The RG flow for the sine-Gordon model is determined by means of the method of Wegner and Houghton in next-to-leading order of the derivative expansion. For small values of the fugacity this agrees with the well-known RG flow of the two-dimensional Coulomb-gas found in the dilute gas approximation and a systematic way of obtaining higher-order corrections to this approximation is given.

I. INTRODUCTION

The well-known phase structure of the two-dimensional Coulomb-gas was investigated in several papers by using the differential RG approach with a smooth cut-off in the momentum space (or equivalently using a sharp cut-off in the coordinate space) (Kosterlitz 1973, Kosterlitz 1974, Jose 1977, Huang 1991, Gersdorff 2000). Our goal is in this work to determine the RG flow for the two-dimensional Coulomb-gas in the framework of the RG approach of Wegner and Houghton (Wegner 1973) taking the field-independent wave-function renormalization into account. Results of two different methods were compared: renormalization by means of the blocking construction in the coordinate space using the dilute gas approximation (real space RG) (Kosterlitz 1973, Kosterlitz 1974, Jose 1977, Huang 1991), and the renormalization of the equivalent sine-Gordon scalar field model, performing the blocking transformations in the momentum space in Wegner-Houghton’s framework with a sharp cut-off and usage of the derivative expansion (Wegner-Houghton RG method) (Wegner 1973, Hasenfratz 1986, Polonyi 2000).

It is believed that several different models, like the sine-Gordon, Thirring, and the X-Y planar models belong to the same universality class, namely to that of the two-dimensional Coulomb-gas. The X-Y model with external field, which is a classical two-component spin-model described by the action,

\[ S = \frac{1}{T} \sum_{x,x'} \cos(\theta_x - \theta_{x'}) + \frac{h}{T} \sum_x \cos(\theta_x) \tag{1} \]

has topological excitations, called vortices, which interact via Coulomb interaction. Therefore, the X-Y model can be mapped by means of the Villain-transformation to a Coulomb-gas (Huang 1991). Such a mapping is, however, only valid up to irrelevant interaction terms. The other example, the sine-Gordon model in dimension \( d = 2 \) is a one-component scalar field theory with periodic self-interaction, which is defined by the Euclidean action:

\[ S = \int d^2x \left[ \frac{1}{2} (\partial \phi)^2 + u \cos(\beta \phi) \right]. \tag{2} \]

The equivalence between the X-Y model and the lattice regulated compactified sine-Gordon model is shown by expressing (2) in terms of the compact variable \( z(x) = e^{i\beta \phi(x)} \) (Huang 1991). This makes the kinetic energy periodic and introduces vortices in the dynamics.

There having been made several efforts during the last two decades to improve the Coulomb-gas results obtained in the dilute-gas approximation, where the analogy with the electrodynamics of polarizable media has been exploited (Amit 1980, Minnhagen 1985, Kupferman 1997). Here we also show that the method of Wegner and Houghton applied to the sine-Gordon model provides a systematic way of obtaining higher-order corrections to the dilute-gas results.

II. REAL SPACE RG

The real space RG approach for the Coulomb-gas in dimension \( d = 2 \) has been investigated in a great detail in the literature (Kosterlitz 1973, Kosterlitz 1974, Jose 1977, Huang 1991). The dilute vortex gas approximation assumes
that all the vortex pairs (charges) are separated with at least to the minimum distance \(a\) in the coordinate space which is increased with infinitesimal steps during the blocking transformation. The contributions of the order higher than \(\delta \sigma /a\) are neglected. The real space RG equations for the dilute vortex-gas are well-known and their derivation is given in the literature (Kosterlitz 1973, Kosterlitz 1974, Jose 1977, Huang 1991):

\[
an \frac{dh}{da} = (2 - \frac{T}{4\pi})\hat{h}, \quad a \frac{dT}{da} = -\pi T^2 \tilde{h}^2
\]

with the dimensionless coupling constants \(\tilde{h}\) and \(T\).

III. WEGNER-HOUGHTON RG APPROACH

The differential RG transformations for the sine-Gordon model are realized in the momentum space integrating out the high-frequency modes above the moving sharp cut-off \(k\) sequentially in infinitesimal steps. In the infinitesimal step, corresponding to moving the cut-off \(k\) to \(k - \delta k\), it is integrated for the high-frequency Fourier-modes of the field variable \(\phi'(x) = \sum_{p=k-\delta k}^k \phi_p e^{i px}\):

\[
e^{-S_{k-\delta k}[\phi]} = \int D[\phi'] e^{-S_k[\phi+\phi']},
\]

Performing the path integral in (4) using the saddle point expansion at \(\phi'=0\), the Wegner-Houghton equation is obtained (Wegner 1973):

\[
\partial_k S_k[\phi] = -\frac{1}{2} \text{tr}' \log(G_{pp}^{-1}[\phi])
\]

with the inverse propagator \(G_{pp}^{-1}[\phi] = \tilde{\partial}^2 S[\phi]/(\tilde{\partial} \phi \tilde{\partial} \phi')\) and the trace \(\text{tr}'\) over the momentum shell \([k - \delta k; k]\). Using the derivative expansion in the form \(S_k = \int d^2x \left[z(k)\frac{1}{2}(\partial \phi)^2 + V(\phi, k)\right]\) this reduces to:

\[
k \partial_k V(\phi, k) = -k^d a \ln \left(\frac{z(k^2 + V''(\phi, k))}{k^2}\right), \quad k \partial_k z(k) = k^d a [V''(\phi, k)]^2 \left[\frac{4z^2 k^2}{DA^4} - \frac{z}{A^3}\right],
\]

with \(A = (z k^2 + V''(\phi, k))\), the potential \(V(\phi, k) = u \cos \phi\), and the field-independent wave-function renormalization \(z(k) = 1/b^2\). Eqs. (6) are obtained by the method described in (Polonyi 2000). The flow equations for the dimensionful coupling constants \(u(k) = -h/T\) and \(z(k) = 1/T\) are obtained by expanding both sides of Eqs. (6) in Fourier series and neglecting the higher harmonics missed on the left hand sides. It is straightforward to use the derivative of the first equation in (6) with respect to \(\phi\). Thus, we find

\[
k \partial_k u(k) = \frac{k^2}{2\pi} \frac{1}{x(k)} \left[1 - (1 - x^2(k))^{1/2}\right], \quad k \partial_k z(k) = -\frac{1}{8\pi} \frac{x^2(k) (1 + x^2(k))}{(1 - x^2(k))^{3/2}},
\]

with \(x = u(k)/k^2 z(k)\). In order to compare Eqs. (7) with Eqs. (3) one should introduce the dimensionless couplings via \(u = -a^{-2}h/T\) and \(z = 1/T\). Choosing the moving distance scale according to \(1/a = (8\pi^2)^{1/4}k\), we get

\[
a \frac{dh}{da} = -\frac{1}{16\pi^2} \frac{T}{h} \left[1 - (1 - 8\pi^2 h^2)^{1/2}\right] + 2\hat{h} - \pi T \tilde{h}^3 \left(\frac{1 + 8\pi^2 \hat{h}^2}{(1 - 8\pi^2 \hat{h}^2)^{5/2}}\right),
\]

\[
a \frac{dT}{da} = -\pi T^2 \tilde{h}^2 \left(\frac{1 + 8\pi^2 \hat{h}^2}{(1 - 8\pi^2 \hat{h}^2)^{5/2}}\right).
\]

Expanding (8) in series of \(\hat{h}(a)\), one finds:

\[
a \frac{d\hat{h}}{da} = \hat{h} \left[2 - \frac{T}{4\pi} \left(1 + 2\pi^2 \hat{h}^2 + ...\right) - \pi T \hat{h}^2 (1 + ...)\right],
\]

\[
a \frac{dT}{da} = -\pi T^2 \hat{h}^2 \left(1 + 2\pi^2 \hat{h}^2 + ...ight).
\]

We see that the leading order terms on the right hand sides are those of Eqs. (3). The terms of higher order in \(\hat{h}\) yield the infinite series of corrections to the dilute-gas result and are summed up in a closed form on the right hand sides of Eqs. (8).
IV. RESULTS

In a previous paper (Nandori 1999), using the local potential approximation \( z(k) \equiv 1 \), it was shown, that the periodicity and the convexity are so strong constraints on the effective potential \( V(\phi, k = 0) \) that it becomes flat. This flattening was tested numerically, as well. In order to determine the flow of the blocked potential in next-to-leading order of the derivative expansion, the flow of the wave-function renormalization \( z(k) \) should also be obtained.

The RG flow of \( z(k) \) and \( u(k) \) (i.e. \( \tilde{h}(k), T(k) \)), obtained by the above described two different blocking transformations (real space and Wegner-Houghton RG) are qualitatively the same. The real space RG method (see Eqs.(3)) yielded the well-known phase-structure for the Coulomb-gas, (see Fig. 1) that has already been obtained in the literature (Kosterlitz 1973, Kosterlitz 1974, Huang 1991) using the dilute gas approximation. There are two phases connected by the Kosterlitz-Thouless transition. In the molecular phase the vortices and anti-vortices form closely bound pairs while in the ionised phase they dissociate into a plasma. Similar results were obtained in (Gersdorff 2000), using momentum space RG with smooth cut-off, without the dilute gas approximation.

The flow diagram for the sine-Gordon model obtained by the Wegner-Houghton RG approach (see Eqs.(8)) using a sharp momentum cut-off, is plotted in Fig. 2. In order to compare the results obtained by the two different RG methods, four RG trajectories calculated by the real space RG method are plotted in Fig. 2, as well. For small values of \( \tilde{h} \) and \( T \), the trajectories for both methods are the same for the same initial conditions. For larger values of \( \tilde{h} \) the dilute gas approximation, for large values of \( T \) the Villain transformation lose their validity, so that the different RG trajectories belonging to the same initial values start to diverge. The flow diagram for the sine-Gordon model is valid for \( \tilde{h}^2 < 8\pi^2 \). At \( \tilde{h}^2 = 8\pi^2 \) a non-trivial saddle point occurs in the path integral (4) and the Wegner-Houghton equation looses its validity.

In order to investigate the flattening of the blocked potential in next-to-leading order of the derivative expansion also the higher harmonics of the periodic potential should be included. Therefore, the generalization to the field dependent, periodic wave-function renormalization should be performed. Since it is unclear how to treat the field-dependent wave-function renormalization in the Wegner-Houghton approach, such a generalization is in progress using Polchinski’s method.

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FIG. 1. Phase structure for the two-dimensional Coulomb-gas (or vortex-gas) obtained by the real space RG method using the dilute-gas approximation. The dimensionless coupling constants are the fugacity of the gas (or the external field) $\tilde{h}$, and the temperature $T$.

FIG. 2. Phase structure for the sine-Gordon model. The full (dashed) lines correspond to the RG trajectories obtained by the Wegner-Houghton (real space) RG methods. The flow diagram for the sine-Gordon model is valid for $\tilde{h}_c^2 < 8\pi^2$ (see horizontal dotted line), above this the Wegner-Houghton equation loses its validity.