We use simple entropy arguments to estimate the possible size of the QGP at the AGS and the SPS. We find that the possibility to form a large volume of QGP at the AGS or the SPS is very small. The size of the QGP at RHIC and the LHC is also predicted.

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One of the main goals of high energy nuclear collisions is to create a quark-gluon plasma (QGP) [1,2] of macroscopic size. The hope of discovering the QGP in heavy-ion collisions is thus to some extent connected to the possibility of measuring the geometric size of the region of secondary particle production. An important tool in accomplishing such measurements is Hanbury Brown-Twiss (HBT) interferometry [3,4]. On the theoretical front, owing to the complexity of high energy heavy-ion collisions, several models to simulate the heavy-ion collisions process [5–10] have been devised. In those, the size of the QGP phase is a vital element, as it may affect significantly the shape of the single parton distributions. In this note, using HBT results from the AGS and the SPS, we infer the possible sizes of QGP in those energy regimes. The sizes of QGP at RHIC and LHC are also predicted.

The idea to predict the size of QGP is simple and is based on the picture given by Lee some years ago [11]. From the second law of thermodynamics we have equation

\[ S_{QGP} \leq S_{Had}. \] (1)

Here \( S_{QGP} \) and \( S_{Had} \) are the total entropy of the QGP phase and of the hadron phase, respectively. As more than eighty percent of final particles are pions, we will calculate the entropy of pions and multiply it by a factor of \( \alpha \) to represent the total particles entropy. For this discussion we take \( \alpha \sim 1.1 \). Eq.(1) can be re-written as

\[ s_{QGP}V_{QGP} \leq \alpha \pi \alpha \pi. \] (2)

Here \( V_{QGP} \) and \( \pi \) are the volume of the QGP phase and of the pion phase respectively, \( s_{QGP} \) and \( s_{\pi} \) represent the entropy density in the QGP phase and the pion phase, and reads

\[ s_{QGP} = \frac{2}{45} \pi^2 T^3 [16 + \frac{21}{2} n_f], \]

\[ s_{Had} = \frac{2}{15} \pi^2 T^3. \] (3)

\( n_f \) is the number of flavors. In the above we assume that the pion, quark and antiquark masses are zero. From Eq. (3), we have

\[ s_{QGP} = \frac{16}{3} + \frac{7}{2} n_f = 12 \sim 16. \] (4)

Using Eq.(2) and Eq.(4), we get

\[ V_{had} \geq 12V_{QGP}. \] (5)

Gaussian functions have been used to fit the two-pion correlation functions at AGS and SPS energy and it has been found that the fitted source radius are \( 5 - 7 fm \) [3]. From Eq.(5), we find that the source radius for QGP is \( 2 - 3 fm \). This result is important but depends strongly on the source radius as measured from pion interferometry. If the QGP size is truly \( 2 - 3 \) fm then we need to consider finite size effects on the spectrum distributions. For example, because of the Heisenberg uncertainty principle, particles in a small volume will have a widespread distribution in momentum space. It has been shown in Ref. [12–15] that for quarks and gluons, finite size effects will become important when the size of QGP is less than 6 fm. For pions, finite size effects set in when the size of the hadron phase is less than 9 fm. For such small QGP sizes at AGS and SPS energy one needs to consider finite size effects on the final state observables, and this fact is bound to affect several of the proposed signatures. Finite size effects on the observables have also been studied by Elze, Greiner and Rafelski [16,17] long ago. If we consider the fact that pion interferometry actually measures the pion source at freezeout time and the freezeout temperature \( (T_f) \) is less than the critical temperature \( (T_c) \), then the QGP source radius should be even smaller than the value given above. Using typical freezeout temperature \( T_f = m_\pi = 138 \) MeV or fitted results \( T_f = 120 \) MeV [18] and assuming the phase transition temperature \( T_c = 200 \) MeV, we have

\[ V_{had} \geq 37 \sim 56 V_{QGP}. \] (6)

This result depends strongly on the freezeout temperature. From Eq. (6), we find that the size of QGP phase is \( 1 - 2 \) fm. This conclusion implies that even if a QGP is formed at the AGS or the SPS, the size of QGP will be very small. Here we also need to mention that this QGP means the QGP at the critical time, i.e. when the system enters the mixed phase. Of course at the freezeout time, the assumption that the pion mass is zero is inappropriate. We will come to this question at the end of the paper. It has been suggested that there is a possibility to form some QGP droplets inside a hadron gas. If this picture is true, we have

\[ S_{had} \geq N_{droplet} S_{droplet}. \] (7)
Here $S_{\text{droplet}}$ is the entropy in each droplet and $N_{\text{droplet}}$ is the total number of droplets. Assuming that the size of all droplets are the same, we have

$$V_{\text{had}} \geq (37 \sim 56)N_{\text{droplet}}V_{\text{droplet}}.$$  \hspace{1cm} (8)

If we assume for example that the droplet number is two, the droplet size will be 0.5-1.2 fm.

The radius of pion sources at RHIC or the LHC are proportional to the pion multiplicity distribution as [19]:

$$R_{\text{had}} \propto (0.6 \sim 1.2)(\frac{dN}{dy})^{1/2\sim 1/3}. \hspace{1cm} (9)$$

From Eq.(9), we find that the hadron phase radius for Au + Au collisions at RHIC is 14-26 fm. First HBT results from RHIC indicate that the HBT radius is larger than the HBT radius at SPS energy, but still less than ten Fermis [20]. This may due to the strong flow that has been observed at RHIC which make the HBT radius less than the true size of hadron phase, $R_{\text{had}}$. From our numbers for the size of the hadron phase at RHIC we get an appreciable QGP size of

$$R_{\text{QGP}} = 4 - 8 \text{ fm}. \hspace{1cm} (10)$$

However this estimation depends strongly on the size of hadron phase. This implies a caveat which we will discuss at the end of this paragraph. We have assumed that at SPS energy, the size of hadron phase is almost the same scale as that of the HBT radius, and this assumption is borne out by numerical simulations. Pion interferometry results from simulator (ARC, RQMD, VENUS) at SPS energy have indicated [21] that pion interferometry gives us the right geometry size of the pion source. A theoretical analysis in Ref. [22] has shown that if we use a box source to fit the pion correlation function of Pb + Pb collisions at SPS energy, the box length is 12 fm, then QGP size at SPS energy will be 3-4 fm which is still very small. So both theory and empirical analyses indicate that even if a QGP is formed at the SPS, it’s size should be very small and it is mandatory to consider finite size effects. But at RHIC energy the correspondence between physical size and HBT radius may not hold anymore, owing to strong flow effects. Present HBT theory shows that HBT only measures part of the whole pion source due to flow which generates a strong correlation between coordinate and momentum. This causes the apparent source radius to become smaller.

At LHC energy, the pion multiplicity becomes large [19] and we estimate the radius of the hadronic phase as

$$R_{\text{had}} = 18 - 36 \text{ fm}, \hspace{1cm} (11)$$

and the corresponding QGP size should be

$$R_{\text{QGP}} = 6 - 11 \text{ fm}. \hspace{1cm} (12)$$

In the following we will use the Bjorken picture to perform an independent estimate of QGP size. In the Bjorken model, QGP is assumed to be produced at first and at time $\tau_0$ it reaches equilibrium then the QGP will evolve according to hydrodynamical equation, that is [1,23]

$$s(\tau) = s(\tau_0) \left(\frac{T_0}{\tau}\right)^{3/2} \hspace{1cm} (\tau > \tau_0). \hspace{1cm} (13)$$

When the temperature drops to $T_c$ at proper time $\tau_c$, a mixed phase occurs, then following equation exists

$$s(\tau) = (\frac{T_c}{\tau})^{4/3} \hspace{1cm} (\tau > \tau_c). \hspace{1cm} (14)$$

According to Ref. [1], after time

$$\tau_h = 6.16\tau_c \hspace{1cm} (15)$$

the whole system will be in a hadronic phase. This phase will expand and temperature will decrease from critical temperature ($T_c$) to freezeout temperature ($T_f$). During the hadron expansion stage, we have

$$s(\tau_f) = s(\tau_h) \left(\frac{\tau_f}{\tau_h}\right)^{4/3} \hspace{1cm} (\tau > \tau_h). \hspace{1cm} (16)$$

Here $\tau_f$ is the freezeout time. The freezeout time can be determined by the following equation [1,23]

$$T(\tau_f) = \left(\frac{T_h}{\tau}\right)^{1/3}. \hspace{1cm} (17)$$

For $T_f = 120$ MeV and $T(\tau_h) = 200$ MeV, we have

$$s(\tau_f) = s(\tau_h)\frac{\tau_h}{\tau_f} = s(\tau_h)\left(\frac{T_f}{T_h}\right)^{3/4}. \hspace{1cm} (18)$$

Finally we get

$$s(\tau_f) \sim \frac{s(\tau_h)}{52}, \hspace{1cm} (19)$$

thus

$$V(\tau_c) = \frac{V_{\text{freezeout}}}{52}. \hspace{1cm} (20)$$

From Eq.(20) we get similar conclusion as above: even if there is QGP formed at AGS or SPS energy, its size should be around the size of nucleon.

In the following we will use energy density of pions calculated from present measurement to estimate the possible QGP energy density at the critical time. Using the Bjorken model and assuming that pion mass is zero, we have

$$\epsilon(\tau_h) = \left(\frac{\tau_f}{\tau_h}\right)^{4/3}. \hspace{1cm} (21)$$

Here $\tau_h$ and $\tau_f$ represent hadron time and freezeout time respectively. In the mixed phase, Eq.(21) is still valid, implying
Assuming the pion mass as zero, we have

$$\frac{\epsilon(\tau_c)}{\epsilon(\tau_h)} = \left(\frac{\tau_h}{\tau_c}\right)^{4/3}. \tag{22}$$

Using $\tau_h = 6.16\tau_c$ and assuming (see Eq. (17))

$$\frac{\tau_h}{\tau_f} = \left(\frac{T(\tau_f)}{T(\tau_h)}\right)^{3}, \tag{23}$$

we have

$$\epsilon(\tau_c) = 11.3 \cdot \epsilon(\tau_f) \frac{T(\tau_h)}{T(\tau_f)} = 87\epsilon(\tau_f). \tag{24}$$

When $T_f = 120$ MeV, it is calculated that the energy density of pions ($\pi^+,\pi^-$ and $\pi^0$) is 0.023 GeV/fm$^3$. According to Eq.(24), we find that the critical density of QGP is

$$\epsilon(\tau_c) = 2.1\text{GeV/fm}^3. \tag{25}$$

Using equation (with $n_f = 2$)

$$\epsilon(\tau_c) = \frac{37\pi^2}{30} T_c^4, \tag{26}$$

we find that the critical temperature $T_c = 191$ MeV, which is consistent with the input value $T_c = 200$ MeV. As we have said earlier the assumption of massless pions is inappropriate at the freezeout time, as the freezeout temperature has almost the same value as the pion mass. It has shown in Ref. [18] that pion data from all heavy-ion reaction are consistent with thermal Bose-Einstein distribution

$$f = \frac{1}{\exp(E/T) - 1} \tag{27}$$

with $T = 120MeV$. Taking the pion mass as 138 MeV, and assuming the freezeout temperature $T_f = 120MeV$, the entropy density for pions reads

$$s = 3 \int \frac{d\mathbf{p}}{(2\pi)^3} [(1 + f)\ln(1 + f) - f \cdot \ln f] = \frac{0.246}{(fm)^3}. \tag{28}$$

Assuming the pion mass as zero, we have $s = 0.297/(fm)^3$. That is the entropy density will decrease if the value of the mass increases. Due to Eq. (28), Eq. (6) changes to

$$V_{had} \geq 69V_{QGP}. \tag{29}$$

Thus the QGP size will be even smaller than in our previous estimates.

Let us end with the following comments:

(1) In the above we have assumed that the chemical potentials of quarks and pions are zero which is a good approximation if the number of quarks and pions are very large. From Fig. 1 or Eq.(28), we find that if the chemical potential decreases the entropy density of pions will decrease too. When chemical potential $\mu = m_{\pi}$, the maximum value of pion chemical potential, we need to consider Bose-Einstein condensate, this is of course beyond the scope of the present work. From Fig.1 we find that the corresponding maximum entropy density of pions is around $0.6/(fm)^3$. Thus we have $V_{had} > 27V_{QGP}$ this is the upper limit of our estimation.

(2) In the above calculations, we have used a simple Bose-Einstein or Fermi-Dirac distribution for pions or quarks. In principle one needs to use the complete source distribution $S(x,K)$ given in Ref. [22,24] to calculate the entropy density which will be used to estimate the possible QGP size at SPS or AGS. To use the function of $S(x,K)$ or $f(x,K)$ to calculate the entropy is very important especially when $x$ and $p$ correlation is essential. In Ref [24], the authors have found that

$$\frac{S}{N_{\pi}} = 3.9 \pm 1.8. \tag{30}$$

Taking $N_{\pi} = 100 \sim 150$, we find that the QGP size is $2 \sim 3fm$, which is consistent with the estimations given above. We can use

$$\frac{S}{N_{\pi}} = \frac{\int d\mathbf{p}[(1 + f)\ln(1 + f) - f \cdot \ln f]}{\int d\mathbf{p}f} \tag{31}$$

to calculate the specific entropy, $S/N$. Using Eq.(27) and choosing $T_f = 120$ MeV, we find that $\frac{S}{N_{\pi}} = 4.32$. This result is consistent with Eq. (30).

(3) In this paper, Eq.(3) has been used to calculate the entropy density of QGP phase in finite volume. This formula certainly should be corrected when QGP are confined in a finite volume. But this is precisely our conclusion: one will not observe a large extent QGP at AGS. To use the function of $S(x,K)$ or $f(x,K)$ to calculate the entropy is very important especially when $x$ and $p$ correlation is essential. In the following, we will calculate de Broglie wavelength of quarks, gluons and pions to show why we need to take into account finite size effects. Using Eq. (27) and taking $T_f = 120$ MeV, we find that the average pion momentum is 372 MeV. When $T_c = 200$ MeV,
The average momentum of gluons is 540 MeV and the average momentum of quarks is 472 MeV. The de Broglie wavelength for pions, quarks and gluons are $\lambda_\pi = 3.3 fm$, $\lambda_q = 2.3 fm$ and $\lambda_g = 2.6 fm$ respectively. It is interesting to notice that the de Broglie wavelength for pion at AGS or SPS energy is still smaller than the size of pion source. But where the de Broglie wavelength for quarks and gluons is almost the same size as the size of QGP source, thus finite size effects for those particles should be important.

(4) In the paper, the total entropy of final state particles is determined by the volume and entropy density of pions. But there are also other ways to calculate the final state entropy. For example, Bialas and Czyz [25] have suggested to use Renyi’s entropy to derive the standard entropy. For a hot pion gas, the degrees of freedom change and the state is a hot pion gas, a quark-dominated gas, or a gluon-dominated gas. Thus the finite size effects on those particles should be important.

(5) All the estimations in this work are based on the assumption that the initial state is QGP. If the initial state is a hot pion gas, a quark-dominated gas, or a gluon-dominated gas, the degrees of freedom change and the volume of the initial state will change too, for a given final state. For example for a hot pion gas, $V_{\text{final}} > 5V_{\text{initial}}$. For a quark gas, $V_{\text{final}} > 35V_{\text{initial}}$. For a gluon gas, $V_{\text{final}} > 26V_{\text{initial}}$. So the initial volume is large in the case of a pion gas, as it should be.

To conclude: it is a key question to know the size of QGP which may be produced at the early time of heavy-ion collisions. We show that if there is a QGP formed at the AGS and/or at the SPS, its radius should only be around 1 – 2 fm. Thus the finite size effects on the QGP distribution functions will be important. We also predict that the size of QGP at RHIC and LHC will be around 4 – 8 fm and 6 – 11 fm respectively. For this size of QGP, the finite volume effects on QGP distribution can be neglected.

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