Cosmological Perturbations in Brane Worlds: Brane Bending and Anisotropic Stresses

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Using a metric–based formalism to treat cosmological perturbations, we discuss the connection between anisotropic stress on the brane and brane bending. First we discuss gauge–transformations, and draw our attention to gauges, in which the brane–positions remain unperturbed. We provide a unique gauge, where perturbations both on the brane and in the bulk can be treated with generality. For vanishing anisotropic stresses on the brane, this gauge reduces to the generalized longitudinal gauge. We further comment on the gravitational interaction between the branes and the bulk.

\section{I. INTRODUCTION}

One outstanding problem in brane world cosmology is to develop a better understanding on the evolution of perturbations. To gain insights is of prime importance. In fact, in order to make predictions regarding the primordial spectrum of perturbations or the anisotropies in the cosmic microwave background, the creation and evolution of perturbations have to be understood. Only then we can hope to test the theory with future cosmological experiments. On the other hand, the question of stability of the global brane world space–time has to be addressed, because some (or most) solutions of Einstein equations could be in principle unstable. To attack these issues, the use of a full five–dimensional description is necessary.

Some work has already addressed fluctuations in brane world theories. Namely, creation of perturbations on the brane was considered in [1]–[3], where specific assumptions were made from the beginning. Other groups have developed a formalism to treat perturbations in brane worlds for rather general situations, see [4] (hereafter Paper I) and [5]–[19]. In Paper I, a metric–based formalism, which represents a straightforward extension of the usual four–dimensional approach [13], was developed for a brane world theory with two branes. Also, some of the evolution equations on the brane were found. However, two assumptions where made, namely: i) it was required that the position of the branes was not affected by a first order perturbation, and ii) a generalized longitudinal gauge (GLG) was used in the bulk. A result of these assumptions was that the anisotropic stresses on the branes had necessarily to vanish. Although, this particular scenario would be adequate to study scalar fields or ordinary matter without anisotropic stresses, it does not represent the most general case. The aim of this paper is precisely to fill this gap by considering the most general case of having anisotropic stresses on the branes and by allowing the brane positions to be perturbed.

We will consider only the case of single extra dimension, which is assumed to be compactified on a circle $S_1$ with a $Z_2$–symmetry. The (unperturbed) branes will be located at the fix points of the $Z_2$–symmetry, which, in our case are taken to be $y_1 = 0$ and $y_2 = R$.

The paper is organized as follows: in the next section we will discuss gauge transformations. After briefly reviewing the construction of gauge invariant variables, we will show that the GLG is not compatible with unperturbed brane positions in the most general case. We will then discuss a novel gauge, which reduces to the GLG for vanishing anisotropic stresses on the branes. We will also discuss the Randall–Sundrum gauge in the context of cosmological perturbations. In Section III, we will derive the perturbed Einstein equations for the gauge introduced in Section II and we will derive the junction conditions. We will also briefly discuss the gravitational coupling between the branes and the bulk in Section IV. Finally, our conclusions can be found in Section V.

\section{II. GAUGE TRANSFORMATIONS AND GAUGE-INVARIANT VARIABLES}

In this section we begin with a brief review of the gauge-invariant formalism for metric perturbations in brane–world models (see Paper I). Then we will discuss gauge–transformations and the inclusion of the anisotropic stresses.

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Throughout this paper, we will consider the case for five-dimensional brane worlds only. The additional spatial coordinate is denoted by \( y^5 = y \). For the purpose of this section, all we need to specify is that the branes are stretched across the usual four-dimensional space-time and that they are located at specific points along the additional dimension. We will be more precise below.

The most general higher-dimensional metric consistent with the maximally symmetric three-dimensional spatial manifold is given by (here the indices \( a \) and \( b \) are either 0 or 5 and \( y^0 = t \))

\[
\begin{align*}
\text{ds}^2 &= a^2 \{ \gamma_{ab} dy^a dy^b - \Omega_{ij} dx^i dx^j \}, \\
\text{where} \quad F &= \eta, \\
\text{function and} \quad F &= \text{three–dimensional maximally symmetric space, given by} \\
\Omega_{ij} &= \delta_{ij} \left[ 1 + \frac{k}{4} x^i x^m \delta_{lm} \right]^{-2}, \\
\text{with} \quad k = 0, 1, -1 \text{ for flat, closed or hyperbolic 3–geometries, respectively. Given this structure of the background, we are able to classify metric perturbations by their three-dimensional tensor properties as in the four-dimensional case [13].} \\
\text{The perturbed metric can be generically written in the form} \\
\text{ds}^2 &= a^2 \left\{ \gamma_{ac} (\delta_c^i + 2 \phi_c^i) dy^a dy^b - \left[ (1 - 2 \psi) \Omega_{ij} + 2E_{[ij} + 2F_{(ij)} + h_{ij} \right] dx^i dx^j - 2W_{ai} dy^a dx^i \right\}, \\
\text{where} \quad F_i \text{ and } h_{ij} \text{ have a vanishing divergence and } h_{ij} \text{ is traceless. Also, the three–vectors } W_{ai} \text{ can be split as follows,} \\
W_{ai} &= B_{a|i} + S_{ai}, \\
\text{where } S_{a|i} &= 0.
\end{align*}
\]  

An infinitesimal coordinate transformation can be written as

\[
x^\alpha \rightarrow \tilde{x}^\alpha = x^\alpha + \xi^\alpha
\]

with (infinitesimal) parameters \( \xi^\alpha \). We split these parameters as \( \xi^\alpha = (\xi^a, \xi^i) \), where \( \xi^i = \eta^i + \xi^i \) being \( \xi \) a scalar function and \( \eta^i \) a divergenceless three–dimensional vector, i.e. \( \eta^i|_i = 0 \). We use the convention that indices of type \( a \) (\( i \)) are lowered, raised and contracted using the metric \( \gamma_{ab} (\Omega_{ij}) \). Furthermore, we take the vertical bar to denote the covariant derivative with respect to \( \gamma_{ab} \) or \( \Omega_{ij} \) depending on the index type. As shown in [4], the transformation of the scalar sector is found to be

\[
\begin{align*}
\delta \phi_{ab} &= -\xi_{(a|b)} - H^c \xi_c \gamma_{ab}, \\
\delta \psi &= H^a \xi_a, \\
\delta E &= -\xi, \\
\delta B_a &= \xi_a - \xi|_a.
\end{align*}
\]

where we have introduced the generalized Hubble parameters

\[
H_c = \frac{a|c}{a}.
\]

The vector perturbations in the metric (3) change according to

\[
\begin{align*}
\delta F_i &= -\eta_i, \\
\delta S_{ai} &= -\eta_i|_a.
\end{align*}
\]

The tensor perturbation \( h_{ij} \) is, of course, invariant under the first order gauge transformation. From these transformation rules one can easily construct gauge invariant variables for scalars and vectors, which we repeat here (see [4] for further details):

**Scalar variables**

\[
\begin{align*}
\Phi_{ab} &= \phi_{ab} + H^c (B_c - E_{[c}) \gamma_{ab} + (B_{(a} - E_{(a})_b \gamma_{ab}, \\
\Psi &= \psi - H^c (B_c - E_{[c}).
\end{align*}
\]
In this paper we shall consider scalar perturbations only, for which the perturbed five-dimensional metric (3) reduces to

\[
\begin{align*}
&ds^2 = a^2 \left\{ b^2 \left[ (1 + 2\phi)dt^2 - 2W dt dy - (1 - 2\Gamma)dy^2 \right] \\
&\quad - \left[ \Omega_{ij} (1 - 2\psi) + 2E_{ij} \right] dx^i dx^j - 2B_{0i} dt dx^i - 2B_{5i} dy dx^i \right\}
\end{align*}
\]

(16)

where we have defined

\[
\phi = \phi^0_0, \quad \Gamma = -\phi^5_5, \quad W = 2\phi^5_0 = -2\phi^0_5.
\]

(17)

It is convenient to use a conformal gauge for the metric \(\gamma_{ab}\), i.e.,

\[
(\gamma_{ab}) = b^2 \text{diag}(1, -1).
\]

(18)

Note that, as it is further explained in Paper I, this gauge for \(\gamma_{ab}\) can always be chosen. Here \(b = b(t, y)\) is a new, independent scale factor.

So far the discussion was independent of the existence of branes. We now introduce brane sources, which are, in general, located at specific points in the fifth dimension. As in Paper I, we are going to consider the setup motivated by heterotic M–theory. The extra dimension is compactified on the orbifold \(S_1/Z_2\). That is, we compactify the fifth dimension on a circle restricting the corresponding coordinate \(y\) to the range \(y \in [-R, R]\) with the endpoints identified. The action of the \(Z_2\) symmetry on the circle is taken to be \(y \rightarrow -y\). Associated to this symmetry there are two fixed points at \(y = y_1 = 0\) and \(y = y_2 = R\), where we assume that two three-branes, stretching across 3 + 1–dimensional space, are located.

We have to truncate the five-dimensional metric in order to make it consistent with the orbifold symmetry. For the unperturbed metric this implies the following conditions at the fix points:

\[
\begin{align*}
g_{\mu\nu}(-y) &= g_{\mu\nu}(y), \\
g_{\alpha\beta}(-y) &= -g_{\alpha\beta}(y), \\
g_{55}(-y) &= g_{55}(y).
\end{align*}
\]

(19)-(21)

When the background geometry is perturbed, however, the branes are in general no longer located at \(y_n = \text{constant}\), but at a perturbed positions which depend on the intrinsic brane coordinates \(x^i\) and \(t\). Thus, the symmetry conditions (19)-(21) cannot directly be applied to the perturbed background. However, by a suitable coordinate change, one can always gauge away these perturbations on the brane positions (see the discussion below). Therefore, using these particular coordinates, (19)-(21) also hold for the perturbed metric.

Note that the conditions (19)-(21) impose some restrictions on the gauge transformation (5) that we are allowed to perform, if the new coordinate system is assumed to have the same symmetries. Indeed, in this case we have to make sure that gauge transformations do not lead out of the class of metrics defined this way. In particular, this means that the parameter \(\xi^a\) for an infinitesimal coordinate transformation has to be restricted by

\[
\begin{align*}
\xi^\mu(-y) &= \xi^\mu(y), \\
\xi^5(-y) &= -\xi^5(y),
\end{align*}
\]

(22)-(23)

which directly follows from (6)-(9) and the symmetry conditions (19)-(21). From these rules we can deduce the \(Z_2\) properties of the various scalar quantities in metric (16): while the background scale factors \(a, b\) as well as the perturbations \(\phi, \psi, \Gamma, E\) and \(B_0\) are \(Z_2\) even, that is, for example, \(a(-y) = a(y)\), the perturbations \(W\) and \(B_5\) are \(Z_2\) odd, that is for example \(W(-y) = -W(y)\). Similarly, for the scalar components in the transformation parameter \(\xi^a\), we find that \(\xi_0\) and \(\xi^4\) are even while \(\xi_5\) is odd.

To be precise, as we did in Paper I, we are going to work in the boundary picture, where instead of working with the whole orbifold, only a half of the circle is considered. We will require that all components of metric (16) are continuous across the full orbicircle except for the odd components \(W\) and \(B_5\) which are allowed to jump at the fixed points (but continuous otherwise). To simplify the notation, we define the value of an odd field on the brane as the one that is approached from within the interval \(y \in [0, R]\). This is precisely the boundary value of the field as viewed in the boundary picture and it represents one half of the jump of this field at the fixed point.
In certain cases, the formalism of brane world perturbations is much simpler if the position of the branes remain fixed at constant $y$. Indeed, if the branes do not bend into the fifth dimension, then the $Z_2$ symmetry along the orbifold will be straightforwardly imposed by the conditions (19)–(21). Furthermore, as explained in Paper I, the low energy–effective action may easier be obtained if the brane positions are unperturbed.

In this subsection we will show that it is always possible to gauge away the perturbation of the brane position. We then motivate a novel gauge, in which brane world perturbations can be treated very generically. We will start considering a gauge for a background with two branes located at fixed $y = y_n$ positions. Then, the most general first order perturbation in this scenario will be described by the metric (16), together with two first order perturbations out of $E_{5|5}$, $B_5$ and $y_b^{(n)}$ ($n=1,2$) by the following simple combination,

$$B_5 - E_{5|5} - y_b^{(n)} ,$$

which directly follow from (8)–(9) and the transformation rule for $y_b$: $\delta y_b = \xi^5$ after a first order gauge transformation (5). Thus, if we choose $B_5 - E_{5|5} = 0$ everywhere in the bulk, i.e. if we choose the GLG, both brane positions are in general perturbed. However, as it will become clear in the following section, only when the anisotropic stresses on the branes vanish, can we use the GLG and yet keep the branes at $y_b = \text{const.}$. Hence, in the GLG the position $y_b^{(n)}(x^\mu)$ of the branes contain only information about the anisotropic stresses.

Observe that, without loosing essential degrees of freedom, it is always possible to perform a gauge transformation that resets the perturbed branes to their initial unperturbed positions $y = y_n$. The procedure is the following:

The general metric (16) contains five degrees of freedom associated to the arbitrary choice of the five quantities $\xi^\alpha$ that describe a five–dimensional gauge transformation. However, in order to relocate the brane positions, we need only to partially fix the arbitrary function $\xi^5$. In fact, since under a general gauge transformation $\xi^\alpha$ the fifth coordinate $y$ changes according to $\tilde{y} = y + \xi^5$, we can take,

$$\xi^5(x^\alpha) = \hat{\xi}^5(x^\alpha) + \tilde{\xi}^5(x^\alpha),$$

being $\hat{\xi}^5(x^\alpha)$ any odd function across the unperturbed brane positions, but smooth otherwise, with the following boundary conditions at the unperturbed brane positions

$$\hat{\xi}^5(t,x^i,y_n) = \mp \lambda_n(t,x^i),$$

where the upper (lower) sign applies when the position $y = y_n$ is approached to the right (left). On the other hand, $\tilde{\xi}^5(x^\alpha)$ is any smooth function with the following boundary conditions at the unperturbed brane positions,

$$\tilde{\xi}^5(t,x^i,y_n) = 0.$$

Note that the boundary condition (27) allows to gauge away the bending of the brane by fixing again the position of the brane at $y = y_n$ (see Fig 1.), while the term $\hat{\xi}^5$ with boundary (28) contains a residual gauge freedom which still remains after fixing the position of the branes. Obviously, any further gauge transformation should satisfy (28). The reason is because the $Z_2$–symmetry applies again to the unperturbed brane positions $y_n$. Thus, $\tilde{\xi}^5$ is an odd function across $y = y_n$ and to avoid the appearance of distributional terms on the metric coefficients, $\hat{\xi}^5$ must indeed be zero at $y = y_n$.

While we relocate the position of the branes, we may choose two smooth functions $\xi = E$ and $\xi_0 = E_0 - B_0$ by means of which $\tilde{E} = \tilde{B}_0 = 0$. Then, we have the following generic set of perturbations $\hat{\phi}$, $\hat{\psi}$, $\hat{\Gamma}$, $\hat{W}$, $\tilde{E} = 0$, $\tilde{B}_0 = 0$ and by (9)

$$\tilde{B}_5 = B_5 + \xi_5 - \xi_{5|5} = B_5 - E_{5|5} + \xi_{5|5} + \xi_5 .$$

However it is not in general possible to choose a quantity $\xi_5$, that vanishes on both branes, by means of which $\tilde{B}_5 = 0$. Indeed, to do so we must choose $\xi_5 = E_{5|5} - B_5 - \xi_{5|5}$. However, the odd quantities $E_{5|5} - B_5 - \xi_{5|5}$ do not in general
vanish on the branes. Therefore, the combinations $\pm (E^{(n)}_{15} - B^{(n)}_{5} + \lambda_{5}/b^{2})$, which represent half of the jump of the odd functions $E_{15} - B_{5} - \xi_{5}$ across the branes, are already gauge invariant. Indeed, we have not further gauge freedom to change them. Obviously, this is a simple consequence of (25) being gauge invariant. On the other hand, as we will explicitly see in the next section, $E^{(n)}_{15} - B^{(n)}_{5} - \xi_{5}$ are directly related to the anisotropic stresses on the branes.

![Diagram](image)

**FIG. 1.** In the most general situation, the brane positions along the fifth dimension $y$ are perturbed (a). However, without affecting the essential gauge freedom, it is always possible to gauge away the perturbation on the brane positions. Once the branes have been relocated, we can use the remaining gauge freedom to set $E = B_{0} = 0$ and simplify $B_{5}$ as much as possible. Observe that, by relocating the brane positions at $y = \text{constant}$, we can straightforwardly impose the $Z_{2}$ symmetry of the orbifold and then the junction conditions for the metric coefficients can be read off directly from Einstein equations.

Although it is not possible in general to switch to a gauge where the odd perturbation $\hat{B}_{5}$ is zero, we can actually eliminate most of it as follows. Since it is always possible to choose $\hat{B}_{5}$ in the form (29), we can then split

$$\hat{B}_{5} = \hat{B}_{5} - \hat{E}_{15} + L_{5} + \hat{\xi}_{5},$$

(30)

where $L_{5}$ is the broken line that connects the values of $\hat{B}_{5}$ on both branes (from within the interval $y \in [y_{1}, y_{2}]$) and
in (30). Then, after this last gauge transformation, the value of the perturbation \( \tilde{B}_{\text{branes}} \) only. Thus, when positions, such a gauge provide great advantages in solving the full five–dimensional dynamics. As it will become clear because, \( \hat{\xi}_5 \) Observe that all the physical information that the perturbation \( \tilde{B} \) broken line \( L \)

denote by \( \tilde{B} \), which for simplicity we will across the branes. On the other hand \( \hat{\xi}_5 = \hat{E}_\text{branes} \) is a smooth function which vanishes on the branes and thus it can be gauged away by the simple \( \tilde{B} \) across the branes, which, as we will see in the next \( \tilde{B}_5 \) carries is actually encoded in \( L_5 \). This is true because, \( \tilde{B}_5 - \hat{E}_\text{branes} \) is a smooth function that vanishes on the branes and thus it can be gauged away by the simple \( \tilde{B}_5 \) carries is actually encoded in \( L_5 \). This is true because, \( \tilde{B}_5 - \hat{E}_\text{branes} \) is a smooth function that vanishes on the branes and thus it can be gauged away by the simple

\[
L_5 = \left[ \tilde{B}_5^{(1)} - E_\text{branes}^{(1)} - \frac{\lambda_1}{b^2} \right] (y - y_1) \theta(y_1 - y) + \left[ \tilde{B}_5^{(2)} - E_\text{branes}^{(2)} - \frac{\lambda_2}{b^2} \right] (y - y_2) \theta(y - y_2)
\]

where the plus (minus) sign in \( \tilde{B}_5^{(n)} \) holds for the value of \( \tilde{B}_5 \) when the brane is approached to the right (left). Now, explicitly using that \( \tilde{B}_5^{(n)} = \pm (B_5^{(n)} - E_\text{branes}^{(n)} - \lambda_n/b^2) \), such a broken line can be easily simplified as follows

\[
L_5 = -\left( B_5^{(1)} - E_\text{branes}^{(1)} - \frac{\lambda_1}{b^2} \right) + \sum_{n=1}^2 2 \left( B_5^{(n)} - E_\text{branes}^{(n)} - \frac{\lambda_n}{b^2} \right) \theta(y - y_n) - \left( B_5^{(n)} - E_\text{branes}^{(n)} - \frac{\lambda_n}{b^2} \right) \frac{y - y_n}{y_2 - y_1}, \tag{31}
\]

Observe that all the physical information that the perturbation \( \tilde{B}_5 \) carries is actually encoded in \( L_5 \). This is true because, \( \tilde{B}_5 - \hat{E}_\text{branes} \) is a smooth function that vanishes on the branes and thus it can be gauged away by the simple gauge transformation

\[
\hat{\xi}_5 = \hat{E}_\text{branes} - \tilde{B}_5 \tag{32}
\]

in (30). Then, after this last gauge transformation, the value of the perturbation \( \tilde{B}_5 \), which for simplicity we will denote by \( B \), would be reduced to its simplest value, namely

\[
B = L_5. \tag{33}
\]

We note here, that this particular gauge reduces to the usual GLG when the jumps of the metric perturbation \( \tilde{B}_5 \) vanish on the branes. We will see below that this is the case when the anisotropic stresses on the branes vanish.

With all this information in mind, we should be careful with the construction of the complete set of gauge invariant variables discussed in the previous section. Of course, in order to avoid distributional terms in the construction of the gauge invariant variables, then (13) should be constructed using \( \tilde{B}_5 - E_\text{branes} \) instead of \( B_5 - E_\text{branes} \). Indeed, that would prevent the appearance of delta–terms coming from the \( y \)-derivative of the odd quantity \( \tilde{B}_5 - E_\text{branes} \), which does not in general vanish on the brane positions.

Observe that, since \( \tilde{B}_5 - E_\text{branes} \) is taken to be zero in this gauge, the metric perturbations \( \tilde{\phi}, \tilde{\psi}, \tilde{\Gamma}, \tilde{W} \) actually correspond to the gauge invariant variables. On the other hand, \( B \) is determined by the broken line (31), whose slope depends on the variables \( B_5^{(n)} - E_\text{branes}^{(n)}, \) which are defined on the branes only and are gauge–invariant under coordinate transformations which respect the symmetry conditions (19) – (21).

In summary, by means of a gauge transformation compatible with leaving the branes at their unperturbed positions, we have been able to eliminate two degrees of freedom \( (E \text{ and } B_5) \) in the metric (16) instead of the three degrees of freedom \( (E, B_0 \text{ and } B_5) \) that can be cancelled with an ordinary gauge transformation. The remaining degrees of freedom that cannot be cancelled are the jumps of \( \tilde{B}_5 - E_\text{branes} + \hat{\xi}_5 \) across the branes, which, as we will see in the next section, are directly related to the anisotropic stresses on the branes.

For simplicity in our notation we will denote the set of gauge invariant variables in the gauge discussed above by \( \phi, \psi, \Gamma, W \text{ and } B \). It is worth noticing that, since it is always possible to keep the two branes at fixed \( y = y_n \) orbifold positions, such a gauge provide great advantages in solving the full five–dimensional dynamics. As it will become clear in the following section, the dynamics of the variable \( B \) is described by the evolution of the anisotropic stresses on the branes only. Thus, when \( B = 0 \), this particular gauge will reduce to the generalized longitudinal gauge introduced in Paper I.
FIG. 2. In general the metric perturbation $\tilde{B}_5$ cannot be gauged away by means of a gauge transformation which is compatible with the $Z_2$ symmetry along the orbifold. The reason is that $\tilde{B}_5$ is an odd quantity and thus it may jump on the brane positions (a). Since the $Z_2$ symmetry conditions (19)–(21) forbid the jumps of $\tilde{B}_5$ from being gauged away, these jumps are already gauge invariant. However, any behavior of the perturbation $\tilde{B}_5$ in the bulk besides the purely linear is totally irrelevant. In fact, it is always possible to perform a gauge transformation compatible with the $Z_2$ symmetry which reduces $\tilde{B}_5$ to the straight line $L_5$ in (a). This is simply possible by choosing a gauge transformation with $\xi_5 = -(\tilde{B}_5 - L_5)$. Observe that such a $\xi_5$ vanishes on both branes (b) and thus is compatible with the $Z_2$ symmetry.

B. Other possible gauges

In the previous subsection we have used the gauge freedom to cancel the perturbations $E$ and $B_0$ and to simplify the perturbation $B_5$ as much as possible and yet keep the branes at their unperturbed positions. We have also said that this particular gauge choice corresponds to the generalization of the four–dimensional longitudinal gauge. On the other hand, another choice for the gauge fixation that is commonly found in the literature is setting the five–dimensional perturbations $\Gamma$, $W$ and $B_5$ to zero, and keep the perturbations $E$ and $B_0$ instead, i.e. the the scalar part of the Randall–Sundrum gauge. However, this choice of gauge is generally incompatible with keeping the branes
at their unperturbed positions. In fact, let us suppose that we start from the most general set of perturbations, i.e \( \phi, \psi, \Gamma, W, E, B_0 \) and \( B_5 \), and we want to set \( \Gamma \) to zero. In order to do that, we will use the transformation equation (6), which can be written as,

\[
\delta T^\Gamma = \xi_5 - \left( \frac{a'}{a} + \frac{b'}{b} \right) \xi_5 + \left( \frac{\dot{a}}{a} - \frac{\dot{b}}{b} \right) \xi_0.
\]  

(34)

Observe that, regardless of the particular value of the gauge fixing function \( \xi_0 \), if we want to set the perturbation \( \Gamma \) to zero we need to solve a differential equation for \( \xi_5 \) with the boundary conditions \( \xi_5^{(n)} = 0 \) on both branes. However, this is not in general possible. As a particular example, consider the Randall–Sundrum case where \( b = 1 \) and \( a = a(y) \), then equation (34) would simply be,

\[
\xi'_5 - \frac{a'}{a} \xi_5 - \Gamma = 0,
\]  

(35)

with general solution,

\[
\xi_5(t, x^i, y) = \left[ C(t, x^i) + \int \Gamma(t, x^i, y) \frac{dy}{a} \right],
\]  

(36)

where \( C(t, x^i) \) is an arbitrary integration constant. Observe, thus, that it is always possible to fix \( C(t, x^i) \) to set \( \xi_5 = 0 \) in one of the branes but not in the two of them at the same time. Thus, suppose that we fix \( C(t, x^i) \) in order to set \( \xi_5^{(2)} = 0 \) on the right brane located at a constant \( y = y_2 \). Then on the left brane \( \xi_5^{(1)} \neq 0 \), which means that this brane cannot be located at a constant \( y = y_1 \) position but, according to (5), its new position should necessarily be

\[
y = y_1 - \frac{1}{b^2} \xi_5(t, x^i, y_1)
\]  

(37)

where \( \xi_5(t, x^i, y_1) \) is given by evaluating (36) at \( y = y_1 \). Then, any gauge transformation (5) from this new position has its fifth component zero in agreement with the \( Z_2 \)–symmetry. A similar situation would happen if we tried to set the perturbation \( W = 0 \).

In summary, although it is possible to use the Randall–Sundrum gauge for brane world scenarios with a single brane located at a fixed \( y \) position, it is not in general possible in this particular gauge to fix the positions of more than one brane. Such an effect was already pointed out in [16].

### III. THE FORM OF THE PERTURBED EINSTEIN EQUATION AND JUNCTION CONDITIONS

In this section, we calculate Einstein equations in the gauge considered in section 2. The metric in this gauge is given by

\[
ds^2 = a^2 \left\{ b^2 \left[ (1 + 2\phi) dt^2 - 2W dt dy - (1 - 2\Gamma) dy^2 \right] - [\Omega_{ij}(1 - 2\psi)] dx^i dx^j - 2B_i dy dx^j \right\}.
\]  

(38)

Note that the only difference between such a gauge and the generalized longitudinal gauge considered in Paper I, is the quantity \( B = B_5 \). As mentioned in the last section, and explicitly seen below, \( B = B_5 \) contains only information about the anisotropic stresses on the branes.

The two branes are located in the orbifold at \( y = \) constant and the Einstein equations can be written as

\[
G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta} + \sum_{n=1}^{2} T^{(n)}_{\alpha\beta} \delta(y - y_n),
\]  

(39)

where we have set the five-dimensional Newton constant to one, for simplicity. The delta-functions in this equation are covariant with respect to the fifth dimension, that is, they include a factor of \( 1/\sqrt{-g_{55}} \). Furthermore, \( T_{\alpha\beta} \) is the bulk stress-energy tensor induced by fields that propagate in the full five-dimensional space time. The brane stress-energy tensors \( T^{(n)}_{\alpha\beta} \), on the other hand, originate from fields that are confined to the branes at the orbifold fix points. We note here, that we are not considering Gaussian normal coordinates.
Next, we specify the stress-energy tensors. As described in Paper I, they can be found using standard methods. For the bulk, the most general form of the unperturbed stress-energy tensor is

\[ T_{\alpha\beta}^{\gamma} = \begin{pmatrix} \rho & 0 & -r \\ -p\delta_{ij} & 0 & -q \\ r & 0 & -q \end{pmatrix}, \]

and the background stress-energy tensor on the branes has the form

\[ T^{(n)\alpha\beta} = \begin{pmatrix} \rho^{(n)} & 0 & 0 \\ 0 & -p^{(n)}\delta_{ij} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

For the perturbed stress-energy tensor we find

\[ \delta T_{\alpha\beta}^{\gamma} = \begin{pmatrix} \delta\rho \delta B_{ij} + (p + p)\delta v_{ij} & -(p + p)b^{-2}\delta v_{ij} & -\delta r \\ \delta r + 2\alpha + \Gamma - (p + q)W & (p + q)b^{-2}B_{ij} - b^{2}\delta v_{ij} & -\delta q \end{pmatrix}, \]

where \( v \) and \( u \) are two potentials for “velocity” fields and \( \sigma \), satisfying \( \sigma_{ij} = 0 \), determines the anisotropic stress. The perturbed brane stress-energy tensors are

\[ \delta T^{(n)\alpha\beta} = \begin{pmatrix} \delta\rho^{(n)} & -(\rho^{(n)} + p^{(n)})b^{-2}\delta v^{(n)}_{ij} & -\delta r^{(n)} \\ \delta r^{(n)} - \rho^{(n)}W & p^{(n)}b^{-2}B_{ij} - b^{2}\delta v^{(n)}_{ij} & 0 \end{pmatrix}, \]

where, unlike in our previous work \([4]\), we have explicitly included an anisotropic stress term \( \sigma_{ij}^{(n)} \) in the brane stress-energy tensors and a non-vanishing component \( \delta T^{(n)\gamma} \). The reason is that, as we will see below, once we include a perturbation \( B \) in our formalism we get new delta terms which should be precisely matched with a brane anisotropic stress and a brane \( \delta T^{(n)\gamma} \) component. We would like to present the equations of motion based on the metric (38) and on the above stress-energy tensors that follow from the Einstein equation (39). The background equations following from (39) have already been given in ref. [4] and thus we should refer to this work for details. However, since the introduction of a \( B \) perturbation will also introduce new terms on the perturbation equations at linear order, which are not present in our previous work [4], we explicitly give them here:

\[
(ab)^2\delta G^0 = \begin{cases} 
2b' + \frac{2a''}{ab} - 2 \frac{a''}{a} - \frac{a''}{a} \frac{a}{a} - \frac{\dot{a}}{\dot{a}} \frac{a}{a} & \Gamma - 3 \left[ \frac{a'}{a} + \frac{2a'\dot{a}}{a^2} + \frac{\dot{a}}{\dot{a}} \frac{a}{a} \right] W - 6 \left[ \frac{2a^2}{a^2} + \frac{\dot{a}}{\dot{a}} \right] \phi \\
+3 \left[ 3\frac{a'}{a} \frac{a}{a} - \frac{a''}{a} - \frac{\dot{a}}{\dot{a}} \frac{a}{a} - 2b' \frac{\dot{a}}{\dot{a}} \frac{a}{a} + 2kb^2 \right] \psi + b^2 (2\psi + \Gamma)_{ij} + 3\psi'' + \left[ \frac{\dot{a}}{\dot{a}} + \frac{a''}{a} \frac{a}{a} - b' \frac{b'}{b} \right] \delta \delta_{ij} \\
= a^2b^2 \left\{ \delta\rho + \sum_{n=1}^{2} (\delta\rho^{(n)} + \rho^{(n)})\delta(y - y_n) \right\},
\end{cases}
\]

\[
(ab)^2\delta G^5 = \begin{cases} 
-6 \left[ \frac{2a^2}{a^2} + \frac{a''}{ab} \right] \Gamma - 3 \left[ \frac{a'}{a} + \frac{2a'\dot{a}}{a^2} + \frac{a'}{a} \frac{a}{a} \right] W + 3 \left[ \frac{a'}{a} \frac{a}{a} - \frac{a''}{a} - \frac{a''}{a} \frac{a}{a} - \frac{a''}{a} \frac{a}{a} \right] \phi \\
+3 \left[ 3\frac{a'}{a} \frac{a}{a} - \frac{a''}{a} - \frac{a''}{a} \frac{a}{a} - \frac{a''}{a} \frac{a}{a} + \frac{\dot{a}}{\dot{a}} \frac{a}{a} + 2kb^2 \right] \psi + b^2 (2\psi - \phi)_{ij} - 3\psi + \left[ \frac{b'}{b} + 3\frac{a''}{a} \frac{a}{a} \right] B_{ij} \\
= -a^2b^2 \delta \phi,
\end{cases}
\]

\[
(ab)^2\delta G^0_5 = \begin{cases} 
3 \left[ \frac{a'}{a} - 2 \frac{a''}{ab} - \frac{a''}{a} \right] W + 3 \left[ \frac{a'}{a} - \frac{a''}{ab} - \frac{a''}{a} \right] \phi \\
+3 \left[ \frac{\dot{a}}{\dot{a}} \frac{a}{a} - \frac{a''}{a} \frac{a}{a} - \frac{\dot{a}}{\dot{a}} \frac{a}{a} + \frac{\dot{a}}{\dot{a}} \frac{a}{a} \right] \phi \\
= a^2b^2 \left\{ -\delta r - \sum_{n=1}^{2} \delta r^{(n)} \delta(y - y_n) \right\},
\end{cases}
\]
\[(ab)^2 \delta G^0_i \equiv \left\{ \left[ \frac{3a'}{a} + \frac{b'}{b} + \frac{1}{2} \frac{\partial}{\partial y} \right] W + \left[ \frac{3\dot{a}}{a} + \frac{\dot{b}}{b} \right] \phi + \left[ \frac{\dot{b}}{b} + \frac{\partial}{\partial t} \right] \Gamma + 2\psi + \frac{1}{b^2} \left[ \frac{1}{2} \frac{\partial}{\partial y} + \frac{3a'}{2a} - \frac{b'}{b} \right] \dot{B} \right\} |_i
\]
\[= a^2 \left\{ -\left(\rho + p\right)v - \sum_{n=1}^{2} \left(\rho^{(n)} + p^{(n)}\right) \psi^{(n)} \delta(y - y_n) \right\} , \tag{47} \]

\[(ab)^2 \delta G^5_i = \left\{ \left[ \frac{3a'}{a} + \frac{b'}{b} \right] \Gamma + \left[ \frac{b'}{b} + \frac{\partial}{\partial y} \right] \phi + \left[ \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{1}{2} \frac{\partial}{\partial t} \right] W - 2\psi + \frac{1}{b^2} \left[ 2kb^2 - \frac{3}{2} \frac{\partial}{\partial t} + \frac{\dot{b}}{b} \frac{\partial}{\partial y} - \frac{1}{2} \frac{\partial^2}{\partial y^2} \right] B \right\} |_i
\]
\[= -a^2 \left\{ (q - p)B + u + \sum_{n=1}^{2} \left( -p^{(n)}B + u^{(n)} \right) \delta(y - y_n) \right\} , \tag{48} \]

\[(ab)^2 \delta G^i_j = \left\{ \left[ \frac{6\dot{a}a}{a} - 2\frac{b''}{b} + 2\frac{b'^2}{b^2} - \frac{3\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{1}{2} \frac{\partial}{\partial y} \right] \Gamma
\]
\[+ \left[ \frac{2b^2}{b} - \frac{6\dot{b}}{b} - \frac{6\dot{a}}{a} \frac{\partial}{\partial y} - \frac{3\dot{a}}{a} \frac{\partial}{\partial t} + \frac{\dot{b}}{b} + \frac{\partial}{\partial y} - \frac{\partial^2}{\partial y^2} \right] \phi
\]
\[+ \left[ \frac{6\dot{a}a}{a} - 2\frac{b''}{b} + 2\frac{b'^2}{b^2} - \frac{3\dot{a}}{a} \frac{\partial}{\partial y} + \frac{\dot{b}}{b} + \frac{\partial}{\partial t} - \frac{\partial^2}{\partial y^2} \right] \psi + 2b^2(\psi - \phi + \Gamma)|_k
\]
\[+ \left[ 3\frac{a'}{a} + \frac{\partial}{\partial y} \right] B^k |_k \right\} \delta^i_j
\[= a^2b^2 \left\{ \delta y \delta^i_j + \sigma^i_j - \sum_{n=1}^{2} \left( \delta p^{(n)} + \Gamma p^{(n)} \right) \delta^i_j - \sigma^{(n)\delta^i_j}_j \right\} \delta(y - y_n) . \tag{49} \]

Here we have defined the delta-function \( \delta \) which incorporates a factor \((-g_{55})^{-1/2} = 1/ab. \)

We write down also the traceless part of eq. (49):

\[(\psi - \phi + \Gamma)^i_j - \frac{1}{3}(\psi - \phi + \Gamma)^k |_k \delta^i_j = -a^2\sigma^i_j - \frac{1}{b^2} \left[ 3\frac{a'}{a} + \frac{\partial}{\partial y} \right] \left( B^i_j - \frac{1}{3}g^i_j B^k |_k \right) . \tag{50} \]

Observe that, for the particular case where the anisotropic stress in the bulk vanishes, i.e. \( \sigma^i_j = 0 \), we may identify

\[\psi - \phi + \Gamma = -\frac{1}{b^2} \left( 3\frac{a'}{a} + \frac{\partial}{\partial y} \right) B \]. \tag{51} \]

As a next step we analyze the background and the perturbed Einstein equations:

**Background equations**

Matching first the delta terms in the background equations we can easily get the junction conditions for the scale factors \( a(t, y) \) and \( b(t, y) \) (see Paper I for details):

\[\frac{a'}{a} = \pm \frac{1}{6} abp^{(n)} , \quad \frac{b'}{b} = \pm \frac{1}{2} ab \left( \rho^{(n)} + p^{(n)} \right) . \tag{52} \]

Here and in the following the upper sign holds for the brane at \( y = y_1 \) and the lower sign for the brane at \( y = y_2 \). On the other hand, from the 05 and 55 component of the background equations, we find
where we have introduced for convenience the traceless quantities
\[ \delta = \frac{\rho}{a^2} - \frac{ab}{ab} + kb^2 = \frac{1}{12} \rho \left( \rho(n) + 3 \rho(n) + q \right), \]
which respectively represent the background energy conservation equation on the brane and a background dynamical equation for the scale factors on the brane.

**Perturbed equations**

Observe first that the components \( \delta G_{0}, \delta G_{i}, \delta G_{j} (i \neq j), \delta G_{0}, \delta G_{5} \) and \( \delta G_{i} \) contain explicit delta-function terms and they should be matched by terms containing first \( y \) derivatives of \( W \) and \( B \) and second \( y \) derivatives of all other quantities. This leads to

\[
\psi' = \frac{\dot{a}}{a} W - \frac{1}{3} \left( B_{5}(n) - E_{5}(n) \right) |_{k} \pm \frac{1}{6} ab \left( \delta \rho(n) - \Gamma \rho(n) \right), \tag{55}
\]

\[
\phi' = -\left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\partial}{\partial t} \right) W \pm \frac{1}{3} ab \left( \delta \rho(n) - \Gamma \rho(n) \right) \pm \frac{1}{2} ab \left( \delta p(n) - \Gamma p(n) \right), \tag{56}
\]

\[
\frac{\pm}{2} a \sigma^{(n)} |_{j} = -\left( B_{5}^{(n)} - E_{5}^{(n)} \right) |_{j} + \frac{1}{3} \delta_{j} \left( B_{5}^{(n)} - E_{5}^{(n)} \right) |_{k}, \tag{57}
\]

\[
W = -\frac{1}{b} \left( B_{5}^{(n)} - E_{5}^{(n)} \right) \pm \frac{a}{b} \left( \rho^{(n)} + p^{(n)} \right) u^{(n)}, \tag{58}
\]

\[
W = \frac{\delta r^{(n)}}{\rho^{(n)}}, \tag{59}
\]

\[
B_{5}^{(n)} = \left( B_{5}^{(n)} - E_{5}^{(n)} \right) |_{i} = u_{i}^{(n)} / p^{(n)}. \tag{60}
\]

In these equations the background junction conditions (52) have been used. From eqns. (58) and (59) we find

\[
\delta r^{(n)} = \pm \frac{a}{b} \rho^{(n)} \left( \rho^{(n)} + p^{(n)} \right) u^{(n)} - \frac{\rho^{(n)}}{b} \left( B_{5}^{(n)} - E_{5}^{(n)} \right). \tag{61}
\]

Recall that the quantities \( \delta r^{(n)} \) and \( u^{(n)} \) are uniquely fixed by the other components and they are, generally, non-vanishing, as already discussed in Paper I. In fact, they are directly related to the anisotropic stresses on the branes. For instance, we can easily transform (61) into an equation involving the anisotropic stresses on the brane as follows:

\[
\delta \Pi^{(n)} |_{j} = \pm \frac{1}{2} \rho^{(n)} \frac{\partial}{\partial t} (ab \sigma^{(n)} |_{j}) \pm \frac{a}{b} \rho^{(n)} \left( \rho^{(n)} + p^{(n)} \right) V^{(n)} |_{j}, \tag{62}
\]

where we have introduced for convenience the traceless quantities \( \delta \Pi^{(n)} |_{j} \) and \( V^{(n)} |_{j} \), which are given by

\[
\delta \Pi^{(n)} |_{j} = \delta r^{(n)} |_{j} - \frac{\delta^{j} \rho^{(n)}}{3} |_{k}, \tag{63}
\]

\[
V^{(n)} |_{j} = \sigma^{(n)} |_{j} - \frac{\delta^{j} \rho^{(n)}}{3} |_{k}. \tag{64}
\]

Finally, we would like to discuss the influence of the new terms discussed in this section (i.e. the quantities \( B = B_{5} \) and \( \sigma^{(n)} \)) for the low-energy effective theory discussed in Paper I. Because \( B \) in the bulk simply connects the values of the anisotropic stresses on the branes (see the line-construction in section 2), the \( y \)-averaged \( B \) along the bulk is zero. However, at the position on the branes, the jumps of the function have to be taken into account, so that the effective four-dimensional anisotropic stress \( \sigma_{4} \) has to be replaced by

\[
\sigma_{4} = e^{-\chi} \langle \sigma \rangle + \frac{1}{2Re^{2\chi}} \sum_{n=1}^{2} \sigma^{(n)}, \tag{65}
\]

where \( < \ldots > \) describes the averaging over the fifth direction. The other terms are not affected (\( R \) is the coordinate distance between the branes and \( \chi \) is a modulus describing the size of the fifth dimension).
In this section, we would like to discuss the gravitational interaction between the branes and the bulk. Since our gauge is well adapted to analyze the whole five–dimensional brane world, we should be able to get some insights on the non-local nature of this particular geometry.

Observe, that the variables $\delta r^{(n)}$ are the components $\delta T^{(n)}_{05}$ of the stress–energy tensor (43) on the branes. Thus, $\delta T^{(n)}_{05} = \delta r^{(n)}/(a^2b^2)$ is connected to an energy flux, i.e. it determines a flux of energy along the fifth dimension. Observe from (63) that the anisotropic part of $\delta r^{(n)}$ is directly related to quantities like $(\rho^{(n)} + p^{(n)})|V^{(n)}$ or $\delta (^{(n)}$, which are naturally related to the third order time–derivative of the reduced quadrupole moment of the matter fields on the brane. Observe, that the emission of gravitational radiation by a cosmological source is directly linked to the third order time–derivative of its reduced quadrupole moment (see [25] for details). Therefore, we may physically interpret the anisotropic part of $\delta r^{(n)}$ as a non–local measure of the energy exchange between the branes and the exchange through the five–dimensional gravitational radiation.

The generalization of the energy–momentum conservation equation (53) for brane perturbations can be found by restricting the $05$ and $5i$–components of the perturbed Einstein equations to the branes. This respectively gives the continuity equation and Euler’s equation. These two equations together with (61) give the full energy–momentum conservation for the brane matter field perturbations under possible interactions with the bulk. In fact, making explicit use of the junctions equations (58)–(60) these three equations may be conveniently written as follows

\[
\frac{\partial}{\partial t} \delta \rho^{(n)} + \frac{\delta \dot{a}}{a} (\delta \rho^{(n)} + \delta p^{(n)}) - \frac{\delta \dot{a}}{a} (\rho^{(n)} + p^{(n)}) v^{(n)} = -(\rho^{(n)} + p^{(n)}) v^{(n)}_{|k} \pm 2ab(p + q) \delta r^{(n)} + 2ab(\delta r + 2\phi r + \Gamma r), \tag{66}
\]

\[
\frac{\partial}{\partial t} \left[ (\rho^{(n)} + p^{(n)}) v^{(n)}_{|i} \right] + 2 \left( \frac{\delta \dot{a}}{a} - \frac{\dot{b}}{b} \right) (\rho^{(n)} + p^{(n)}) v^{(n)}_{|i} =
\]

\[-b^2 \left[ (\rho^{(n)} + p^{(n)}) \phi_{|i} + \delta \phi_{|i} \right] + \frac{4}{ab} \left[ 2b^2 (p - q) B^{(n)}_{|i} + b^2 k B^{(n)}_{|i} + \frac{1}{3} b^2 B^{(n)}_{ik} + \frac{1}{2} a^2 b^2 u_{|i} \right], \tag{67}
\]

\[
\frac{\partial}{\partial t} B^{(n)} = \mp ab \left( \rho^{(n)} + p^{(n)} \right) v^{(n)} - b^2 \frac{\delta r^{(n)}}{\rho^{(n)}} \tag{68}
\]

These equations provide a coupled system of differential equations for the matter perturbations on the brane. In particular observe that:

i) Equation (66) describes the growth of pressure and density perturbations sourced by $r$, $\delta r$, $\delta r^{(n)}$ and a Poisson–like term $v^{(n)}_{|k}$ from the velocity perturbations.

ii) Equation (67) describes the growth of velocity perturbations, which are sourced by $\delta p^{(n)}$, $\phi$, the anisotropic stresses on the branes and the momentum flux $u_{|i}$ from the bulk matter.

iii) Equation (68) describes the growth of anisotropic stresses on the branes sourced by $\delta r^{(n)}$ and velocity perturbations on the branes. This equation states that anisotropic stresses can be smoothed out by changing into velocity perturbations and gravitational radiation. For the particular case that $\delta r^{(n)} = 0$, then equation (61) states that decays on the anisotropic stresses can induce velocity fluctuations on the matter fields only (reference [22] provides details on this type of effects for ordinary cosmology).

Let us collect the physical meaning of the quantities $r$, $\delta r$ and $\delta r^{(n)}$:

i) $r$ determines a non-perturbed energy flow in the bulk arising from the $05$–component of the background stress–energy tensor (40). For the particular case that the bulk is empty, i.e. $\rho + q = 0$, this term is zero.

\[1\]The reduced quadrupole moment of the matter fields on the brane is defined as, $\tilde{f}^{ij} = \int \rho \left[ x^i x^j - \frac{1}{3} \delta^{ij} x^k x_k \right] d^3 x$. Thus, $\tilde{f}^{ij}$ is related to the non–spherical part of the kinetic energy of the source which is related to the anisotropic stress.
ii) $\delta r$ determines a first order perturbation on the energy flow in the bulk arising from the 05–component of the perturbed stress–energy tensor (42). Also, for the particular case that the bulk is empty this term is zero. Recall, however, that being $\delta r = 0$ does not mean that there is not a flow of energy through the bulk. In fact, it is possible to have pure gravitational radiation propagating in the bulk and yet have $\delta r = 0$. This is true because $\delta r$ comes from a purely local stress–energy tensor for first order bulk perturbations. However, the energy associated to gravitational waves is non–local. Indeed, first order gravitational waves are vacuum solutions of Einstein equations and their energy can be only properly computed by non–local space–time averages of quadratic wave amplitudes (see for instance [25]).

iii) $\delta r^{(n)}$ determines a first order perturbation of the “projected” energy flow on the branes, which may have two contributions: a local one (due to the matter distribution) and a non–local one (due to gravitational waves). While the local projection of $\delta r^{(n)}$ on the brane is zero when the bulk is empty, in general $\delta r^{(n)} \neq 0$ due to the non–local contributions. It is worth noticing that for domain walls we have $\delta r^{(n)} = 0$, because the equation of state is $\rho + p = 0$ and the anisotropic stress vanishes. This suggest, that, at least for the scalar part discussed here, domain walls do not radiate energy (at first order), see also [23].

Finally, the restriction of the 55 component of the perturbed Einstein equations, eq. (49), to the brane leads to the following evolution equation

\[
\begin{align*}
 b^2(2\psi - \phi)_{\left|_t\right.} - 3\dot{\psi} - 3\dot{\phi} + 3\left(\frac{\dot{b}}{b} - \frac{\dot{a}}{a}\right)\dot{\psi} + 6kb^2(\psi + \phi) + a^2b^2\rho^{(n)} [\frac{1}{6}(1 + 3w^{(n)})\phi \\
 + \frac{\delta q + 2q\phi}{\rho^{(n)}a} + \frac{1}{6} \left(1 + 3w^{(n)}\right) \delta^{(n)} + \frac{\delta p^{(n)} a}{\delta \rho^{(n)} a} \delta^{(n)} (n) \pm \frac{a}{b} \frac{r}{\rho^{(n)}} (1 + w^{(n)}) \omega^{(n)}] + a^2r \left(\dot{B}_{5}^{(n)} - \dot{E}_{5}^{(n)}\right) = 0.
\end{align*}
\]

V. DISCUSSION

Using the formalism presented in Paper I, we have discussed how anisotropic stresses on the branes can be easily included. We have also argued, that if we use the generalized longitudinal gauge in the bulk, the branes no longer remain at fixed $y = \text{const.}$ orbifold positions. On the other hand, the brane positions are related to the anisotropic stresses only. Also, we have introduced a novel gauge, in which the positions of the branes may be fixed at $y = \text{const.}$ while the information on the anisotropic stresses is encoded into the metric perturbation $B$. Finally, we have discussed the gravitational interaction between the branes, which, in our formalism, is described by the 05–component of the brane energy momentum tensor.

An outstanding problem is to find solutions of the bulk Einstein equations (44)–(49). It is worth noticing that some work has already addressed this issue from the point of view of a brane–observer only. However, in the presence of perturbations, this is not enough. In fact, one needs to solve the full five–dimensional Einstein equations in order to fully understand the non–local interaction between the two branes. Some work along this direction was already done in Paper I for the linear M–theory background. A derivation of the low-effective perturbation equations in Randall–Sundrum brane world [26] (with two branes) is still lacking. However, even without specifying the details of the theory, one is able to derive rather general statements about the evolution of perturbations and the consequences of the dynamics of the perturbations, (see [1], [7], [9], [14], [15] and [19] for a discussion on scalar perturbations, [12] for a discussion on vector perturbations and [18] for tensor perturbations). For example, in [14] it was shown that due to the energy–flow, onto or away from the branes, will modify the evolution of super–horizon amplitudes on the brane. In fact, in contrast to the usual results in four dimensions [13], adiabatic perturbations will not be constant in general. As discussed in [14], this effect is a consequence of the statement that super–horizon amplitudes are constant if the energy–conservation holds [21]. However, energy–conservation can easily be violated in brane worlds.

Large scale anisotropies for a brane in an Anti–de Sitter bulk were discussed in [19]. Also, using the gauge presented in this paper, the full set of equations on the brane for this scenario was derived in [20]. Hence, there is the hope,

2The propagation of linear gravitational waves is determined by the first order perturbation of the vacuum Einstein equations. However, the backreaction of the gravitational waves over the background geometry is determined by Einstein equations with a non–zero stress–energy tensor associated to the waves. Such a tensor can be obtained by non–local averages of quadratic combinations of the first order gravitational wave amplitudes.
that the formalism presented here can be used to fully understand the interaction between the bulk gravitational field and the brane matter.

Note added: While this paper was prepared for publication, ref. [11] appeared, which addressed some of the questions in section 2.

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