In the quest for the realization of a low noise quantum computing device, geometric phases [1] are now getting considerable attention because of their intrinsic tolerance to area preserving noise [2–5]. So far, only the adiabatic geometric phase (i.e. Berry phase) was discussed in this context.

While it is indeed useful because of its intrinsic tolerance to noise, the application of adiabatic geometric phases to quantum computing has several drawbacks. First, while adiabaticity does not necessarily mean slow evolutions, it would nevertheless be advantageous, while retaining the tolerance to noise, to remove the adiabaticity constraint in order to take full advantage of the short coherence times of the envisioned quantum computers.

Moreover, another drawback of the adiabatic phase gate is that during the adiabatic evolution, both geometric and dynamic phases are acquired. The later is not tolerant to area preserving noise and must be removed. This could be done using spin-echo like refocusing schemes which require going over the adiabatic evolution twice [2,3,5]. However, if the second pass does not retrace exactly the first one, the dynamic phase will not completely cancel, thereby introducing errors. In other words, leaving aside experimental errors, the ‘random’ noise in the classical fields controlling the quantum evolution should be the same on each pass for the tolerance to noise to be preserved.

A third difficulty is that adiabatic geometric phases are only possible if non-trivial loops are available in the space of parameters controlling the qubit’s evolution. In other words, the single qubit Hamiltonian must be of the form

\[ H = -\frac{1}{2}B_x(t)\sigma_x - \frac{1}{2}B_y(t)\sigma_y - \frac{1}{2}B_z(t)\sigma_z, \]

where external control over all three (effective) fields \( B_i(t) \) is possible. Such control is not possible in most of the current quantum computer architectures proposals. Control over only two fields, \( B_x \) and \( B_z \), is usually the norm. In this case, all loops in parameter space are limited to the \( x-z \) plane and the (relative) Berry phase is limited to integer multiples of \( 2\pi \). It therefore cannot be observed and is of no use for computation. Note that control over fields in all three directions is possible in NMR where the Berry phase as been observed experimentally [2]. More recently, Falci et al. [5], extended the original charge qubit proposal [6] from a symmetric to an asymmetric dc-SQUID design to allow a non-zero \( B_y \) and therefore non-trivial closed paths in parameter space.

As we shall see, all of the above issues, namely slow evolutions, need of refocusing and control over many fields, seems to be resolved when one considers the non-adiabatic generalization of Berry’s phase, the Aharonov-Anandan (AA) phase [7]. This was noticed very recently by Xiangbin and Keiji [8]. In this note, we point out that, while being an attractive idea, the application of the AA phase to quantum computation is not straightforward.

Let us start by recalling the main ideas related to the AA phase and then comment on it’s use as a geometric phase gates for quantum computation.

Consider a system whose Hamiltonian \( H \) is controlled by a set of external parameters \( \mathbf{R}(t) \). Upon varying adiabatically the control parameters \( \mathbf{R}(t) \) around a closed loop \( C \) such that \( \mathbf{R}(\tau) = \mathbf{R}(0) \), if the system is initially in an eigenstate of \( H \) it will remain, by adiabaticity, an eigenstate of the instantaneous Hamiltonian. The final state will therefore differ only by a phase factor from the initial state. Berry showed that this phase factor has a dynamic and geometric contribution, the later depending solely on the loop \( C \) in parameter space [1]. For a Hamiltonian (which is non-degenerate on \( C \)), starting the evolution with a superposition of eigenstates, each eigenstate will acquire it’s corresponding Berry phase and it can be observed by interference.

Adiabaticity was invoked by Berry to ensure that the evolution of each eigenstate is cyclic, i.e. that the final and initial state differ only by a phase factor:

\[ |\psi(\tau)\rangle = U(\tau)|\psi(0)\rangle = e^{i\phi}|\psi(0)\rangle, \]

for some real phase \( \phi \). It is then possible to generalize Berry’s phase to non-adiabatic evolutions by choosing, for a given \( H(t) \), the particular initial states for which eq. (2) holds. For non-adiabatic evolutions, these so-called cyclic initial states [9] are generally not eigenstates of the system’s Hamiltonian. Aharonov and Anandan [7] showed that the total phase acquired by such a cyclic initial state in the interval \([0, \tau]\) on which it is cyclic is given by the sum of a dynamic

\[ \delta = -\int_0^\tau dt \langle \psi(t)|H(t)|\psi(t)\rangle, \]
\[ \beta = \phi - \delta. \] (4)

The later is the AA phase. This result is exact, it does not rest on an adiabatic approximation but is restricted to cyclic initial states where eq. (2) holds.

The AA phase is not associated to a closed loop in parameter space, as in Berry’s case, but rather to a close loop \( C \) in projective Hilbert space \([7]\). For a (pseudo) spin-1/2, which is the system of interest for quantum computation, \( \beta \) is related to half of the solid angle enclosed by the Bloch vector \( \mathbf{b}(t) \) on the Bloch sphere. Recall that the Bloch vector is defined through the density matrix as

\[ \rho(t) = |\psi(t)\rangle \langle \psi(t)| = \frac{1}{2} \left( 1 + \mathbf{b}(t) \cdot \sigma \right). \] (5)

The first of the above mentioned issues with the adiabatic phase is already solved as the adiabaticity constrain has been relaxed by choosing an appropriate cyclic initial state (which depends on the particular evolution we are interested in).

The second drawbacks of the adiabatic phase is solved by choosing evolutions such that

\[ \langle \psi(t)|H(t)|\psi(t)\rangle = 0 \] (6)

at all time. The dynamic contribution is thus zero and only a geometric AA phase is acquired over \( C \). For (6) to be zero at all time, the axes of rotations must always be orthogonal to the state vector. The corresponding paths are then spherical polygons where each segment lays along a great circle on the Bloch sphere. It is a clear advantage of the AA-phase for computation that such path exist since there is then no need for cancellation of the dynamical phase using spin-echo techniques.

To address the third issue, we restrict our attention to Hamiltonians for which only two fields are non-zero

\[ H = -\frac{1}{2} B_x(t) \sigma_x - \frac{1}{2} B_z(t) \sigma_z. \] (7)

In implementations where the fields \( B_x \) and \( B_z \) cannot be non-zero simultaneously, only simple spherical polygons (more precisely, corresponding to geometrical phases which are integer multiple of \( \pi/2 \)) are possible. Hence, while it is possible to generate SU(2) in that situation and therefore reach any points on the Bloch sphere, it is not possible for the Bloch vector to follow any paths on this sphere.

However, if one can turn on and tune the coefficients of \( \sigma_x \) and \( \sigma_z \) simultaneously, the following evolution becomes possible

\[ R_x(\pi/2)R_n(\pi)R_x(\pi/2) \langle 0 | = -i e^{-i\theta} |0 \rangle, \] (8)

with \( n = (\sin \theta, 0, \cos \theta) \). Figure 1 is a plot of this path on the Bloch sphere. It is straightforward to verify that the dynamic phase is zero for this evolution and as a result the geometric AA phase is just \(- (\theta + \pi/2)\). By varying \( \theta \) it is then possible to obtain any geometric phases.

The rotation \( R_n(\alpha) \) can be realized, for example, with a symmetric DC-SQUID charge qubit. This is equivalent to what was suggested recently by Xiangbin and Keiji [8].

However, even if for this path the geometric phase is the only contribution, it is not observable. Indeed, one could conclude from the discussion on the Berry phase that by applying the above procedure on a superposition \( a|0 \rangle + b|1 \rangle \), each logical states will accumulate it’s corresponding geometric AA phase and that, as a result, there will be an observable AA phase difference at the end of the evolution. Indeed, for the adiabatic geometric phase, starting with a superposition will yield an observable phase difference because cyclicity is imposed on each members of the superposition by the cyclic and adiabatic variation of the external parameters.

In the non-adiabatic case, since the AA phase depends critically on the state (and not so much on the Hamiltonian) this is not the case: a superposition of cyclic initial states is not necessarily a cyclic initial state itself. This can be clearly seen from figure 2 where the evolution of \((|0 \rangle + |1 \rangle) / \sqrt{2}\) under the sequence of rotations (8) is shown. This evolution is clearly non-cyclic and is not on great circles. This points out to the fact that the AA phase is non-linear. This is a result of the indistinguishability of global phase factors [10,11].
FIG. 2. The sequence of rotations (8) applied on a superposition of states is shown from an angle showing the south pole.

That the AA phase is not observable on a single qubit is also clear from its definition. The AA phase was defined only for cyclic evolutions and observable properties are, again since global phase factors are not physical, unchanged for such evolutions. The AA phase is therefore not observable by itself for any evolutions of an isolated qubit. This is in sharp contrast to Xiangbin et al. claims [8].

To get a physically significant (and therefore computationally significant) AA phase, one can use a second qubit to ‘monitor’ the phase on the first one. The point here is that for the geometric phase to be observable, the evolution of the total system must not be cyclic (because under cyclic evolutions, by definition, the observable properties of a system are left unchanged), however, for the AA phase to be defined, part of the total system must undergo a cyclic evolution.

In the language of quantum computation, a possible procedure is to start with the two-qubit state $|00\rangle$ and apply a Hadamard transformation on the first one to obtain:

$$\frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) |0\rangle. \quad (9)$$

Then, apply the sequence (8) on the second qubit conditionally on the first qubit to be $|1\rangle$:

$$CR_x(\pi/2)CR_n(\pi)CR_x(\pi/2) \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|00\rangle - ie^{-i\theta}|10\rangle) \quad (10)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle - ie^{-i\theta}|1\rangle)|0\rangle.$$

The $CR_x(n)$'s are controlled rotations. The geometric phase can then be observed from the first qubit by interference. Of course, this qubit could initially be in an arbitrary superposition $a|0\rangle + b|1\rangle$ and the above procedure therefore corresponds to a non-adiabatic geometric phase gate.

Before concluding that AA phase gates are possible (and perhaps useful), we still need to give a prescription for the application of the above gate. Indeed, these controlled-rotations will most probably not, by themselves, be in the repertory of any quantum computer design and we need to implement them using sequences of elementary operations.

Following ref. [12], any controlled-operations on two qubits can be obtained from controlled-nots and single bit gates as shown in figure 3. For this network to have the desired effect, the single-bit gates $A$, $B$ and $C$ must be chosen such that:

$$ABC = 1, \quad (11)$$

$$A \sigma_x \sigma_y C = U. \quad (12)$$

Hence, if qubit one (top most in figure 3) is $|0\rangle$ then the second qubit is left unchanged. In the opposite case, $U$ is applied. In the case of interest here, the $U$s we need to implement are simple rotations and a simple form of the network of figure 3 will do. To implement $CR_n(\pi)$ we choose $A = R_x(\pi/2)$, $B = R_y(-\pi/2)$ and $C = 1$ and replace the CNOTs by $C\sigma_y$ gates

$$C\sigma_y = \begin{pmatrix} 1 & 1 \\ 0 & -i \\ i & 0 \end{pmatrix}. \quad (13)$$

This operation can be implemented by the following sequence of elementary operations:

$$C\sigma_y = e^{i\pi/4} R_x(3\pi/2) CP(3\pi/2) R_x(\pi/2) R_z(\pi/2) \quad (14)$$

This particular sequence is specific to quantum computer implementations having the conditional phase shift gate

$$CP(\gamma) = e^{-i\gamma \sigma_z \otimes \sigma_z/2}, \quad (15)$$

in their repertory but similar sequences can be found for other implementations. That the above gates do implement a controlled-$R_n(\pi/2)$ gate is straightforward to verify.

For $CR_x(\pi/2)$ a simpler sequence based on a variation from (14) is possible:

$$CR_x(\pi/2) = e^{-i\pi/4} CP(\pi/2) R_x(\pi/2) R_x(\pi/4) \quad (16)$$

Using the above sequence and (14), it is possible, by inspection, to 'compile' the total sequence $CR_x(\pi/2)CR_n(\pi)CR_x(\pi/2)$ from $2 * 7 + 12 = 26$ down to 22 elementary operations.
We now have a complete prescription for the implementation of (10) but it involves the application of 22 elementary gates, a number that is quite large for a gate whose purpose is to implement a noiseless phase-shift gate. However, this number should not be compared to the dynamical version of this gate (which of course requires only one elementary operation) but rather to its adiabatic counterpart. As shown by Falci et al. [5] for an asymmetric dc-SQUID charge qubit, a single purely adiabatic geometric evolution requires about 10 elementary operations (the loop in parameter space in this case has four segments, see figure 1 of reference [5], which can be interpreted as elementary operations. This loop must be repeated twice accompanied by two \( \pi \) pulses to refocus the dynamical phase). Therefore, in the present non-adiabatic case, the complexity of a single gate is about twice as that in the adiabatic case.

What is more important than the complexity of sequences here, is that the extra gates will bring the state vector out of great circles and bring in a dynamic phase. A direct calculation of the dynamic component shows that it is non-zero and as the value

\[
\delta = \pi \left( 1 + \cos \theta \right). \tag{17}
\]

The evolution is thus not purely geometric and we are back to the initial problem of removing a dynamical phase. Of course, it would be too hasty to conclude here that the AA phase is not useful for quantum computing as a more clever sequence of gates, one which is keeping the state vector on great circles or on which the dynamic contribution cancels itself, could be found.

Another important issue to address is tolerance to imperfections. Adiabatic geometric phases are interesting for computational purposes because they are tolerant to area preserving errors. Such tolerance should also be present in the non-adiabatic geometric gates if they are to be useful for computation. Let’s consider here an error \( \epsilon \) in the application of the first gate of the sequence (8) (we will not consider the extra gates (14) and (16) at the moment):

\[
R_x(\pi/2)R_n(\pi)R_x(\pi/2 + \epsilon). \tag{18}
\]

Of course, this is not an area preserving error and from our understanding of the adiabatic case, one should not expect the AA phase to be invariant in this circumstance. In any case, it is useful to consider this simple error here as it illustrates, as we will see, the sensitivity of the AA phase.

Applying the erroneous sequence (18) on the originally cyclic initial state \( |0\rangle \) one obtains

\[
\begin{align*}
-i \cos(\epsilon/2)(\cos \theta - i \sin \theta) \langle 0 | + i \sin(\epsilon/2)(\cos \theta + i \sin \theta) \langle 1 |
\end{align*}
\]

The evolution is not cyclic anymore and we cannot define the AA phase in this situation. Figure 4 shows this evolution on the Bloch sphere (open gray path). Note that to first order in \( \epsilon \), the non-cyclicity remains and this procedure is therefore not tolerant to small imperfections in the generating gates. The small errors can take the state vector out of great circles and bring in a dynamical contribution to the phase. In worst cases, as above, the evolution is no longer cyclic and the AA phase can no longer be defined.

We notice from (19) that, to first order, there is some tolerance on the phase of the \( |0\rangle \) state but there is a partial bit flip error. While a bit flip error can sometimes be easier to correct than a phase error, this procedure, when considering the extra gates (14) and (16), requires much more rotations than a simple dynamic \( R_z(\theta) \) operation and there are, therefore, more possible sources of errors.

With the fact that AA-phase gates appears difficult to implement, this intolerance to small imperfections reduces the interest of using geometric AA phase gates for quantum computation.

Helpful discussions with S. Lacelle and A.M. Zagoskin are acknowledged. This work was partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC), by D-Wave Systems inc., by the Fonds pour la Formation de Chercheurs et l’Aide la Recherche (FCAR) from the Québec government, the Canadian Institute for Advanced Research, the Canada Research Chair program and in part, at the Institute for Theoretical Physics, Santa Barbara, by the National Science Foundation under grand No. PHY94-07194.
