimes^4 T \), defined by
\[ \tilde{R} = R_{23}R_{12}R_{34}R_{23}. \]
This interchanges the first and second pairs of factors in the tensor product, allowing one to extend the Yang-Baxter operator \( R \) for twistor space to a Yang-Baxter operator \( \tilde{R} \) for the conformal space-time \( \wedge^2 T \),
and, for consistency, this should agree with that found above. The primary task is to see whether any such \( R \) exist, and to find them.

It is easy to see that \( \tilde{R} \) inherits various properties of \( R \), but in the Appendix we show that two crucial properties of \( \tilde{R} \) must be shared by \( R \), a fact which substantially simplifies the task of finding it.

**Theorem 3.** If a family of Yang-Baxter operators \( R \) which is a deformation of the flip \( \Phi \) satisfies 
\[ [R_{23}R_{12}R_{34}R_{23}]^2 = 1, \] 
then \( R^2 = 1 \), and if, in addition, \( R_{23}R_{12}R_{34}R_{23} \) commutes with \( \Lambda(4) = \Lambda \otimes \Lambda \otimes \Lambda \otimes \Lambda \) for invertible \( \Lambda \), then \( R \) commutes with \( \Lambda(2) = \Lambda \otimes \Lambda \).

We have already observed that for conformal space-time \( \tilde{R}^2 = 1 \), and the invariance of the commutation relations under the infinitesimal rotation \( \Theta \) means that \( \tilde{R} \) commutes with \( \Theta \) and so with the invertible operator \( \Lambda = \exp(s\Theta) \). This suggests that the non-commutative twistor algebra is defined by an operator \( \tilde{R} \) which we shall now find. We first notice that the determinant of \( \tilde{R} \) is only defined on the subspace \( \wedge^2_\Theta T^* \otimes \wedge^2 \Theta T^* \). There is no problem provided that \( \tilde{R} \) has an extension to the whole of \( \otimes^4 T^* \) with the same involutory and intertwining properties. Our argument will start by assuming that there is such an extension, and then justifying this once \( \tilde{R} \) has been found. Once established we can deduce that \( R \) also commutes with the spin action \( \Theta_1 + \Theta_2 \). In fact \( \tilde{R} \) enjoys an additional, less obvious symmetry arising from its compatibility with the exact sequence: \( 0 \to \mathcal{T}_- \to \mathcal{T} \to \mathcal{T}_+ \to 0 \).

**Proposition 4.** Let \( N \) be a nilpotent operator on \( \mathcal{T} \) such that \( \text{im}(\Theta) = \ker(N) = \mathcal{T}_- \) and let \( K = 1 + sN \). Then \( K^{(4)} \) commutes with \( \Theta \).

**Proof:** By definition \( K \) acts as the identity on \( \mathcal{T}_- \) and so \( K^{(2)} = \epsilon_- \). By contrast \( K^{(2)} = \epsilon_+ \) must have the general form \( \epsilon_+ = n + \epsilon_- \) for scalar \( n \) and \( n \in T_+ \wedge T_- \). For \( u_0 \in T_+ \wedge T_- \) we must have \( K^{(2)} = u_0 + s\lambda \epsilon_- \) for a suitable scalar \( \lambda \) which we shall now find. We first notice that the determinant of \( K \) is clearly 1, so that for any \( \xi \) and \( \eta \in T_+ \wedge T_- \) we have \( \xi \wedge \eta = K^{(2)} \xi \wedge K^{(2)} \eta \), and hence \( B(\xi, \eta) = B(K^{(2)} \xi, K^{(2)} \eta) \). In particular, taking \( \epsilon_+ = \epsilon_+ \) and \( n = u_0 \) gives

\[
0 = B(\epsilon_+, u_0) = B(\epsilon_+ + s\nu \epsilon_-, u_0 + s\lambda \epsilon_-) = s\lambda + sB(n, u_0),
\]

so that \( \lambda = -B(n, u_0) = \frac{1}{2}G(n, u_0) \), and \( K^{(2)} = u_0 + \frac{1}{2}sG(n, u_0)\epsilon_- \).

We now calculate that

\[
K^{(4)}(u \otimes v) = K^{(4)}(u_+ \epsilon_- \otimes \Theta v_0 - \Theta u_0 \otimes v_+ \epsilon_- + G(u_0, \Theta v_0)\epsilon_- \otimes \epsilon_-)
\]

\[
= u_+ \epsilon_- \otimes (\Theta v_0 + \frac{1}{2}sG(n, \Theta u_0)\epsilon_- - (\Theta u_0 + \frac{1}{2}sG(n, \Theta u_0)\epsilon_-) \otimes v_+ \epsilon_- + G(u_0, \Theta v_0)\epsilon_- \otimes \epsilon_-.
\]

Since \( G(n, \Theta u_0) = -G(u_0, \Theta n) \) and, in particular, \( G(n, \Theta n) = 0 \), this reduces to

\[
u_+ \epsilon_- \otimes \Theta v_0 - \Theta u_0 \otimes v_+ \epsilon_- + G(u_0 + \frac{1}{2}s\gamma u_0, \Theta(v_0 + \frac{1}{2}s\gamma v_0))\epsilon_- \otimes \epsilon_- = \tilde{R}(K^{(4)}(u \otimes v)),
\]

and since \( \tilde{R} \) automatically commutes with any operator \( \Lambda^{(4)} \) this gives the stated result.

We also want \( R \) to satisfy the Yang-Baxter equation, but there is another requirement which is much stronger, namely that the restriction of \( \tilde{R} = R_{23}R_{12}R_{34}R_{23} \) to \( \wedge^2_\Theta T^* \otimes \wedge^2 \Theta T^* \) contains only first order terms in \( \Theta \) (that is, in \( R \)). With the requirements that \( R \) be an involution and have the same symmetries as \( \tilde{R} \), this limits the possibilities considerably.

**Theorem 5.** For non-degenerate \( \Theta \), any Yang-Baxter operator \( R = \Phi + R' \) satisfying \( R^2 = 1 \), commuting with \( \Theta \) and giving only first order corrections to \( \tilde{R} \) on the conformal space-time is in the two-parameter family

\[
R' = \alpha\Theta_1|\epsilon_-\rangle\langle\epsilon_+| + \beta|\epsilon_-\rangle\langle\epsilon_+|\Theta_1,
\]

for some scalars \( \alpha, \beta \in \mathbb{C} \). These operators automatically satisfy the Yang-Baxter equation.
\textbf{Proof:} Since $R'$ commutes with the action of $\Theta$ it must preserve eigenspaces for the action. The formula for $\Gamma(\Theta)$ shows that generically (unless $a, b$ or $a \pm b$ vanishes) each symmetric tensor $e_j \otimes e_j$ is in a one-dimensional eigenspace of its own, and since $R'$ maps symmetric tensors to antisymmetric tensors we must have $e_j \otimes e_j \in \ker(R')$. However, $\ker(R')$ must be invariant for operators of the form $K_{\pm}(2)$ and by suitable choices of $N$, $K_s e_j \otimes K_s e_j \ (j = 1, 2)$ generate the whole of $T \otimes T$. which must therefore also lie in the kernel.

Generically, there is also a four-dimensional null space for the action of $\Theta$ spanned by $e_1 \otimes S e_3$, $e_+ = e_1 \wedge e_2$, $e_3 \otimes S e_4$, and $e_- = e_3 \wedge e_4$, and the other eigenspaces are two-dimensional spanned by tensors of the form $e_j \otimes S e_k$ and $e_j \wedge e_k$ for various distinct $\{ j, k \} \notin \{ \{1, 2\}, \{3, 4\} \}$. We have just seen that the symmetric tensors are in the kernel of $R'$. Since $R'$ maps antisymmetric to symmetric tensors each $e_j \wedge e_k$ must map to a multiple of $e_j \otimes S e_k$. In fact, this multiple must also vanish otherwise overlapping terms such as $R_{12} R_{23} (e_i \otimes e_j \otimes e_k)$, give quadratic contributions to $\tilde{R}$, so we may assume that $R'(e_i^* \otimes S e_k^*) = 0$ and $R'(e_i^* \wedge e_j) = 0$ for the given range of $j$ and $k$.

The only interesting deformations therefore occur in the four-dimensional null space. Here it is useful to note that $\Theta e_+ = \frac{1}{2} \hbar(b-a) e_1 \otimes S e_2$, and $\Theta e_- = \frac{1}{2} \hbar(b+a) e_3 \otimes S e_4$. (Since $e_\pm$ and their duals are invariant under the appropriate actions of $\Theta_1 + \Theta_2$, these could equally have been written in terms of $\Theta_2$.) We already know that $R'$ kills $e_3 \otimes S e_4$ and $e_3 \wedge e_4$, so, again discarding the deformations which mix the symmetric and antisymmetric products of the same vectors and lead to quadratic contributions to $\tilde{R}$, we are left with $R'$ of the form

$$R' e_+ = \alpha \Theta_1 e_- \quad R' \Theta_1 e_+ = \alpha' e_-,$$

or, equivalently, $R' = \alpha \Theta_1 |e_-| e_+^* + \beta |e_-| e_+ |e_+|$. The condition that $R'^2 = 0$ holds automatically as do the Yang-Baxter equations. $\blacksquare$

We shall next show that with an appropriate choice of constants $\alpha$ and $\beta$ this is compatible with $\tilde{R}$. Since by construction $R_{23} R_{12} R_{34} R_{23}$ is an involution commuting with the same operators as $\tilde{R}$ this verifies that there is indeed an extension of $\tilde{R}$ to $\otimes^4 T^*$, so justifying our working assumption.

**THEOREM 6.** The only Yang-Baxter operator for $T$ compatible with the known Yang-Baxter operator $\tilde{R}$ on the non-commutative conformal space-time is

$$R = \Phi + 2(\Theta_1, |e_+^*| |e_-| +.$$

\textbf{Proof:} We have noted that there are only first order corrections so the Yang-Baxter operator takes the form $\tilde{R} = \Phi + R'$, where $\Phi$ is the flip and

$$\tilde{R}' = \Phi_{23} \Phi_{12} \Phi_{34} R_{23} + \Phi_{23} \Phi_{12} R_{34} \Phi_{23} + \Phi_{23} R_{12} \Phi_{34} \Phi_{23} + R_{23} \Phi_{12} \Phi_{34} \Phi_{23}.$$

Since, for example $\Phi_{23} R_{24} = R_{24}' \Phi_{23}$, this can be rewritten as

$$\tilde{R}' = R_{14}^* \Phi_{12} \Phi_{23} + R_{24}^* \Phi_{13} + R_{13}^* \Phi_{24} + R_{23}^* \Phi_{12} \Phi_{34} \Phi_{23}.$$

Applying this to $(u \wedge x) \otimes (v \wedge y)$ we get

$$R_{14}^* (x \otimes y \otimes u \otimes v - u \otimes y \otimes x \otimes v - x \otimes v \otimes u \otimes y + u \otimes v \otimes x \otimes y) + R_{24}^* (v \otimes x \otimes u \otimes y - v \otimes u \otimes x \otimes y - y \otimes u \otimes x \otimes v + y \otimes u \otimes x \otimes v) + R_{13}^* (u \otimes y \otimes v \otimes x - x \otimes y \otimes u \otimes v - u \otimes v \otimes y \otimes x + x \otimes v \otimes y \otimes u) + R_{23}^* (v \otimes u \otimes y \otimes x - v \otimes x \otimes y \otimes u - y \otimes u \otimes v \otimes x + y \otimes x \otimes u \otimes v).$$

If we take $u, v \in T_+$ and $x, y \in T_-$ and use the form of $R'$ most of these terms disappear, leaving

$$-R_{14}^* u \otimes y \otimes x \otimes v + R_{24}^* y \otimes u \otimes x \otimes v + R_{13}^* u \otimes y \otimes v \otimes x - R_{23}^* y \otimes u \otimes v \otimes x.$$
It will be sufficient to consider the case of $u = e_1$ and $v = e_2$, where
\[
R(e_1 \otimes e_2) = \frac{1}{\hbar}(a + b)ae_3 \otimes e_4 + \frac{1}{\hbar}(b - a)\beta e_3 \wedge e_4
= \frac{1}{\hbar}(a + b) + \beta(b - a)e_3 \otimes e_4 + \frac{1}{\hbar}(a(a + b) - \beta(b - a))e_4 \otimes e_3.
\]
Writing $\sigma = \frac{1}{\hbar}(a(a + b) + \beta(b - a))$ and $\delta = \frac{1}{\hbar}(a(a + b) - \beta(b - a))$, we reduce $R'(e_1 \wedge x \otimes e_2 \wedge y)$ to
\[
\begin{align*}
\sigma(-e_3 \otimes y \otimes x \otimes e_4 + y \otimes e_3 \otimes x \otimes e_4 + e_4 \otimes y \otimes e_3 \otimes x - y \otimes e_4 \otimes e_3 \otimes x) \\
+ \delta(-e_4 \otimes y \otimes x \otimes e_3 + y \otimes e_4 \otimes x \otimes e_3 + e_4 \otimes y \otimes e_3 \otimes x - y \otimes e_4 \otimes e_3 \otimes x)
= \sigma y \otimes e_3 \otimes x \wedge e_4 + \delta y \wedge e_4 \otimes x \wedge e_3.
\end{align*}
\]
This is non-vanishing if $x = e_3$ and $y = e_4$, or vice versa. The first possibility gives
\[
R'(e_1 \wedge x \otimes e_2 \wedge y) = \sigma e_4 \wedge e_3 \otimes e_4 = -\sigma \epsilon_- \wedge \epsilon_-,
\]
whilst the second gives
\[
R'(e_1 \wedge e_4 \otimes e_2 \wedge e_3) = -\delta \epsilon_- \otimes \epsilon_-.
\]
On the other hand $e_1 \wedge e_3 = e_2 + i\epsilon_3$ and $e_2 \wedge e_4 = e_2 - i\epsilon_3$ gives
\[
R'(e_1 \wedge e_3 \otimes e_2 \wedge e_4) = G(e_1 \wedge e_3, \Theta(e_2 \wedge e_4))\epsilon_- \otimes \epsilon_- = -2h\epsilon_- \wedge \epsilon_-.
\]
This gives $\frac{1}{\hbar}(a(a + b) + \beta(b - a)) = 2b$, so that we take $\alpha = \beta = 2$. This is also the choice appropriate to the other choice of $x$ and $y$ and leads to the stated solution, which expressed in terms of coordinates agrees with that of [4]. We can also check that this form of $R'$ works on tensors of the form $\epsilon_+ \otimes u_0$. However, at this point we must mention a subtlety which has so far been suppressed, that we should really be working not with $\wedge^2 T$ but with $\wedge^2 T_\% = (1 - R) \otimes^2 (T)$. Fortunately, the correction $R'$ has a large kernel, and the only difference is for multiples of $\epsilon_+$, which should be replaced by $\frac{1}{\hbar}(1 - R)\epsilon_+ = \epsilon_+ - \Theta \epsilon_-$. Fortunately this does not cause any serious complications.

3. Commutation relations for twistors

We can immediately combine Theorem 6 with Proposition 1 to give an explicit form for the twistor commutation relations.

COROLLARY 7. The commutation relations for non-commutative twistor space can be written as
\[
\begin{align*}
\frac{1}{2}\langle e|Au, \langle f|Av \rangle &= \langle f \otimes e|A(2)\Theta_1 \epsilon_- \rangle(Au \otimes v) + \langle f \otimes e|A(2)\epsilon_- \rangle(Au \otimes v) \\
- \langle f \otimes e|\Theta_1 \epsilon_- \rangle(Au \otimes Av) - \langle f \otimes e|\epsilon_- \rangle(Au \otimes Av).
\end{align*}
\]
Recalling that $\ker(R') \supset T_- \otimes T + T \otimes T_-$, we see that the first term on the right-hand side vanishes whenever $u$ or $v$ lies in $T_-$. Similarly, the fact that $\text{im}(R') \subseteq T_- \otimes T_-$ means that the second term on the right vanishes for $e$ or $f$ in $T_+$. Taken together we see that $\langle e|Au, \langle f|Av \rangle = 0$ whenever $e$ or $f \in T_+^*$ and $u$ or $v \in T_-$, and, in particular, elements of the form $\langle e|Au$ with $e \in T_+$ and $u \in T_-$ are central in the algebra.

More generally it is sufficient for our purposes to use only $e$ and $f$ in $T_+$, and then we derive the commutation relation
\[
\begin{align*}
\frac{1}{2}\langle e|Au, \langle f|Av \rangle &= \langle f \otimes e|A(2)\Theta_1 \epsilon_- \rangle(Au \otimes v) + \langle f \otimes e|A(2)\epsilon_- \rangle(Au \otimes v) \\
- \langle f \otimes e|\Theta_1 \epsilon_- \rangle(Au \otimes Av) - \langle f \otimes e|\epsilon_- \rangle(Au \otimes Av).
\end{align*}
\]
Introducing the symplectic forms $\omega_\pm(u, v) = \langle \epsilon_\pm | u \otimes v \rangle$ on $T_\pm$, and using the self-adjointness of $\Theta_1$, we may rewrite this as
\[
\begin{align*}
\frac{1}{2}\langle e|Au, \langle f|Av \rangle &= \langle f \otimes e|A(2)\Theta_1 \epsilon_- \rangle(\omega_+(u, v) + \omega_-(\Gamma(\Theta)u, v) \\
= \langle f \otimes e|A(2)[\omega_+(u, v)\Theta_1 + \omega_-(\Gamma(\Theta)u, v)]\epsilon_- \rangle.
\end{align*}
\]
The most interesting case arises for \( u = e_1 \) and \( v = e_2 \) when \( \omega_+(u,v) = 1 \) and \( \omega_+(\Gamma(\Theta)u,v) = \frac{1}{\hbar}(b-a) \), so that

\[
[\langle e | A e_1 \rangle, \langle f | A e_2 \rangle] = \hbar(f \otimes e | A^{(2)}((b + a)e_3 \otimes_S e_4 + (b - a)e_3 \wedge e_4))
\]

Setting \( A_{jr} = \langle e_j | A e_r \rangle \) and taking \( e = e_r \) and \( f = e_s \), we deduce that whenever \( r \) and \( s \) are 1 or 2, the commutation relation gives

\[
[A_{11}, A_{22}] = 2\hbar (bA_{s4}A_{r4} + aA_{s4}A_{r3}).
\]

Setting \( z_j = A_{1j} \) and recalling that we already know that \( z_3, z_4, w_3, \) and \( w_4 \) are central, we obtain the relations

\[
[z_1, z_2] = 2\hbar(a + b)z_3z_4 \quad [z_1, w_2] = 2\hbar(az_3w_4 + bz_4w_3) \quad [w_1, w_2] = 2\hbar(a + b)w_3w_4.
\]

The first of these is essentially the relation of [4] for the twistor algebra. These provide an alternative way of constructing the Minkowski algebra by rewriting the matrix elements \( \langle e^*_s | A^{(2)} \xi \rangle \) for \( \xi \in \wedge^2 \mathbb{R}^* \) in terms of \( z_{jk} = w_j z_k - w_k z_j \), we get the formulae of [4]. For example, \( T = \langle e^*_s | A^{(2)} \xi \rangle = z_3w_4 - w_3z_4 \) is central to the algebra.

**Appendix**

This Appendix is devoted to a proof of Theorem 3. This is done in stages through a chain of propositions. We start by rewriting the conditions that \( R^2 = 1 \), and that \( R \) commutes with \( \Lambda^{(4)} = \Lambda \otimes \Lambda \otimes \Lambda \otimes \Lambda \). The first condition is equivalent to \( R = R^{-1} = [R^{-1}] = [R^{-1}] \), and the second to \( R = \Lambda^{(4)}R\Lambda^{(4)} = [\Lambda^{(2)}R\Lambda^{(2)}]^{-1} \). We should like to see whether these two statements, together with the Yang-Baxter equations, are sufficient to give \( R^{-1} = R = \Lambda^{(2)}R\Lambda^{(2)} \).

To this end we ask whether two Yang-Baxter operators \( P \) and \( Q \) satisfying

\[
P_{23}P_{12}P_{34}P_{23} = Q_{23}Q_{12}Q_{34}Q_{23}
\]

must necessarily be identical, that is \( P = Q \).

**Lemma 8.** Yang-Baxter operators \( P \) and \( Q \) satisfy \( P_{23}P_{12}P_{34}P_{23} = Q_{23}Q_{12}Q_{34}Q_{23} \) if and only if there exist \( Y \) and \( Z \) such that

\[
P_{23}P_{12} = Q_{23}Q_{12}Z_{23}, \quad Q_{23}Q_{12} = Z_{12}P_{23}P_{12},
\]

\[
P_{12}P_{23} = Q_{12}Q_{23}Y_{12}, \quad Q_{12}Q_{23} = Y_{23}P_{12}P_{23}.
\]

**Proof:** Our main tool is separation of variables, which we shall use repeatedly. We start with our assumed identity, which can be rearranged as

\[
Q_{12}^{-1}Q_{23}^{-1}P_{23}P_{12} = Q_{34}Q_{23}P_{23}^{-1}P_{34}^{-1}.
\]

Since the left hand side is independent of 4 and the right is independent of 1, we see that each side must act as the identity on those factors and so can be written in the form \( Z_{23} \). Rearranging this gives

\[
P_{23}P_{12} = Q_{23}Q_{12}Z_{23}, \quad Q_{34}Q_{23} = Z_{23}P_{34}P_{23},
\]

and by dropping the indices by 1 the second equation becomes

\[
Q_{23}Q_{12} = Z_{12}P_{23}P_{12}.
\]

Similarly, since \( P_{12} \) and \( P_{34} \) commute (and similarly for \( Q \)) the assumed identity can be rearranged as

\[
Q_{34}^{-1}Q_{23}^{-1}P_{23}P_{34} = Q_{12}Q_{23}P_{12}^{-1}P_{34}^{-1},
\]

and setting both sides equal to \( Y_{23} \) gives (with an index shift)

\[
P_{12}P_{23} = Q_{12}Q_{23}Y_{12}, \quad Q_{12}Q_{23} = Y_{23}P_{12}P_{23}.
\]
It is clearly possible to recover the original equation by eliminating either $Y$ or $Z$.  

**Lemma 9.** Yang-Baxter operators $P$ and $Q$ satisfy $P_{23}P_{12}P_{34}P_{23} = Q_{23}Q_{12}Q_{34}Q_{23}$ if and only if there exist $Y, Z, C$ and $D$ satisfying

\[
Q_{12}Z_{12} = C_2 P_{12}, \quad Q_{12}Y_{12} = C_1 P_{12},
\]

\[
Z_{12}P_{12} = Q_{12}D_1, \quad Y_{12}P_{12} = Q_{12}D_2,
\]

and $P_{12}P_{23}P_{12} = Q_{12}Q_{23}Q_{12}D_2 = C_2^{-1}Q_{12}Q_{23}Q_{12}$.

**Proof:** Using the Yang-Baxter equation for $Q$ and the second equation of each pair in Lemma 8 we obtain

\[
Q_{12}Z_{12}P_{23}P_{12} = Q_{12}Q_{23}Q_{12} = Q_{23}Q_{12}Q_{23} = Q_{23}Y_{12}P_{12}P_{23}.
\]

Now using the Yang-Baxter equation for $P$, we deduce that

\[
Q_{12}Z_{12}P_{12}^{-1} = Q_{23}Y_{23}P_{23}^{-1},
\]

and, since one side is the identity on the first factor and the other on the third, we deduce that both sides are of the form $C_2$ for some operator $C$. Rearranging this gives

\[
Q_{12}Z_{12} = C_2 P_{12}, \quad Q_{23}Y_{23} = C_2 P_{23},
\]

and by lowering the indices the second equation gives

\[
Q_{12}Y_{12} = C_1 P_{12}.
\]

Similarly using the first equation of each pair

\[
Q_{23}Q_{12}Z_{23}P_{23} = P_{23}P_{12}P_{23} = P_{12}P_{23}P_{12} = Q_{12}Q_{23}Y_{12}P_{12},
\]

from which we see that

\[
Q_{23}^{-1}Z_{23}P_{23} = Q_{12}^{-1}Y_{12}P_{12},
\]

and both sides must be of the form $D_2$, giving

\[
Z_{12}P_{12} = Q_{12}D_1, \quad Y_{12}P_{12} = Q_{12}D_2.
\]

Substituting these expressions into the previous identities we obtain $P_{12}P_{23}P_{12} = Q_{12}Q_{23}Q_{12}D_2$, and similarly $P_{12}P_{23}P_{12} = C_2^{-1}Q_{12}Q_{23}Q_{12}$. The argument easily reverses.

The next two results cover the cases of real interest to us. For these we limit ourselves to Yang-Baxter operators $P$ and $Q$ which are deformations of the standard flip operators, from which one readily checks that $Y, Z, C$ and $D$ are all deformations of the identity operator. The main observation which we need is that each deformation of the identity operator, such as $C$, has a unique square root which is also a deformation of the identity. It is given by the binomial expansion of $\sqrt{1 + x}$ with $C - 1$ substituted for $x$. (In fact, since $T$ is finite-dimensional, this can be rewritten as a polynomial in $C$).

**Lemma 10.** If Yang-Baxter operators $P$ and $Q$ satisfy $P_{23}P_{12}P_{34}P_{23} = Q_{23}Q_{12}Q_{34}Q_{23}$ and $Q = P^{-1}$, then $Q = P$, that is $P^2 = 1$.

**Proof:** When $Q = P^{-1}$ we immediately get the equations

\[
(P_{12}P_{23}P_{12})^2 = D_2 = C_2^{-1},
\]
so that \( C \) and \( D \) are inverses. Now we also have

\[
D_2 = (P_{12}P_{23}P_{12})^2 = (P_{12}P_{23}P_{12})(P_{23}P_{12}P_{23}) = (P_{12}P_{23})^3,
\]

and similarly

\[
D_2 = (P_{23}P_{12})^3.
\]

Consequently

\[
P_{12}D_2 = P_{12}(P_{23}P_{12})^3 = (P_{12}P_{23})^3P_{12} = D_2P_{12},
\]

so that \( D_2 \), and therefore \( C_2 \), commute with \( P_{12} \). They similarly commute with \( P_{23} \), which, by lowering the indices, means that \( D_1 \) and \( C_1 \) commute with \( P_{12} \). (Since \( C_j \) clearly commutes with \( P_{12} \) for \( j > 2 \) this shows that all the \( C_j \) and \( P_{rs} \) commute.)

The remaining equations of Lemma 9 now give

\[
Q_{12}^2D_1 = Q_{12}Z_{12}P_{12} = C_2P_{12}^2,
\]

and

\[
Q_{12}^2D_2 = Q_{12}Y_{12}P_{12} = C_1P_{12}^2.
\]

Given that \( C \) and \( D \) are inverse and commute with \( P_2 \), these reduce to \( P_{12}^2 = C_1C_2 \), or \( P^4 = C \otimes C \). As noted above \( C \) has a unique square root \( K \) deformed from the identity, and given that the undeformed value of \( P^2 \) is 1 we take square roots to see that \( P^2 = K \otimes K \).

Since \( K \) is given by a polynomial in \( C \) it enjoys the same commutation properties as \( C \), and in particular \( K_j \) commutes with each of the \( P_{rs} \). This means that

\[
K_{rs}^{-2} = C_{rs}^{-1}
\]

\[
= (P_{12}P_{23}P_{12})^2
\]

\[
= P_{12}P_{23}P_{12}P_{23}P_{12}
\]

\[
= P_{12}P_{23}K_1K_2P_{23}P_{12}
\]

\[
= K_1K_2P_{12}P_{23}^2P_{12}
\]

\[
= K_1K_2^2K_3P_{12}^2
\]

\[
= K_1K_2^3K_3.
\]

Separating the indices we see that \( K = 1 \), and so \( P^2 = K \otimes K = 1 \).

We now look at the second case where \( Q_{12} = \Lambda^{(2)}P_{12}\Lambda^{(2)}^{-1} \). and assume that \( P^2 = 1 = Q^2 \).

**Lemma 11.** If Yang-Baxter operators \( P \) and \( Q \) satisfy \( P_{23}P_{12}P_{34}P_{23} = Q_{23}Q_{12}Q_{34}Q_{23} \) and \( P^2 = 1 = Q^2 \) then \( P = Q \).

**Proof:** Since \( P \) and \( Q \) are involutions our earlier identity \( C_2P_{12}^2 = Q_{12}^2D_1 \) reduces to \( C_2 = D_1 \), so that \( C \) and \( D \) must both be multiples of 1, indeed the same multiple, \( \lambda I \). The equation

\[
Q_{12}Q_{23}Q_{12}D_2 = C_{rs}^{-1}Q_{12}Q_{23}Q_{12}.
\]

then shows that \( \lambda^2 = 1 \), and using the undeformed value of \( C = 1 \) we deduce that \( C = 1 = D \). This means that

\[
Q_{12}Z_{12} = P_{12}, \quad Q_{12}Y_{12} = P_{12}.
\]

\[
Z_{12}P_{12} = Q_{12}, \quad Y_{12}P_{12} = Q_{12},
\]

and

\[
P_{12}P_{23}P_{12} = Q_{12}Q_{23}Q_{12}.
\]
The first sets of equations are now consistent and can be regarded simply as definitions of \(Y\) and \(Z\). Using them in the second set we have

\[
Y_{23}P_{23} = Q_{23} = Q_{12}^{-1}P_{12}P_{23}Q_{12}^{-1} = Y_{12}P_{23}Y_{12}^{-1}.
\]

We now write \(P_{jk} = \widehat{P}_{jk}\Phi_{jk}\), where \(\Phi_{jk}\) is the flip which is the undeformed value of \(P\). Then

\[
Y_{23}\widehat{P}_{23}\Phi_{23}Y_{12} = Y_{12}\widehat{P}_{23}\Phi_{23},
\]

and since \(\Phi_{23}Y_{12} = Y_{13}\Phi_{23}\), this gives

\[
Y_{23}\widehat{P}_{23}Y_{13} = Y_{12}\widehat{P}_{23},
\]

Similarly we obtain

\[
Y_{12}\widehat{P}_{12}Y_{13} = Y_{23}\widehat{P}_{12},
\]

and from these we deduce that

\[
(\widehat{P}_{23}Y_{13}^{-1}\widehat{P}_{23}^{-1})(\widehat{P}_{12}Y_{13}^{-1}\widehat{P}_{12}^{-1}) = (Y_{23}^{-1}Y_{12})(Y_{12}^{-1}Y_{23}) = 1,
\]

and thence that

\[
Y_{13}\widehat{P}_{23}^{-1}\widehat{P}_{12}Y_{13} = \widehat{P}_{23}^{-1}\widehat{P}_{12}.
\]

Setting \(V = \widehat{P}_{23}^{-1}\widehat{P}_{12}\), this can be rewritten as \(Y_{13}VY_{13} = V\), which leads to

\[
Y_{13}V^2 = Y_{13}VY_{13}VY_{13} = V^2Y_{13},
\]

showing that \(Y_{13}\) and \(V^2\) commute. Our standard square root argument shows that \(Y_{13}\) commutes with \(V\), which in turn gives the identity \(Y_{13}^2 = 1\), and so \(Y_{13} = 1\), and \(Q = P\). In the case in hand this shows that \(R\) commutes with \(\Lambda\) and so with \(\Theta\).}

References