AdS/CFT correspondence and quantum induced dilatonic multi-brane-worlds

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ABSTRACT

d\text{5} dilatonic gravity action with surface counterterms motivated by AdS/CFT correspondence and with contributions of brane quantum CFTs is considered around AdS-like bulk. The effective equations of motion are constructed. They admit two (outer and inner) or multi-brane solutions where brane CFTs may be different. The role of quantum brane CFT is in inducing of complicated brane dilatonic gravity. For exponential bulk potentials the number of AdS-like bulk spaces is found in analytical form. The correspondent flat or curved (de Sitter or hyperbolic) dilatonic two branes are created, as a rule, thanks to quantum effects. The observable early Universe may correspond to inflationary brane. The found dilatonic quantum two brane-worlds usually contain the naked singularity but in couple explicit examples the curvature is finite and horizon (corresponding to wormhole-like space) appears.

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1 Introduction

Recent booming activity in the study of brane-worlds is caused by several reasons. First, gravity on 4d brane embedded in higher dimensional AdS-like Universe may be localized [1, 2]. Second, the way to resolve the mass hierarchy problem appears[1]. Third, the new ideas on cosmological constant problem solution come to game [7, 8]. Very incomplete list of references[3, 4] (and references therein) mainly on the cosmological aspects of brane-worlds is growing every day.

The essential element of brane-world models is the presence in the theory of two free parameters (bulk cosmological constant and brane tension, or brane cosmological constant). The role of brane cosmological constant is to fix the position of the brane in terms of tension (that is why brane cosmological constant and brane tension are almost the same thing). Being completely consistent and mathematically reasonable, such way of doing things may look not completely satisfactory. Indeed, the physical origin (and prediction) of brane tension in terms of some dynamical mechanism may be required.

The ideology may be different, in the spirit of refs.[6, 5]. One considers the addition of surface counterterms to the original action on AdS-like space. These terms are responsible for making the variational procedure to be well-defined (in Gibbons-Hawking spirit) and for elimination of the leading divergences of the action. Brane tension is not considered as free parameter anymore but it is fixed by the condition of finiteness of spacetime when brane goes to infinity. Of course, leaving the theory in such form would rule out the possibility of consistent brane-world solutions existence. Fortunately, other parameters contribute to brane tension. If one considers that there is quantum CFT living on the brane (which is more close to the spirit of AdS/CFT correspondence[9] ) then such CFT produces conformal anomaly (or anomaly induced effective action). This contributes to brane tension. As a result dynamical mechanism to get brane-world with flat or curved (de Sitter or Anti-de Sitter) brane appears. The curvature of such 4d Universe is expressed in terms of some dimensional parameter $l$ which usually appears in AdS/CFT set-up and of content of quantum brane matter. In other words, brane-world is the consequence of the fact (verified experimentally by everybody life) of the presence of matter on the brane! For example, sign of conformal anomaly terms for usual matter is such that in one-brane case the de Sitter (ever expanding, inflationary ) Universe is preferrable solution of
brane equation\(^3\).

The scenario of refs.[6, 5] may be extended to the presence of dilaton(s) as it was done in ref.[10] or to formulation of quantum cosmology in Wheeler-De Witt form [11]. Then whole scenario looks even more related with AdS/CFT correspondence as dilatonic gravity naturally follows as bosonic sector of d5 gauged supergravity. Moreover, the extra prize-in form of dynamical determination of 4d boundary value of dilaton-appears. In ref.[10] the quantum dilatonic one brane Universe has been presented with possibility to get inflationary or hyperbolic or flat brane with dynamical determination of brane dilaton. The interesting question is related with generalization of such scenario in dilatonic gravity for multi-brane case. This will be the purpose of present work.

In the next section we present general action of d5 dilatonic gravity with surface counterterms and quantum brane CFT contribution. This action is convenient for description of brane-worlds where bulk is AdS-like spacetime. There could be one or two (flat or curved) branes in the theory. As it was already mentioned the brane tension is fixed in our approach, instead of it the effective brane tension is induced by quantum effects. In section three, the explicit analytical solution of bulk equation for number of exponential bulk potentials is presented. There is the possibility to have two (inner and outer) branes associated with each of above bulk solutions. It is interesting that quantum created branes can be flat, or de Sitter (inflationary) or hyperbolic. The role of quantum brane matter corrections in getting of such branes is extremely important. Nevertheless, there are few particular cases where such branes appear on classical level, i.e. without quantum corrections. We also briefly describe how to get generalization of above solutions for quantum dilatonic multi-brane-worlds with more than two branes. Brief summary of results is given in final section where also the result of studying the character of singularities for proposed two-brane solutions is presented. In most cases, as usually occurs in AdS dilatonic gravity, the solutions contain the naked singularity. However, in couple cases the scalar curvature is finite and there is horizon. The corresponding 4d branes may be interpreted as wormhole.

\(^3\)Similar mechanism for anomaly driven inflation in usual 4d world has been invented by Starobinsky[15] and generalized for dilaton presence in refs.[17]
2 Dilatonic gravity action with brane quantum corrections

Let us present the initial action for dilatonic AdS gravity under consideration. The metric of (Euclidean) AdS has the following form:

\[ ds^2 = dz^2 + \sum_{i,j=1}^{4} g^{(4)}_{ij} dx^i dx^j, \quad g^{(4)}_{ij} = e^{2\tilde{A}(z)} \hat{g}_{ij}. \]  

(1)

Here \( \hat{g}_{ij} \) is the metric of the Einstein manifold, which is defined by \( r_{ij} = k \hat{g}_{ij} \), where \( r_{ij} \) is the Ricci tensor constructed with \( \hat{g}_{ij} \) and \( k \) is a constant. One can consider two copies of the regions given by \( z < z_0 \) and glue two regions putting a brane at \( z = z_0 \). More generally, one can consider two copies of regions \( \tilde{z}_0 < z < z_0 \) and glue the regions putting two branes at \( z = \tilde{z}_0 \) and \( z = z_0 \). Hereafter we call the brane at \( z = \tilde{z}_0 \) as “inner” brane and that at \( z = z_0 \) as “outer” brane.

Let us first consider the case with only one brane at \( z = z_0 \) and start with Euclidean signature action \( S \) which is the sum of the Einstein-Hilbert action \( S_{EH} \) with kinetic term and potential \( V(\phi) = \frac{1}{2}l^2 + \Phi(\phi) \) for dilaton \( \phi \), the Gibbons-Hawking surface term \( S_{GH} \), the surface counter term \( S_1 \) and the trace anomaly induced action \( W \):

\[ S = S_{EH} + S_{GH} + 2S_1 + W, \]

(2)

\[ S_{EH} = \frac{1}{16\pi G} \int d^5 x \sqrt{|g(5)|} \left( R(5) - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi + \frac{12}{l^2} + \Phi(\phi) \right), \]

(3)

\[ S_{GH} = \frac{1}{8\pi G} \int d^4 x \sqrt{|g(4)|} \nabla_\mu n^\mu; \]

(4)

\[ S_1 = -\frac{1}{16\pi Gl} \int d^4 x \sqrt{|g(4)|} \left( \frac{6}{l} + \frac{l}{4} \Phi(\phi) \right), \]

(5)

\[ W = b \int d^4 x \sqrt{\tilde{g}} \tilde{F} A + b' \int d^4 x \sqrt{\tilde{g}} \left\{ A \left[ 2 \square - \tilde{R}_{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu - \frac{4}{3} \tilde{R} \square - \frac{2}{3} (\tilde{\nabla}_\mu \tilde{R}) \tilde{\nabla}_\mu \right] A \right\}. \]

\[ \text{For the introduction to anomaly induced effective action in curved space-time (with torsion), see section 5.5 in [12]. This anomaly induced action is due to brane CFT living on the boundary of dilatonic AdS-like space.} \]
\begin{equation}
\left\{ b'' + \frac{2}{3} (b + b') \right\} \int d^4 x \sqrt{\tilde{g}} \left[ \tilde{R} - 6 \Box A - 6 (\tilde{\nabla}_\mu A)(\tilde{\nabla}^\mu A) \right]^2 
\end{equation}

\begin{equation}
+ C \int d^4 x \sqrt{\tilde{g}} A \phi \left[ \Box - 2 \tilde{R}_{\mu \nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu - \frac{2}{3} \tilde{R} \Box + \frac{1}{3} (\tilde{\nabla}^\mu \tilde{R}) \tilde{\nabla}_\mu \right] \phi .
\end{equation}

Here the quantities in the 5 dimensional bulk spacetime are specified by the suffices (5) and those in the boundary 4 dimensional spacetime are specified by (4). The factor 2 in front of $S_1$ in (2) is coming from that we have two bulk regions which are connected with each other by the brane. In (4), $n^\mu$ is the unit vector normal to the boundary. In (4), (5) and (6), one chooses the 4 dimensional boundary metric as $g_{(4) \mu \nu} = e^{2 A} \tilde{g}_{\mu \nu}$. We should distinguish $A$ and $\tilde{g}_{\mu \nu}$ with $\tilde{A}(z)$ and $\hat{g}_{ij}$ in (1). The metric $\tilde{g}_{ij}$ is given by $\tilde{g}_{\mu \nu} d x^\mu d x^\nu \equiv l^2 (d s^2 + d \Omega^2)$. We also specify the quantities given by $\tilde{g}_{\mu \nu}$ by using $\tilde{\nabla} A$. $G(\tilde{G})$ and $F(\tilde{F})$ are the Gauss-Bonnet invariant and the square of the Weyl tensor.

In the effective action (6) induced by brane quantum matter, with $N$ scalar, $N_{1/2}$ spinor, $N_1$ vector fields, $N_2 (= 0$ or $1$) gravitons and $N_{\text{HD}}$ higher derivative conformal scalars, $b$, $b'$ and $b''$ are

\begin{equation}
b = \frac{N + 6 N_{1/2} + 12 N_1 + 611 N_2 - 8 N_{\text{HD}}}{120 (4 \pi)^2} ,
b' = \frac{- N + 11 N_{1/2} + 62 N_1 + 1411 N_2 - 28 N_{\text{HD}}}{360 (4 \pi)^2} ,
b'' = 0 .
\end{equation}

Usually, $b''$ may be changed by the finite renormalization of local counterterm in gravitational effective action. As it was the case in ref.[10], the term proportional to $\left\{ b'' + \frac{2}{3} (b + b') \right\}$ in (6), and therefore $b''$, does not contribute to the equations of motion. Note that CFT matter induced effective action may be considered as brane dilatonic gravity.

For typical examples motivated by AdS/CFT correspondence[9] one has: a) $\mathcal{N} = 4$ $SU(N)$ SYM theory: $b = - b' = \frac{C}{4} = \frac{N^2 - 1}{4 (4 \pi)^2}$, b) $\mathcal{N} = 2$ $Sp(N)$ theory: $b = \frac{12 N^2 + 18 N - 2}{24 (4 \pi)^2}$, $b' = - \frac{12 N^2 + 12 N - 1}{24 (4 \pi)^2}$. One can write the corresponding expression for dilaton coupled spinor matter [14] which also has non-trivial (slightly different in form) dilatonic contribution to CA than in case of holographic conformal anomaly[13] for $\mathcal{N} = 4$ super Yang-Mills theory.
We can also consider the case where there are two branes at \( z = \tilde{z}_0 \) and \( z = z_0 \), adding the action corresponding to the brane at \( z = \tilde{z}_0 \) to the action in (2).

3 Dilatonic quantum brane-worlds

Let us consider the solution of field equations for two-branes model. First of all, one defines a new coordinate \( z \) by \( z = \int dy \sqrt{f(y)} \) and solves \( y \) with respect to \( z \). Then the warp factor is \( e^{2A(z,k)} = y(z) \). Here one assumes the metric of 5 dimensional spacetime as follows:

\[
ds^2 = g_{(5)\mu \nu} dx^\mu dx^\nu = f(y) dy^2 + y \sum_{i,j=1}^{4} \hat{g}_{ij}(x^k) dx^i dx^j.
\] (8)

Here \( \hat{g}_{ij} \) is the metric of the 4 dimensional Einstein manifold as in (1).

Here we only summarize the obtained results (for more details, see [21]). Generally the obtained bulk solutions have the form:

\[
\phi(y) = p_1 \ln (p_2 y), \quad \Phi(\phi) = -\frac{12}{l^2} + c_1 \exp (a\phi) + c_2 \exp (2a\phi).\]

(9)

Case 1
(a) bulk solution

\[
c_1 = \frac{6kp_2p_1^2}{3 - 2p_1^2}, \quad c_2 = 0, \quad a = -\frac{1}{p_1}, \quad p_1 \neq \pm \sqrt{6}, \quad f(y) = \frac{3 - 2p_1^2}{4ky}.\]

(10)

(b) When \( k \neq 0 \) and \( p_1^2 < 2 \), there is an outer brane solution if \( F_1(y_+) \geq -8b' \), and there is an inner brane solution if \( F_1(y_-) \leq 8b' \). Here \( F_1 \) is defined by

\[
F_1(y_0) \equiv \frac{3}{16\pi G} \left( \frac{q}{2} \frac{y_0^2}{2} - \frac{1}{2l} y_0^2 - \frac{q^2 p_1^2 l y_0}{16} \right).
\] (11)

and \( y_{\pm} \) is given by

\[
y_{\pm}^{1/2} = \frac{3ql}{8} \left( 1 \pm \sqrt{1 - \frac{4p_1^2}{9}} \right).\]

(12)
(c) Solution for $k = 0$

$$p_1^2 \rightarrow \frac{3}{2}, \quad y_0^\frac{1}{4} = \frac{3ql}{4}, \quad ql = \frac{4}{4}.$$ \hspace{1cm} (13)

Case 2

(a) bulk solution

$$c_1 = -6kp_2, \quad a = \pm \frac{1}{\sqrt{3}}, \quad p_1 = \mp \sqrt{3}, \quad f(y) = \frac{3}{p_2 - 4ky}.$$ \hspace{1cm} (14)

(b) In case of $k > 0$, $\tilde{c}_2 \equiv \frac{c_2}{p_2}$ should be positive and there is an outer brane solution, at least if $F_2 \left( \frac{\tilde{c}_2}{p_2} \right) \geq -8b'$, where

$$F_2(y_0) \equiv \frac{3}{16\pi G} \left( \frac{y_0}{2\sqrt{2\tilde{c}_2 - 4ky_0}} - \frac{y_0^2}{2l} + \frac{kl y_0}{4} - \frac{l\tilde{c}_2}{24} \right).$$ \hspace{1cm} (15)

(c) In case of $k < 0$, $F_2(y)$ has at least one minimum if $\tilde{c}_2 < k^2 l^2 \left( \sqrt{\frac{1}{2}} - 1 \right)$ or $\tilde{c}_2 > 0$. If the value of $F_2(y)$ at the maximum is larger than $-8b'$, there is an outer brane solution. If $\tilde{c}_2 > 0$ and $F_2(0) < 8b'$ or $\tilde{c}_2 < 0$ and $F_2 \left( \frac{\tilde{c}_2}{p_2} \right) < 8b'$, there can be an inner brane solution.

(d) In case of $k = 0$, if $\tilde{c}_2 > 0$, the solution is given by

$$y_0 = \frac{l\sqrt{\tilde{c}_2}}{2} \left( \sqrt{\frac{2}{3}} \pm \frac{1}{\sqrt{3}} \right).$$ \hspace{1cm} (16)

Case 3

(a) bulk solution

$$c_2 = 3kp_2, \quad a = \pm \frac{1}{\sqrt{3}}, \quad p_1 = \mp \sqrt{3}, \quad f(y) = \frac{21\sqrt{p_2}}{8\sqrt{y} \left( c_1 y + 7k \sqrt{p_2} y \right)}.$$ \hspace{1cm} (17)

(b) When $\tilde{c}_1 \equiv \frac{c_1}{\sqrt{p_2}} < 0$, $k > 0$ and there can be outer brane solution if $F_3 \left( \frac{49k^2}{c_1^2} \right) > -8b'$, where

$$F_3(y_0) \equiv \frac{3}{16\pi G} \left\{ \frac{y_0}{2y_0} \sqrt{\frac{8\sqrt{y_0} \left( \tilde{c}_1 y_0 + 7k \sqrt{y_0} \right)}{21}} \right. \left. - \frac{y_0^2}{2l} - \frac{l}{24} \sqrt{y_0} \left( \tilde{c}_1 y_0 + 3k \sqrt{y_0} \right) \right\}.$$ \hspace{1cm} (18)
(c) When $\tilde{c}_1 > 0$ and $k < 0$, there always exists outer brane solution if $F_3 \left( \frac{4a^2}{\tilde{c}_1^2} \right) > -8b'$.

From the above results in case 1~3, we find there very often appear two (inner and outer) branes solution as in the first model by Randall and Sundrum [1]. Moreover, the branes may be curved as de Sitter or hyperbolic space which gives the way for ever expanding inflationary Universe. Such solutions often can exist even if there is no any quantum effect, i.e., $b' = 0$.

Let us make few remarks on the form of metric. If one considers the metric in the form (1), the warp factor $e^{2\tilde{A}(z)}$ does not behave as an exponential function of $z$ but as a power of $z$. This would require that we need a region (of complete spacetime) where, the potential and the dilaton become almost constant. It results in difficulties when one tries to explain the hierarchy using this model.

Hence, we presented number of dilatonic (inflationary, flat or hyperbolic) two brane-world Universes which are created by quantum effects of brane matter. Sometimes, such Universes may be realized due to specific choice of dilatonic potential even on classical level.

In some papers (for example in [18]), the solution with many branes was proposed. In such model, there are two AdS spaces with the different radii or different values of the cosmological constants. They are glued by a brane, whose tension is given by the difference of the inverse of the radii. In the solution, the value of $\frac{dA}{dz}$ in the metric of the form (1) jumps at the brane, which tells the value of $f(y)$ in the metric choice in (8) jumps on the brane since $\sqrt{f(y)} = \frac{dz}{dy} = \frac{1}{2y \frac{dA}{dz}}$. Imagine one includes the quantum effects on the brane. Then one can, in general, glue two AdS-like spaces with same values of the cosmological constant. Let us assume that there is a brane at $y = \hat{y}_0$ and there are two AdS-like spaces in $y > \hat{y}_0$ and $y < \hat{y}_0$ glued by the brane. One now denotes the quantity in the AdS-like space in $y > \hat{y}_0$ ($y < \hat{y}_0$) by the suffix $+$ ($-$). Then we can often generalize two brane-world for multi-brane case.

4 Discussion

In summary, we presented the generalization of quantum dilatonic brane-world[10] where brane is flat, spherical (de Sitter) or hyperbolic and it is
induced by quantum effects of CFT living on the brane. In this generalization one may have two brane-worlds or even multi-brane-worlds which proves general character of scenario suggested in refs. [5, 6] where instead of arbitrary brane tension added by hands the effective brane tension is produced by boundary quantum fields. What is more interesting the bulk solutions have analytical form, at least, for specific choice of bulk potential under consideration.

In classical dilatonic gravity the variety of brane-world solutions has been presented in ref. [16] where also the question of singularities has been discussed. The fine-tuned example of bulk potential where one gets bulk solution which is not singular has been presented. In our solutions, the curvature singularity appears at \( y = 0 \).

In case 1, when \( y \sim 0 \) and the coordinates besides \( y \) are fixed, the infinitesimally small distance \( ds \) is given by \( ds = \sqrt{f} dy \sim \frac{dy}{\sqrt{y}} \), which tells that the distance between the brane and the singularity is finite. Then in cases of \( k = 0 \) and \( k < 0 \), the singularity is naked when we Wick re-rotate spacetime to Lorentzian signature. When \( k > 0 \), the singularity is not exactly naked after the Wick re-rotation since the horizon is given by \( y = 0 \), i.e. the horizon coincides with the curvature singularity.

In case 2, the situation is not changed for \( k = 0 \), \( k > 0 \) and \( k < 0 \) with \( \tilde{c}_2 > 0 \) from that in case 1 and the distance between the brane and the singularity is finite since \( ds \sim \frac{dy}{\sqrt{y}} \sqrt{\frac{f}{2k}} \) when \( y \) is small. When \( k < 0 \) with \( \tilde{c}_2 < 0 \), however, the singularity is not naked since there is a kind of horizon at \( y = \frac{\tilde{c}_2}{2k} \), where \( \frac{1}{f(y)} = 0 \). We should note the scalar curvature \( R_\text{(5)} \) is finite.

This tells that \( y \) is not proper coordinate when \( y \sim \frac{\tilde{c}_2}{2k} \). If new coordinate \( \eta \) is introduced : \( \eta^2 \equiv 2\left(y - \frac{\tilde{c}_2}{2k}\right) \), the metric in (8) is rewritten as follows,

\[ ds^2 = -\frac{3}{4k} d\eta^2 + \left(\frac{\tilde{c}_2}{2k} + \frac{\eta^2}{2}\right) \sum_{i,j=1}^{4} \hat{g}_{ij}(x^k) dx^i dx^j. \]  

(19)

The radius of 4d manifold with negative \( k \), whose metric is given by \( \hat{g}_{ij} \), has a minimum \( \frac{\tilde{c}_2}{2k} \) at \( \eta = 0 \), which corresponds to \( y = \frac{\tilde{c}_2}{2k} \). The radius increases when \( |\eta| \) increases. Therefore the spacetime can be regarded as a kind of wormhole, where two universes corresponding to \( \eta > 0 \) and \( \eta < 0 \), respectively, are joined at \( \eta = 0 \).

In case 3, the singularity is naked (the singularity is not exactly naked when \( k > 0 \) as in case 1) in general and the distance between the brane and
the horizon is finite except $k > 0$ and $\tilde{c}_1 < 0$ case since there is a horizon at $\sqrt{y} = -\frac{2k}{\tilde{c}_1}$ where the scalar curvature is finite.

The price for having analytical bulk results (exactly solvable bulk potential) is the presence of (naked) singularity. One can, of course, present the fine-tuned examples of bulk potential as in refs.\[10, 16\] where the problem of singularity does not appear. Moreover, bulk quantum effects may significantly modify classical bulk configurations \[6, 19, 20\] which presumably may help in resolution of (naked) singularity problem. However, in such situation there are no analytical bulk solutions in dilatonic gravity.

There are various ways to extend the results of present work. First of all, one can construct multi-brane dilatonic solutions within the current scenario for another classes of bulk potential. However, this requires the application of numerical methods. Second, it would be interesting to describe the details of brane-world anomaly driven inflation (with non-trivial dilaton) at late times when it should decay to standard FRW cosmology. Third, within similar scenario one can consider dilatonic brane-world black holes which are currently under investigation.

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