Abstract

We advance a method used to analyse the neutrino properties (masses and mixing) in the seesaw mechanism. Assuming the hierarchical Dirac and light neutrino masses we establish rather simple relations between the light and the heavy neutrino parameters in the favored region of the solar and the atmospheric neutrino experiments. A empirical condition satisfied by the RH mixing angles is obtained.

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1 Introduction

Whether the neutrinos have nonzero masses? How large are the mixing angles? Are they like that in the quark sector? Those are among the pressing questions in particle physics. The solar [1] and atmospheric [2] neutrino data suggest that neutrinos do have masses and the recent results from Super-Kamiokande (SK) [2] implies a nearly maximal mixing of $\nu_\mu$ and $\nu_\tau$. In another hand, the fact that neutrinoless double-$\beta$ decay and other lepton number nonconserving processes are not observed experimentally reflects the smallness of the neutrino masses [3]. The seesaw mechanism has a natural explanation for the small neutrino masses and may enhance lepton mixing up to maximal [4][5][6].

According to the seesaw mechanism the effective neutrino mass matrix is given by [4]

$$m_{\text{eff}} = m_D M^{-1} m_D^T$$

at $M \gg m_D$. Here $M$ is the Majorana mass matrix of the right handed (RH) neutrino components and $m_D$ is the neutrino Dirac mass matrix which could be equal to the mass matrix of the up quarks: $m_D = m_{\text{up}}$ according to some kind of quark-lepton symmetry [4][5][6]. In the basis where $M^{-1}$ is diagonal,

$$M^{-1} = M_i^{-1} \delta_{ij} \equiv R_i^2 \delta_{ij} \ (i, j = 1, 2, 3).$$

$m_D$ can be written as [7]

$$m_D = U_0 m_D^{\text{diag}} V_0.$$  

Here $U_0$ and $V_0$ are left-handed (LH) and right-handed (RH) rotations respectively and $m_D^{\text{diag}} = \text{diag} \{m_1, m_2, m_3\}$.

In this paper, we study a problem what we can know about the masses and mixing of the right handed neutrinos from the low energy neutrino data.
The paper is organized as follows. In Sec. II a parametrization is introduced and the seesaw mechanism is expressed in two formulas: one involves only the neutrino masses and the other the mass ratios as well as mixing angles. Then the approach to the RH neutrino masses and mixing is present. In Sec. III we get rather simple relations between the masses and mixing angles entering the seesaw formula in the favored regions of the solar and atmospheric experiments. The numerical results they infer are given after that. We summarize and discuss our main results in Sec. IV.

2 General framework

2.1 Parametrizing

Since the CP-violating effects in neutrino oscillations should be small [8], we shall therefore assume \( U_0 \) and \( V_0 \) are real. We also set \( U_0 \sim I \). That is, the left-handed rotations that diagonalize the charged lepton and neutrino Dirac mass matrices, \( m_l \) and \( m_D \), are the same or nearly the same and the large lepton mixing results from the seesaw transformation [4]. Under these assumptions, Eq.(1) is rewritten as:

\[
m_D^{\text{diag}} V_0 M^{-1} V_0^T m_D^{\text{diag}} = U_0^T U (N^{\text{diag}})^2 U^T U_0 \approx U (N^{\text{diag}})^2 U^T,
\]

or

\[
(m_D^{\text{diag}})^{-1} U (N^{\text{diag}})^2 U^T m_D^{\text{diag}} = V_0 M^{-1} V_0^T.
\]

Where \( U \) is LH rotation induced by \( M^{-\frac{1}{2}} \) and \( N^{\text{diag}} = \text{diag} \{n_1, n_2, n_3\} \) with \( n_i^2 = m_i^{\text{eff}} \) \((i = 1, 2, 3)\), the eigenvalues of \( m^{\text{eff}} \).

We introduce the following parametrization analogy with that in the two flavors case [9],

\[
\begin{align*}
\xi_3 &= \frac{1}{2} \ln \frac{m_2^2}{m_1^2}, & \xi_8 &= \frac{1}{6} \ln \frac{m_3^2}{m_1 m_2}; \\
\eta_3 &= \frac{1}{2} \ln \frac{R_1}{R_2}, & \eta_8 &= \frac{1}{6} \ln \frac{R_1 R_2}{R_3^2}.
\end{align*}
\]
\[ \kappa_3 = \frac{1}{2} \ln \frac{n_2}{m_1}, \quad \kappa_8 = \frac{1}{6} \ln \frac{n_3^2}{m_1 n_2} \]  

(8)

and [10]

\[ U = \exp(i \theta_{23} \lambda_7) \exp(i \theta_{13} \lambda_5) \exp(i \theta_{12} \lambda_2), \]  

(9)

\[ V_0 = \exp(i \beta_{23} \lambda_7) \exp(i \beta_{13} \lambda_5) \exp(i \beta_{12} \lambda_2). \]  

(10)

Here, \( \lambda_2, \lambda_5, \lambda_7 \) are Gell-mann matrix. One can see that \( \eta_3, \eta_8 \geq 0 \) and the significances of these parameters are apparent, for example, \( \eta_3 = 0 \) implies \( M_1 = M_2 \) and \( \eta_3 = 3 \eta_8 \) implies \( M_2 = M_3 \). Using the diagonal Gell-mann matrix \( \lambda_3 \) and \( \lambda_8 \), the mass matrices involving now can be rewritten as

\[ m_D^{diag} = (m_1 m_2 m_3)^{\frac{1}{2}} e^{-\xi_3 \lambda_3 - \sqrt{3} \xi_8 \lambda_8}, \]  

(11)

\[ M^{-1} = (R_1^2 R_2^2 R_3^2)^{\frac{1}{2}} e^{2 \eta_3 \lambda_3 + 2 \sqrt{3} \eta_8 \lambda_8}, \]  

(12)

\[ (\Lambda^{diag})^2 = (n_1^2 n_2^2 n_3^2)^{\frac{1}{2}} e^{-2 \kappa_3 \lambda_3 - 2 \sqrt{3} \kappa_8 \lambda_8}. \]  

(13)

This parametrization shows clearly that the relevant variables in the diagonalization of \( M^{-1} \) are \( \theta_{12}, \theta_{13}, \theta_{23}, \kappa_3, \kappa_8, \xi_3 \) and \( \xi_8 \). Of these, it is usually assumed that \( \xi_3 \) and \( \xi_8 \) can be identified with the corresponding quantities of the up sector of quarks and \( \theta_{12}, \theta_{13}, \theta_{23}, \kappa_3, \kappa_8 \) can be obtained from the low energy neutrino data. Now let us denote

\[ \overline{X}(\kappa, \xi, \theta) = (R_1^2 R_2^2 R_3^2)^{\frac{1}{2}} V_0 e^{2 \eta_3 \lambda_3 + 2 \sqrt{3} \eta_8 \lambda_8} V_0^T \]  

(14)

\[ = (m_1 m_2 m_3)^{-\frac{1}{2}} (n_1^2 n_2^2 n_3^2)^{\frac{1}{2}} X(\kappa, \xi, \theta). \]

Here

\[ X(\kappa, \xi, \theta) = e^{\xi_3 \lambda_3 + \sqrt{3} \xi_8 \lambda_8} U e^{-2 \kappa_3 \lambda_3 - 2 \sqrt{3} \kappa_8 \lambda_8} U^T e^{\xi_3 \lambda_3 + \sqrt{3} \xi_8 \lambda_8} \]  

(15)

where \( \kappa, \xi \) and \( \theta \) refer to \( \kappa_3, \kappa_8 \); \( \xi_3, \xi_8 \) and \( \theta_{12}, \theta_{13}, \theta_{23} \) respectively that will not introduce confusion. Eq.(14) is equivalent with the following two equations:

\[ R_1^2 R_2^2 R_3^2 = (m_1 m_2 m_3)^{-2} (n_1^2 n_2^2 n_3^2), \]  

(16)
\[ X(\kappa, \xi, \theta) = V_0 e^{2\eta_3 \lambda_3 + 2\sqrt{3} \eta_8 \lambda_8} V_0^T . \] (17)

Eq.(16) is just the equality of the determinations of both sides of Eq.(14). Taking a total term \((R_1^2 R_2^2 R_3^2)^+ = (m_1 m_2 m_3)^- \) out we get Eq.(17). For late use, we present here the expression of the inverse of \(X(\kappa, \xi, \theta)\). It is easy to know from Eq.(15) that
\[ X^{-1}(\kappa, \xi, \theta) = e^{\xi_3 \lambda_3 - \sqrt{3} \xi_8 \lambda_8} U e^{2\kappa_3 \lambda_3 + 2\sqrt{3} \kappa_8 \lambda_8} U^T e^{-\xi_3 \lambda_3 - \sqrt{3} \xi_8 \lambda_8} . \] (18)

So
\[ X^{-1}(\kappa, \xi, \theta) = X(-\kappa, -\xi, \theta) \equiv Y(\kappa, \xi, \theta) . \] (19)

and we have
\[ Y(\kappa, \xi, \theta) = V_0 e^{-2\eta_3 \lambda_3 - 2\sqrt{3} \eta_8 \lambda_8} V_0^T . \] (20)

We will start from Eqs.(17)(20) to derive the expressions of \(\eta_3, \eta_8\) and \(V_0\). Then from Eq.(16) \(M_i (i = 1, 2, 3)\)can be obtained.

In following discussion, we shall omit the variables \(\kappa, \xi, \theta\) in \(X\) and \(Y\) if no more confusion would be introduced.

### 2.2 Determination of the Majorana masses

In this subsection we deduce two equations about \(\eta_3\) and \(\eta_8\). Taking the trace of both sides of Eq.(17) we obtain
\[ \text{tr} \left( V_0 e^{2\eta_3 \lambda_3 + 2\sqrt{3} \eta_8 \lambda_8} V_0^T \right) = \text{tr} e^{2\eta_3 \lambda_3 + 2\sqrt{3} \eta_8 \lambda_8} = \text{tr} X, \] (21)

that is,
\[ A \equiv e^{2\eta_3 + 2\eta_8} + e^{-2\eta_3 + 2\eta_8} + e^{-4\eta_8} = X_{11} + X_{22} + X_{33} . \] (22)

Similarly, taking the trace of both sides of Eq.(20) we get,
\[ B \equiv e^{-2\eta_3 - 2\eta_8} + e^{2\eta_3 - 2\eta_8} + e^{4\eta_8} = Y_{11} + Y_{22} + Y_{33} . \] (23)
It is sufficient for solving $\eta_3$ and $\eta_8$ from Eqs.(22)(23) since $X_{ii}$ and $Y_{ii}$ ($i = 1, 2, 3$) are known. Once $\eta_3$ and $\eta_8$ are solved, inserting $M_1 = M_3 e^{-2\eta_3 - 6\eta_8}$, $M_2 = M_3 e^{2\eta_3 - 6\eta_8}$, $n_1^2 = n_3^2 e^{-2\kappa_3 - 6\kappa_8}$, and $n_2^2 = n_3^2 e^{2\kappa_3 - 6\kappa_8}$ in Eq.(16) and identifying $m_i$ ($i = 1, 2, 3$) with the masses of up sector of quarks, we obtain the following expressions of the RH neutrino masses,

$$M_1 = Fe^{-2\eta_8 - 2\eta_3}, \quad M_2 = Fe^{-2\eta_8 + 2\eta_3}, \quad M_3 = Fe^{4\eta_8}. \quad (24)$$

Here $F = \frac{m^2}{m_3} e^{4\kappa_8 - 4\xi}$. All the above results are exact but formal. We need to decouple $\eta_3$ and $\eta_8$ in Eqs.(22)(23). From Eq.(22), we have

$$A = e^{2\eta_3 + 2\eta_8} + e^{-2\eta_3 + 2\eta_8} + e^{-4\eta_8} \geq 3 \left( e^{2\eta_3 + 2\eta_8} e^{-2\eta_3 + 2\eta_8} e^{-4\eta_8} \right)^\frac{1}{3} = 3. \quad (25)$$

The equality is satisfied when $\eta_3 = \eta_8 = 0$, which indicates the three degenerate Majorana masses, $M_1 = M_2 = M_3$. We will consider another more interesting case where $A \gg 3$ (then $B \gg 3$ is also true) corresponding to at most two degenerate Majorana masses. And if $A, B \gg 3$, Eq.(22) and Eq.(23) can be approximated as follows,

$$e^{2\eta_3 + 2\eta_8} + e^{-2\eta_3 + 2\eta_8} \approx A, \quad (26)$$

$$e^{4\eta_8} \approx B. \quad (27)$$

($I$) $A > B$

It is easy to know from Eqs.(26)(27) that $A > B$ implies $\eta_3 > \eta_8$. So $e^{-2\eta_3 + 2\eta_8} (< 1)$ may be omitted in Eq.(26). Eqs.(26)(27) are simplified as follows:

$$e^{2\eta_3 + 2\eta_8} = e^{2\eta_3 - 2\eta_8} e^{4\eta_8} \approx A, \quad (28)$$

$$e^{2\eta_3 - 2\eta_8} + e^{4\eta_8} \approx B. \quad (29)$$
$e^{2\eta_3 - 2\eta_8}$ and $e^{4\eta_8}$ are two roots of a quadratic equation:

$$x^2 - Bx + A = 0.$$  \hfill (30)

The solution to it is direct,

$$e^{2\eta_3 + 2\eta_8} \approx A,$$  \hfill (31)

$$e^{-2\eta_3 + 2\eta_8} \approx \frac{2}{B - \sqrt{B^2 - 4A}},$$  \hfill (32)

$$e^{-4\eta_8} \approx \frac{2}{B + \sqrt{B^2 - 4A}}.$$  \hfill (33)

(II) $A < B$

In this case, we have $\eta_3 < \eta_8$. Omitting the term $e^{2\eta_3 - 2\eta_8}$ in Eq.(27), we have

$$e^{2\eta_3 + 2\eta_8} + e^{-2\eta_3 + 2\eta_8} \approx A,$$  \hfill (34)

$$e^{4\eta_8} = e^{2\eta_3 + 2\eta_8} e^{-2\eta_3 + 2\eta_8} \approx B.$$  \hfill (35)

Now $e^{2\eta_3 + 2\eta_8}$ and $e^{-2\eta_3 + 2\eta_8}$ are two roots of the following quadratic equation:

$$x^2 - Ax + B = 0.$$  \hfill (36)

and the solution is

$$e^{2\eta_3 + 2\eta_8} \approx \frac{A + \sqrt{A^2 - 4B}}{2},$$  \hfill (37)

$$e^{-2\eta_3 + 2\eta_8} \approx \frac{A - \sqrt{A^2 - 4B}}{2},$$  \hfill (38)

$$e^{-4\eta_8} \approx \frac{1}{B}.$$  \hfill (39)

In (I) we obtain $e^{-2\eta_3 + 2\eta_8} \sim e^{-4\eta_8}$ (or $M_2 \sim M_3$) when $B^2 \sim 4A$ and in (II) $e^{2\eta_3 + 2\eta_8} \sim e^{-2\eta_3 + 2\eta_8}$ (or $M_2 \sim M_3$) when $A^2 \sim 4B$. However, far beyond these regions, both (I) and (II) give the same asymptotic solution: $e^{2\eta_3 + 2\eta_8} \approx A$,
and $e^{-4\eta_8} \approx \frac{1}{B}$. Then $e^{-2\eta_3 + 2\eta_8} = e^{-2\eta_3 - 2\eta_8} e^{4\eta_8} \approx \frac{B}{A}$. The solutions are also useful for rough estimation of the Majorana masses even when two of them are degenerate. This can be seen from $e^{2\eta_3 + 2\eta_8} < e^{2\eta_3 + 2\eta_8} + e^{-2\eta_3 + 2\eta_8} < 2 e^{2\eta_3 + 2\eta_8}$ and $e^{4\eta_8} < e^{2\eta_3 - 2\eta_8} + e^{4\eta_8} < 2 e^{4\eta_8}$. The maximal deviations for $e^{2\eta_3 + 2\eta_8}$ and $e^{4\eta_8}$ are both 2 times.

Usually one should have to solve a cubic characteristic equation to obtain the eigenvalues. In seesaw model, however, one usually encounters such case where $e^{2\eta_3 + 2\eta_8} \gg 1$ and $e^{-4\eta_8} \ll 1$ simultaneously. This is a practical difficulty in calculation. More worse, the solution of a cubic equation is too ugly to see any relation between various physical quantities. By taking the trace of $X$ and its inverse, we decompose the eigen-equation in two equations and each contains the main term of $e^{2\eta_3 + 2\eta_8}$ and $e^{4\eta_8}$ respectively. In concrete calculation, the expressions of $A$ and $B$ can be simplified to such a great extent that the dependence on the parameters can be obtained explicitly and we will discuss it later.

### 2.3 Determination of the RH angles

Once one have the three eigenvalues solved, then the three eigenvectors (and then the three rotation angles) of $M^{-1}$, can be found by the standard procedure of the linear algebra. However, we find that it can be simplified if expressing the inverse of $X$ with its adjoint matrix. All the elements of $V_0$ can be expressed in a more symmetric form containing only the linear combination of the elements of $X$ and $Y$, not involving the quadratic terms of $X_{ij}$ any longer. We will use $Q_i$ ($i = 1, 2, 3$) ($Q_1 > Q_2 > Q_3$) to denote the three eigenvalues of $X$ in the deduction of the RH angles.

The eigen-equation of $X$ is

$$
(X - Q_iI) \begin{pmatrix} V_{i1} \\ V_{i2} \\ V_{i3} \end{pmatrix} = 0 \quad (i = 1, 2, 3),
$$

(40)
where \( V_{ij} = (V_0)_{ij} \). From which one can easily deduce

\[
V_{21} = \frac{(X_{12}X_{33} - X_{13}X_{23}) - Q_1X_{12}}{(X_{23}^2 - X_{33}X_{22}) + (X_{33} + X_{22})Q_1 - Q_1^2}V_{11}, \quad (41)
\]

\[
V_{31} = \frac{(X_{13}X_{22} - X_{12}X_{23}) - Q_1X_{13}}{(X_{23}^2 - X_{33}X_{22}) + (X_{22} + X_{33})Q_1 - Q_1^2}V_{11} \quad (42)
\]

and etc. We know that

\[
X^{-1} = \frac{1}{\det X} \text{adjoint} X. \quad (43)
\]

Notice \( \det X = 1 \), the inverse of \( X \) is just its adjoint matrix,

\[
Y_{11} = X_{22}X_{33} - X_{23}^2, \quad Y_{22} = X_{11}X_{33} - X_{13}^2, \quad Y_{33} = X_{11}X_{22} - X_{12}^2, \quad (44)
\]

\[
Y_{12} = X_{13}X_{23} - X_{12}X_{33}, \quad Y_{13} = X_{12}X_{23} - X_{13}X_{22}, \quad Y_{23} = X_{12}X_{13} - X_{11}X_{23}. \quad (45)
\]

and \( Y_{ij} = Y_{ji} \). The quadratic terms in Eq.(37) are just the elements of \( Y \).

Substituting the expressions of \( Y_{ij} \) \((i, j = 1, 2, 3)\) in Eq.(37) we have

\[
V_{21} = \frac{Y_{12} + Q_1X_{12}}{(Y_{11} + Q_1X_{11}) - (Q_2^{-1} + Q_3^{-1})}V_{11}, \quad (46)
\]

\[
V_{31} = \frac{Y_{13} + Q_1X_{13}}{(Y_{11} + Q_1X_{11}) - (Q_2^{-1} + Q_3^{-1})}V_{11} \quad (47)
\]

Here we have used \( \text{tr}X = X_{11} + X_{22} + X_{33} = Q_1 + Q_2 + Q_3 \) and \( Q_1Q_2Q_3 = 1 \).

We can express all the non-diagonal elements of \( V_0 \) in a unified form:

\[
V_{ij} = \frac{Y_{ij} + Q_jX_{ij}}{(Y_{jj} + Q_jX_{jj}) - Q_j^{-1}}V_{jj} \quad (i, j = 1, 2, 3 \quad \text{and} \quad i \neq j). \quad (48)
\]

Here \( \hat{Q}_j^{-1} = \text{tr}Y - Q_j^{-1} \). Considering the normalization condition (or unitarity of \( V_0 \)) \( V_0V_0^T = V_0^TV_0 = I \), all the elements can be gotten from Eq.(40). Then the three RH angles can be expressed as \( \tan \beta_{23} = \frac{V_{23}}{V_{33}}, \cos \beta_{13} \sin \beta_{12} = V_{12}, \) and \( \sin \beta_{13} = V_{13} \).
In the above approach, we do not need to assume the RH mixing angles to be small as in Ref. [7]. All the relations obtained, including the masses and the angles, can be easily transformed to express the light neutrino parameters in $M^{-1}$, $m_D$ and $V_0$. The approach is just to make the following exchange $\kappa \leftrightarrow -\eta$, $\xi \leftrightarrow -\xi$ and $\theta_{ij} \leftrightarrow \beta_{ij}$ ($1 \leq i < j \leq 3$).

We know that the expressions of $A$ and $B$ are really complicated. However, up to now we have not given any simplification for them. We will discuss it in Sec. III.

3 Analysis and result

The deficit of muon neutrino observed by Super-Kamiokande Collaboration and the zenith angle distributions of the data can be explained by oscillation between $\nu_\mu$ and $\nu_\tau$ with the best fit parameters at [2]

$$\left(\sin^2 2\theta_{23}, \Delta m^2_{atm}\right) = (0.95, 5.9 \times 10^{-3} \text{eV}^2).$$

The $\nu_e - \nu_\mu$ explanation to the solar neutrino problem requires one of the following sets of parameters (the best fit values) [11]:

<table>
<thead>
<tr>
<th></th>
<th>$\Delta m^2_{solar}[\text{eV}^2]$</th>
<th>$\sin^2 2\theta_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>VO</td>
<td>$6.5 \times 10^{-11}$</td>
<td>0.75</td>
</tr>
<tr>
<td>MSW(LMA)</td>
<td>$1.8 \times 10^{-5}$</td>
<td>0.76</td>
</tr>
<tr>
<td>MSW(LOW)</td>
<td>$7.9 \times 10^{-8}$</td>
<td>0.96</td>
</tr>
<tr>
<td>MSW(SMA)</td>
<td>$5.4 \times 10^{-6}$</td>
<td>$6.0 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

(50)

Here MSW and VO refer to Mikheyev-Smirnov-Wolfenstein matter-enhanced oscillations [12] and vacuum oscillations (so-called just-so oscillation) respectively. LMA (SMA) stands for a large (small) mixing angle and LOW refers to low probability (or low mass). We assume the effective neutrino masses have hierarchical property, that is, $m^{eff}_1 \ll m^{eff}_2 \ll m^{eff}_3$. So $n_3 = m^{eff}_3 \approx \sqrt{\Delta m^2_{atm}}$ and $n_2^2 = m^{eff}_2 \approx \sqrt{\Delta m^2_{solar}}$. We have little knowledge about the value of $m^{eff}_1$ for which we use the parameter $r = \frac{m^{eff}_1}{m^{eff}_3} \gg 1$. There are only upper limits on the
value of the remaining mixing angle $\theta_{13}$ [13]:

$$|\sin \theta_{13}|^2 \equiv |U_{e3}|^2 \leq 0.015 - 0.05. \quad (51)$$

For simplicity, we set $\theta_{13} = 0$. The Dirac masses of neutrino are taken at the scale $\mu = 10^9 GeV$ [14]:

$$m^{diag}_D (\mu) = diag \{ m_u (\mu), m_c (\mu), m_t (\mu) \} = diag \{ 1.47 MeV, 427 MeV, 149 GeV \}. \quad (52)$$

These are the whole values entering $A$ and $B$.

Instead of calculating the Majorana mass matrix by just inserting the values of these parameters in, we give a more general analysis in two cases according to whether $\theta_{12}$ is large (VO, LMA and LOW) or small (SMA) and derive the corresponding relations between the masses and mixing of the RH neutrino and the other neutrino parameters.

In this section we start from Eqs.(22)(23) to get the two mass ratios, $\eta_3$ and $\eta_8$. Then using Eq.(41), the elements (and then the mixing angles) of the RH mixing matrix would be obtained. The Majorana masses can be obtained from Eq.(24).

Although we have decoupled the Majorana masses and the RH mixing, the expressions of these parameters would be so complicated due to the complicated structure of $X$ that it is not easy to see explicitly the relations of different parameters. The hierarchical properties of the Dirac and effective masses of neutrinos make it possible to drop the smaller terms in $A$ and $B$. In the following, only the leading order terms of $X_{ij}$ ($Y_{ij}$) and $A$ ($B$) will be reserved respectively.
### 3.1 Case I: large $\theta_{12}$

#### 3.1.1 mass

In this case, all the elements of $U$ have the same order except that $U_{e3} = 0$.

Reserving the leading order terms in $A$ and $B$, we find

\begin{align*}
A & \approx U_{e2}^2 \exp (2\xi_3 + 2\xi_8 + 2\kappa_3 - 2\kappa_8) + U_{\mu 3}^2 \exp (4\kappa_8 - 2\xi_3 + 2\xi_8), \quad (53) \\
B & \approx U_{\tau 1}^2 \exp (2\kappa_3 + 2\kappa_8 + 4\xi_8). \quad (54)
\end{align*}

It is easy to see that both $A$ and $B$ are far larger than 3. Noticing that in the regions concerned we also have $A < B$. Then from Eq.(33) one has

\begin{align*}
Q_1 &= e^{2\eta_3 + 2\eta_8} \approx U_{e2}^2 \exp (2\kappa_3 - 2\kappa_8 + 2\xi_3 + 2\xi_8), \quad (55) \\
Q_2 &= e^{-2\eta_3 + 2\eta_8} \approx U_{\mu 3}^2 \exp (4\kappa_8 - 2\xi_3 + 2\xi_8), \quad (56) \\
Q_3 &= e^{-4\eta_8} \approx \frac{1}{U_{\tau 1}^2} \exp (-2\kappa_3 - 2\kappa_8 - 4\xi_8). \quad (57)
\end{align*}

Here we have used the relation $U_{e2}^2 U_{\mu 3}^2 = U_{\tau 1}^2$ which is satisfied when $\theta_{13} = 0$.

We would point out that our results would be right as long as $\theta_{13}$ is small enough.

Substituting the eigenvalues in Eq.(24), we have

\begin{align*}
M_1 & \approx \frac{1}{\sin^2 \theta_{12}} m_u^2, \quad M_2 \approx \frac{1}{\sin^2 \theta_{23}} \frac{m_\mu^2}{m_1^{\text{eff}}}, \quad M_3 \approx \sin^2 \theta_{23} \sin^2 \theta_{12} \frac{m_\tau^2}{m_1^{\text{eff}}}. \quad (58)
\end{align*}

The formula are the same as given in Ref. [7] $M_1$ and $M_2$ scale as $1/m_2^{\text{eff}}$ and $1/m_3^{\text{eff}}$ respectively while $M_3$ scales as $1/m_1^{\text{eff}}$, which gives scales for the two lighter masses, $M_1$ and $M_2$, lower and the heaviest one, $M_3$, higher than one would expect when no mixing occurs.
3.1.2 angles

Reserving the leading order terms of the numerators and denominators in Eq.(41) respectively, we obtain

\[ V_{21} \approx \frac{U_{\mu 2}}{U_{\tau 2}} e^{-2\xi_3} V_{11}, \quad V_{31} \approx \frac{U_{\tau 2}}{U_{\tau 2}} e^{-\xi_3-3\xi_3} V_{11}, \] (59)

\[ V_{12} \approx -\frac{U_{\mu 2}}{U_{\tau 2}} e^{-2\xi_3} V_{22}, \quad V_{32} \approx -\frac{U_{\mu 1}}{U_{\tau 1}} e^{\xi_3-3\xi_3} V_{22}, \] (60)

\[ V_{13} \approx \frac{U_{\tau 1}}{U_{\tau 1}} e^{-\xi_3-3\xi_3} V_{33}, \quad V_{23} \approx \frac{U_{\mu 1}}{U_{\tau 1}} e^{\xi_3} V_{33}. \] (61)

Exploiting the unitarity of \( V_0 \), it is appropriate to set \( V_{ii} \approx 1 \). Then the three RH angles are

\[ \beta_{12} \approx V_{12} \approx -\frac{m_u}{m_c} \cos \theta_{23} \cot \theta_{12}, \] (62)

\[ \beta_{13} \approx V_{13} \approx \frac{m_u \cot \theta_{12}}{m_t \sin \theta_{23}}, \] (63)

\[ \beta_{23} \approx V_{23} \approx -\frac{m_c}{m_t} \cot \theta_{23}. \] (64)

All of the RH angles are small and independent of the effective neutrino masses. Note that, not like the LH quark mixing where \( \tan \theta \approx \sqrt{\frac{m_u}{m_d}} \) in two-generation case [15], the RH mixing angles scale linearly with the ratios of the Dirac neutrino masses.

3.1.3 VO

Inserting the parameters in Eq.(47), we have

\[ M_1 \approx 8.0 \times 10^8 GeV, \quad M_2 \approx 4.6 \times 10^9 GeV, \quad M_3/r \approx 1.5 \times 10^{17} GeV. \] (65)

The mixing angles are easy to obtain from Eq.(49),

\[ \beta_{12} \approx -4.6 \times 10^{-3}, \quad \beta_{13} \approx 3.2 \times 10^{-5}, \quad \beta_{23} \approx -4.3 \times 10^{-3}. \] (66)
3.1.4 LMA

In this case we have nearly the same RH angles as in VO and we find

\[ M_1 \approx 1.5 \times 10^6 \text{GeV}, \quad M_2 \approx 4.6 \times 10^6 \text{GeV}, \quad M_3/r \approx 2.8 \times 10^{14} \text{GeV}. \]  \hspace{1cm} (67)

3.1.5 LOW

The results for this case are as follows:

\[ M_1 \approx 1.5 \times 10^7 \text{GeV}, \quad M_2 \approx 4.6 \times 10^9 \text{GeV}, \quad M_3/r \approx 6.6 \times 10^{15} \text{GeV}; \]  \hspace{1cm} (68)

\[ \beta_{12} \approx -3.3 \times 10^{-3}, \quad \beta_{13} \approx 2.3 \times 10^{-5}, \quad \beta_{23} \approx -4.3 \times 10^{-3}. \]  \hspace{1cm} (69)

3.2 Case II: small \( \theta_{12} \) (SMA)

In this case we have

\[ A \approx U_{e1}^2 \exp(-2\kappa_3 - 2\kappa_8 + 2\xi_3 + 2\xi_8) + U_{e2}^2 \exp(2\kappa_3 - 2\kappa_8 + 2\xi_3 + 2\xi_8) \approx X_{10}, \]  \hspace{1cm} (70)

\[ B \approx U_{\tau2}^2 \exp(-2\kappa_3 + 2\kappa_8 + 4\xi_8) + U_{\tau1}^2 \exp(2\kappa_3 + 2\kappa_8 + 4\xi_8) \approx Y_{33}. \]  \hspace{1cm} (71)

Again, they satisfy \( B > A \gg 3 \) and \( A^2 \gg 4B \). So

\[ Q_1 \approx A \approx f U_{e2}^2 \exp(2\kappa_3 - 2\kappa_8 + 2\xi_3 + 2\xi_8), \]  \hspace{1cm} (72)

\[ Q_2 \approx \frac{B}{A} \approx U_{\mu3}^2 \exp(e \kappa_3 + 4\kappa_8 - 2\xi_3 + 2\xi_8), \]  \hspace{1cm} (73)

\[ Q_3 \approx \frac{1}{B} \approx f U_{\tau1}^2 \exp(-2\kappa_3 - 2\kappa_8 - 4\xi_8). \]  \hspace{1cm} (74)

Here \( f = \frac{r}{r + \cot^2 \theta_{12}} \) and it cannot be omitted since \( \cot \theta_{12} \gg 1 \). Similar with case I, we have

\[ M_1 \approx f \frac{1}{\sin^2 \theta_{12}} \frac{m_2^2}{m_{eF}^2}, \quad M_2 \approx f^{-1} \frac{1}{\sin^2 \theta_{23}} \frac{m_\tau^2}{m_{eF}^2}, \quad M_3 \approx f^{-1} \sin^2 \theta_{23} \sin^2 \theta_{12} \frac{m_\mu^2}{m_1^2 \sin^2 \theta_{12}}. \]  \hspace{1cm} (75)
For the mixing angles, we have

\[ \beta_{12} \approx V_{12} \approx -f \frac{m_u}{m_c} \cos \theta_{23} \cot \theta_{12}, \]  
(76)

\[ \beta_{13} \approx V_{13} \approx f \frac{m_u}{m_t} \cot \theta_{12} \]  
(77)

\[ \beta_{23} \approx V_{23} \approx -\frac{m_c}{m_t} \cot \theta_{23}. \]  
(78)

Again the factor \( f \) appears. Note that the expressions of \( M_2 \) and \( \beta_{23} \) are the same as that when \( \theta_{12} \) is large. Moreover, the SK data suggests strongly that \( \theta_{23} \approx \frac{\pi}{4} \). So both \( M_2 \) and \( \beta_{23} \) have the same values in all the favored regions considered. When \( r \gg \cot^2 \theta_{12} \) (then \( f \approx 1 \)), we have the same expressions of the RH masses and the mixing angles no matter whether \( \theta_{12} \) is large or not.

Substituting the values of the parameters in, from Eq.(56) we have

\[ M_1 \approx 4.7 \times 10^8 f \text{GeV}, \quad M_2 \approx 4.6 \times 10^9 \text{GeV}, \quad M_3 \approx 3.0 \times 10^{12} \frac{f}{f} \text{GeV}. \]  
(79)

and from Eq.(57),

\[ \beta_{12} \approx -7.0 \times 10^{-2} f, \quad \beta_{13} \approx 4.9 \times 10^{-4} f, \quad \beta_{23} \approx -4.3 \times 10^{-3}. \]  
(80)

Here, with the value of \( \theta_{12} \) substituted in, \( f \approx \frac{f}{r + 6.6 \times 10^{-2}} \).

Comparisons with the exact numerical results are given in Tables I-III and from which we can see they fit well. In calculation we take \( m^\text{diag}_D (\mu) \) at \( \mu = 10^9 \text{GeV} \). Note that the ratios of its diagonal elements are almost fixed when \( \mu \) varies. They satisfy the approximate relation

\[ \frac{m_u (\mu) m_t (\mu)}{m_c^2 (\mu)} \approx 1. \]  
(81)

So the deviation is mainly resulted from \( F (= \frac{m_s^2}{m_t^2} e^{4\kappa_s - 4\xi_s}) \) when \( m^\text{diag}_D (\mu) \) at different scale is taken.
4 Summary and Discussion

We introduce a parametrization which transforms all the involving masses in the seesaw formula to the mass ratios. Then by taking the traces of $X$ and its inverse, we derive the equations of the Majorana mass ratios, $\eta_3$ and $\eta_8$. The solutions to these equations are obtained under some conditions. The elements of $V_0$ are expressed in a unified form. Finally, assuming the hierarchical Dirac and the effective neutrino masses, we have deduced the rather simple relations among the various neutrino parameters entering the seesaw formula in the favored regions and the numerical results are also given.

Now let us give a combined analysis of the results obtained and list our main points as follows:

- $M_2 (\approx 4.6 \times 10^9 \text{GeV})$ and the product of $M_1$ and $M_3$ are nearly independent of $\theta_{12}$.

- The three RH neutrino masses are hierarchical and $\frac{M_3}{M_2} \left( \propto \frac{m_{\text{eff}}}{m_1} \right) \gg \frac{M_2}{M_1} \left( \propto \frac{m_{\text{eff}}}{m_2} \right)$.

- $\beta_{23} (\approx -4.3 \times 10^{-3})$ and $\beta_{12}/\beta_{13} \approx \frac{1}{2} \frac{m_t}{m_c} \sin 2\theta_{23} \approx -\frac{1}{2} \frac{m_t}{m_c}$ are also independent of $\theta_{12}$. Moreover, the RH mixing angles satisfy the following condition

$$\frac{\beta_{12} \beta_{23}}{\beta_{13}} \approx \cos^2 \theta_{23} \approx \frac{1}{2}$$

which is independent of not only $\theta_{12}$ and the effective neutrino masses but also the Dirac masses of neutrinos. Considering the smallness of $\theta_{13}$, which together with the condition $\Delta m^2_{\text{solar}} \ll \Delta m^2_{\text{atm}}$ makes the solar and atmospheric neutrino oscillations approximately decoupled [16], it is interesting to notice that the (13) elements ($U_{e3}$, $V_{13}$ and $U_{u3}$) determined by the third mixing angles of the three corresponding mixing matrices are all small. It is also noteworthy that the third mixing angles in both the

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CKM matrix the RH mixing matrix are of orders of the products of the other two angles respectively. In the former, we have $\left| \frac{U_{us}}{U_{cb}} \right| \approx 2$.

- Numerically, the lightest right-handed neutrino mass can lie from $10^6 GeV$ to $10^8 GeV$ while the heaviest right-handed neutrino mass range from about $10^{12} GeV$ to far larger than $10^{17} GeV$. $M_3$ lies at around the unification scale of SUSY, $M_X \approx 2 \times 10^{16} GeV$ in LMA and LOW. Note that the VO solution $M_3/r \sim 10^{17} GeV$ is close to the reduced Planck scale [17] and new physics might affect our results at this scale.

- Numerically, all the three RH angles are small although they may contain the contribution from the diagonalization of $M^{-1}$. The absolute values of $\beta_{12}$ and $\beta_{13}$ are about $10^{-3} \sim 10^{-2}$ and $10^{-6} \sim 10^{-4}$ respectively.

- SMA solution seems natural in the sense that $M_3 \sim 10^{15} GeV$ for a wide range of $r$ while $M_3$’s for the other three regions (VO, LMA and LOW) increase rapidly with $r$ and become too large to be viable. Both the two mass squared differences splittings (of the order $10^{-3} eV^2$ and $10^{-11} eV^2$ respectively) and the scale of the heaviest RH neutrino mass $M_3 (\gg 10^{17} GeV)$ make the VO solution to the solar neutrino problem look very unnatural.

In this work, we set $\theta_{13} = 0$. However $\theta_{13}$ may become important in the seesaw mechanism especially in the SMA region if it is not exactly equal to zero since in this case $\theta_{13}$ is comparable with $\theta_{12}$. It may lead to large RH mixing angles owing to the contribution from the diagonalization of $M^{-1}$ as well as degenerate masses. This can also be seen from that the coefficient of $U_{e3}$ in $A$ are much larger than that of the other elements of $U$. We point out that the method is even valid in such case while more skills are needed. We will discuss it in more details in later paper.

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