Quantum correlation functions and the classical limit

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Abstract

We study the transition from the full quantum mechanical description of physical systems to an approximate classical stochastic one. Our main tool is the identification of the closed-time-path (CTP) generating functional of Schwinger and Keldysh with the decoherence functional of the consistent histories approach. Given a degree of coarse-graining in which interferences are negligible, we can explicitly write a generating functional for the effective stochastic process in terms of the CTP generating functional. This construction gives particularly simple results for Gaussian processes. The formalism is applied to simple quantum systems, quantum Brownian motion, quantum fields in curved spacetime. Perturbation theory is also explained. We conclude with a discussion on the problem of backreaction of quantum fields in spacetime geometry.

I Classical vs. quantum probability

I.1 Introduction

The emergence of classical behaviour in quantum systems is a very important question on the foundations of quantum theory. An explanation of how the classical world emerges is absolutely essential for any scheme that has ambitions to go beyond the operational description of the Copenhagen interpretation. In recent years the programme of decoherence has provided some insight on how this transition is effected and suggested branches of physics, where relevant phenomena are important, like quantum optics and mesoscopic physics.

From another perspective the issue of classicalisation is of significance in cosmology. We want to know how the perceived classical world is obtained from a (presumably) quantum underlying description: in the early universe processes

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are assumed to be governed by quantum field theory, but later a classical hydro-
dynamic description suffices to capture all relevant physics. The same question
is asked for quantum gravity: only now the focus is on the emergence of clas-
sical spacetime rather than the matter fields. At a more technical level one is
interested to know, when the semiclassical gravity approximation is valid, i.e.
the coupling of classical metric variables to quantum fields.

In all such discussions, the first step is to establish what is meant by classical
behaviour. The notion of classicality can be defined in different ways, according
to the context. For instance:

- the absence of interferences in a given basis: in other words an approximate
diagonalisation of the density matrix [1, 2, 3].
- determinism or approximate determinism or some form of predictability [4, 5,
6].
- the validity of a hydrodynamic or thermodynamic description for a many-body
system [5, 7, 8].
- the existence of exact or approximate superselection rules [9].

Whatever the definition of classicality might be, there is a consensus about
how it appears. Coarse-graining is necessary. Since the underlying theory is as-
sumed to be quantum theory (which is by definition non-classical), one can get
a different behaviour only by examining a truncated version of the theory. The
intuitive picture for emergent classicality is that of a random phase approxima-
tion: the coarser the description of the system the more the interference phase
will cancel out when averaged within the coarse-grained observable. The general
question is then, which types of coarse-graining can regularly lead to classical
behaviour.

In this paper, we take the attitude that a system exhibits classical behaviour
if it admits an approximate description in terms of classical probability theory.
Since we are interested in systems changing in time, we ask that the evolution
of coarse-grained observables is described by probability theory: in other words
that it should be modeled by a stochastic process. Quantum processes have
an important difference from stochastic processes: their correlation functions
are complex-valued rather than real-valued. This is equivalent to the fact that
quantum mechanical evolution cannot be described by a probability measure.

We will focus on how classical correlation functions can be constructed from
the quantum mechanical ones through coarse-graining. We shall then identify a
prescription that allows us to implement this effective description for a general
quantum system. A part of the relevant material has appeared previously in
[10]. This presentation is simultaneously an elaboration and a simplification
of the mathematical constructions performed in this reference, with an eye to
possible applications.

We shall then apply this formalism in various cases. We will show that in
Gaussian systems, the classical limit is mostly determined by the real part of the quantum two-point function. We shall verify this in a number of examples: simple harmonic oscillators, the Caldeira-Leggett model of quantum Brownian motion, scalar fields in Minkowski and general spacetimes. We shall then discuss the perturbation expansion, from which we shall infer that a perturbation expansion of the quantum theory does not imply a perturbation expansion for the corresponding stochastic one. We conclude with a discussion of the validity of the semiclassical approximation in quantum gravity. This is a topic, for which our formalism is particularly adequate to address.

The first step is, however, a brief summary of classical probability theory.

I.2 Classical probability

In classical probability one assumes that at a single moment of time the possible elementary alternatives lie in a space \( \Omega \), the sample space. Observables are functions on \( \Omega \), and are usually called random variables.

The outcome of any measurement can be phrased as a statement that the system is found in a given subset \( C \) of \( \Omega \). Hence the set of certain well-behaved (measurable) subsets of \( \Omega \) is identified with the set of all coarse-grained alternatives of the system. To each subset \( C \), there corresponds an observable \( \chi_C(x) \), the characteristic function of the set \( C \). It is defined as \( \chi_C(x) = 1 \) if \( x \in C \) and \( \chi_C(x) = 0 \) otherwise. It is customary to denote the characteristic function of \( \Omega \) as 1 and of the empty set as 0.

Note that if an observable \( f \) takes values \( f_i \) in subsets \( C_i \) of \( \Omega \), we have that

\[
f(x) = \sum_i f_i \chi_{C_i}(x)
\]

A state is intuitively thought of as a preparation of a system. Mathematically it is represented by a measure on \( \Omega \), i.e. a map that to each alternative \( C \) it assigns its probability \( p(C) \). It has to satisfy the following properties

- for all subsets \( C \) of \( \Omega \), \( 0 \leq p(C) \leq 1 \)
- \( p(0) = 0; p(1) = 1 \).
- for all disjoint subsets \( C \) and \( D \) of \( \omega \), \( p(C \cup D) = p(C) + p(D) \)

Due to (1.1) one can define \( p(f) = \sum_i f_i p(C_i) \); \( p(f) \) is then clearly the mean value of \( f \). In the case that \( \Omega \) is a subset of \( \mathbb{R}^n \), the probability measures are defined in terms of a probability distribution, i.e. a positive function on \( \Omega \), which we shall (abusingly) denote as \( p(x) \).

\[
p(f) = \int dx p(x)f(x)
\]

Operationally, one interprets probabilities through ensemble frequencies. The probability distribution is then viewed as corresponding to an ensemble
of a large number \( (N) \) of identically prepared systems. The preparation procedure determines the state of the system. An alternative \( C \) is a filter out of which an individual system can either pass or not. If \( n_C(N) \) denotes the number of systems that have passed from \( C \), then according to the frequency interpretation of probability \( n_C(N)/N \rightarrow \mu(C) \) as \( N \rightarrow \infty \).

We also have a conditional probability interpretation: after an ensemble corresponding to a probability distribution \( p(x) \) has passed from a filter \( C \), the subensemble is described by the probability distribution \( p(x) \chi_C(x)/p(C) \).

Assume now that we have prepared a system in a state \( p \) and we want to perform a series of measurements of an observable \( f = \sum_i f_i \chi_{C_i}(x) \) at time \( t_1 \) and of \( g = \sum_j g_j \chi_{D_j}(x) \) at time \( t_2 > t_1 \). For simplicity, we shall ignore any self-dynamics of the physical systems as it evolves from \( t_1 \) to \( t_2 \). We can consider a number of measurement situations, labeled by \( i \) and \( j \), corresponding to an arrangement where the filter \( C_i \) is placed at time \( t_1 \) and the filter \( D_j \) at time \( t_2 \). From a series of measurements one will establish the number of systems in the ensemble that pass both filters and hence identify the probability

\[
p(i, t_1; j, t_2) = \int dx p(x) \chi_{D_j}(x) \chi_{C_i}(x) \tag{I.3}
\]

Performing this experiment for all different choices of \( i \) and \( j \), we can construct the statistical correlation function

\[
\langle f_{t_1} g_{t_2} \rangle = \sum_{ij} f_i g_j p(i, t_1; j, t_2) = p(\text{BA}) \tag{I.4}
\]

More generally the system might have intrinsic dynamics. This would correspond to a map \( \tau_{t_1, t_2} \) that takes the state \( p(x) \) at time \( t_1 \) to the state \( \tau_{t_1, t_2}[p](x) \) at time \( t_2 \), in such a way as to preserve normalisation and positivity. The correlation function should read

\[
\langle f_{t_1} g_{t_2} \rangle = \int dx g(x) \tau_{t_1, t_2}[fp](x) \tag{I.5}
\]

When we want to study properties of the system at more than one moment of time, we need to introduce a sample space for histories. If we denote by \( T \) the set of all possible time instants, we can identify \( \Omega^T \) as a suitable subset of \( \times_{t \in T} \Omega_t \), where \( \Omega_t \) is a copy of the system’s sample space labeled by a moment of time \( t \). The elements of \( \Omega^T \) are paths \( t \rightarrow x_t \) and will be denoted as \( x(\cdot) \). A history observable is a function on \( \Omega^T \). Given a function \( f \) on \( \Omega_t \), we can define a family of history observables \( F_t \) defined as

\[
F_t[x(\cdot)] = f(x(t)) \tag{I.6}
\]

The state is represented by a probability measure \( P \) on \( \Omega^T \). It contains information about both initial condition and the dynamics and for any function
\[ F \text{ on } \Omega^T \text{ it gives its mean value } P(F). \text{ We can, abusively, write it in terms of a probability distribution as} \]
\[ P(F) = \int Dx(\cdot)P[x(\cdot)]F[x(\cdot)] \quad (I.7) \]

The correlation functions \( \langle f_{t_1} g_{t_2} \rangle \) can then be written as \( P(F_{t_1} G_{t_2}) \) in terms of the functions \( F_t \) and \( G_t \) defined by (1.6). The information of the correlation functions of a single observable \( f \) is contained in the generating functional

\[ Z_f[J(\cdot)] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dt_1 \ldots dt_n \langle f_{t_1} \ldots f_{t_n} \rangle J(t_1) \ldots J(t_n) = \int Dx(\cdot)P[x(\cdot)] \exp \left( i \int dt F_t[x(\cdot)]J(t) \right) \quad (I.8) \]

The generating functional is essentially the Fourier transform of the probability measures. The definition can be extended for families of observables \( f_i \) rather than a single one. Since correlation functions can be operationally determined, it is possible in principle to determine the probability measure with arbitrary accuracy.

### I.3 Quantum correlations

The brief summary of probability theory and the role of the temporal correlations was important in order to establish notation. It also enabled to identify the operational meaning of correlation functions in classical probability, in a way that can be transferred in the quantum context.

The corresponding structures for a single moment of time are well known in standard quantum theory. Elementary alternatives are rays on a complex Hilbert space \( H \), observables are self-adjoint operators on \( H \), a general property (a filter) corresponds to a projection operator and a state to a density matrix.

Let us now consider an ensemble of quantum systems prepared in a state described by a density matrix \( \hat{\rho} \) and try to operationally construct the correlation function of two observables \( \hat{A} = \sum_i \hat{P}_i \) and \( \hat{B} = \sum_j \hat{Q}_j \) at times \( t_1 \) and \( t_2 > t_1 \) respectively. Here \( \hat{P}_i \) are an exhaustive (\( \sum_i \hat{P}_i = 1 \) ) and exclusive (\( \hat{P}_i \hat{P}_j = \hat{P}_i \delta_{ij} \) ) set of projectors, and so is \( \hat{Q}_j \).

Let the Hamiltonian of the system be \( H \) and \( \hat{\rho}_0 \) the state of the system at time \( t = 0 \). Then a series of measurements will enable us to identify the probability that both \( \hat{P}_i \) and then \( \hat{Q}_j \) where found true. According to the rules of quantum theory this will be

\[ p(i,t_1;j,t_2) = \text{Tr} \left( \hat{Q}_j e^{-iH(t_2-t_1)} \hat{P}_i e^{-iHt_1} \hat{\rho}_0 e^{iHt_1} \hat{P}_i e^{iH(t_2-t_1)} \right) \]

\[ = \text{Tr} \left( \hat{Q}_j(t_2) \hat{P}_i(t_1) \hat{\rho}_0 \hat{P}_i(t_1) \right), \quad (I.9) \]
where we used the Heisenberg picture notation for operators 
\[ \hat{A}(t) = e^{i\hat{H}t}Ae^{-i\hat{H}t}. \]
If we now vary over all possible values of \(i\) and \(j\), we can construct the statistical correlation function between \(\hat{A}\) and \(\hat{B}\)
\[ \langle \hat{A}_{t_1}\hat{B}_{t_2} \rangle_S = \sum_{ij} a_i b_j p(i,t_1; j, t_2) \quad (I. 10) \]
But this correlation function is not what one usually calls correlation function in quantum theory. This name is usually employed for the expectation of a product of operators
\[ \langle \hat{A}(t_1)\hat{B}(t_2) \rangle_Q = \text{Tr} \left( \rho \hat{A}(t_1)\hat{B}(t_2) \right) = \sum_{ij} a_i b_j \text{Tr} \left( \rho \hat{P}_i(t_1)\hat{Q}_j(t_2) \right). \quad (I. 11) \]
This is a complex-valued object, in contrast to (1.10) that was constructed using frequencies of events and can only be real-valued. What does then the quantum mechanical correlation correspond to? Clearly it is unlike classical correlations. The fact that it is complex-valued suggests that it has something to do with quantum mechanical quantities such as interference phases. This remark turns out to be accurate. In [11] a scheme was described, in terms of which the quantum mechanical correlation functions can be operationally measured. It proceeds essentially by measuring interference phases between different states. It is a measurement procedure similar to ones used for the Aharonov-Bohm effect or the Berry phase [12]. This is natural in a sense, since the Berry phase is the irreducible element for which quantum theory necessitates the use of complex numbers [13]. However, in the present paper we are interested in the classical limit rather than the full structure of quantum theory and we shall not pursue this topic. The interested reader is referred to [11].
We now want to check the possibility that the quantum and the statistical correlation functions coincide. An easily discernible case is when \([\hat{A}(t_1), \hat{B}(t_2)] = 0\). More generally, it can be checked that a necessary and sufficient condition is
\[ \text{Re} \text{Tr} \left( \hat{Q}_j(t_2)\hat{P}_i(t_1)\rho\hat{P}_{i'}(t_1) \right) = 0, \quad (I. 12) \]
for all \(i, j\) and \(i' \neq i\). In this case the following property is satisfied
\[ \sum_i p(i, t_1; j, t_2) = \text{Tr}(\rho\hat{Q}_j(t_2)) = p(j, t_2) \quad (I. 13) \]
for all \(j\). This implies that the probabilities assigned to the set of all possible histories satisfy the additivity condition. They therefore define a classical probability measure. It is evident that in this case the quantum and the statistical correlation functions would coincide.
This condition for classicality is exactly the one upon which the formalism of consistent histories is based. This formalism is an indispensable part of our analysis and we therefore proceed to examine it next.
II Quantum processes

II.1 Consistent histories

The consistent histories approach to quantum theory was developed by Griffiths [14], Omnés [4], Gell-Mann and Hartle [15, 5, 6]. The basic object is a history, which corresponds to properties of the physical system at successive instants of time. A discrete-time history $\alpha$ will then correspond to a string $\hat{P}_t_1, \hat{P}_t_2, \ldots, \hat{P}_t_n$ of projectors, each labeled by an instant of time. From them, one can construct the class operator

$$\hat{C}_\alpha = \hat{U}^\dagger(t_1)\hat{P}_t_1\hat{U}(t_1)\ldots\hat{U}^\dagger(t_n)\hat{P}_t_n\hat{U}(t_n)$$  \hspace{1cm} (II. 1)

where $\hat{U}(s) = e^{-i\hat{H}s}$ is the time-evolution operator. The probability for the realisation of this history is

$$p(\alpha) = \text{Tr} \left( \hat{C}_\alpha^\dagger \hat{\rho}_0 \hat{C}_\alpha \right),$$  \hspace{1cm} (II. 2)

where $\hat{\rho}_0$ is the density matrix describing the system at time $t = 0$.

But this expression does not define a probability measure in the space of all histories, because the Kolmogorov additivity condition cannot be satisfied: if $\alpha$ and $\beta$ are exclusive histories, and $\alpha \lor \beta$ denotes their conjunction as propositions, then it is not true that

$$p(\alpha \lor \beta) = p(\alpha) + p(\beta).$$  \hspace{1cm} (II. 3)

The histories formulation of quantum mechanics does not, therefore, enjoy the status of a genuine probability theory.

However, an additive probability measure is definable, when we restrict to particular sets of histories. These are called consistent sets. They are more conveniently defined through the introduction of a new object: the decoherence functional. This is a complex-valued function of a pair of histories given by

$$d(\alpha, \beta) = \text{Tr} \left( \hat{C}_\alpha^\dagger \hat{\rho}_0 \hat{C}_\beta \right).$$  \hspace{1cm} (II. 4)

A set of exclusive and exhaustive alternatives is called consistent, if for all pairs of different histories $\alpha$ and $\beta$, we have

$$\text{Re} \ d(\alpha, \beta) = 0.$$  \hspace{1cm} (II. 5)

In that case one can use equation (2.2) to assign a probability measure to this set. The consistent histories interpretation then proceeds by postulating that any prediction or retrodiction, we can make based on probabilities has always to make reference to a given consistent set. This leads to counter-intuitive situations, of getting mutually incompatible predictions, when reasoning within different consistent sets. The predictions of this theory are therefore contextual:
but in any case, this is a general feature of all realist interpretations of quantum theory.

Except for trivial cases, it is only coarse-grained observables that satisfy an exact (or approximate) consistency condition. This means that the histories are constructed out of projectors $P$, whose trace is much larger than unity.

II.2 The Closed-Time-Path generating functional

We saw that in quantum theories probabilities and statistical correlations are contained in the decoherence functional; in fact, in its diagonal elements. We shall now show that the same is true for the quantum correlation functions.

Recall that in the decoherence functional projectors enter in a time-ordered series. This suggests that it would be best to use time-ordered correlation functions. Let $A^n$ denote a family of commuting operators. Then the time-ordered two-point correlation function is defined as

$$G^{2,0}(a_1, t_1; a_2, t_2) = \theta(t_2 - t_1)\text{Tr}[\hat{\rho}_0 \hat{A}^{a_1}(t_1) \hat{A}^{a_2}(t_2)] + \theta(t_1 - t_2)\text{Tr}[\hat{\rho}_0 \hat{A}^{a_2}(t_2) \hat{A}^{a_1}(t_1)]$$ (II. 6)

One can similarly define time-ordered $n$-point functions, or anti-time-ordered

$$G^{0,2}(a_1, t_1; a_2, t_2) = \theta(t_1 - t_2)\text{Tr}[\hat{\rho}_0 \hat{A}^{a_1}(t_1) \hat{A}^{a_2}(t_2)] + \theta(t_2 - t_1)\text{Tr}[\hat{\rho}_0 \hat{A}^{a_2}(t_2) \hat{A}^{a_1}(t_1)]$$ (II. 7)

In general, one can define mixed correlation functions $G^{r,s}$, with $r$ time-ordered and $s$ anti-time-ordered entries, as for instance

$$G^{2,1}(a_1, t_1; a_2, t_2|b_1, t_1') = \theta(t_2 - t_1)\text{Tr}[\hat{A}^{b_1}(t_1') \hat{\rho}_0 \hat{A}^{a_1}(t_1) \hat{A}^{a_2}(t_2)] + \theta(t_1 - t_2)\text{Tr}[\hat{A}^{b_1}(t_1') \hat{\rho}_0 \hat{A}^{a_2}(t_2) \hat{A}^{a_1}(t_1)]$$ (II. 8)

These correlation functions are generated by the Closed-Time-Path (CTP) generating functional associated to the family $A^n$

$$Z_A[J_+, J_-] = \sum_{n,m=0}^{\infty} \frac{i^n(-i)^m}{n!m!} \int dt_1 \ldots dt_n dt_1' \ldots dt_m'$$

$$G^{n,m}(a_1, t_1; \ldots; a_n, t_n|b_1, t_1'; \ldots; b_m, t_m')\delta^{a_1}(t_1) \ldots J^{a_n}(t_n) J^{b_1}(t_1') \ldots J^{b_m}(t_m')$$ (II. 9)

The name closed-time arose, because in the original conception (by Schwinger [16] and Keldysh [17] the time path one follows is from some initial time $t = 0$ to $t \rightarrow \infty$, thus covering all time-ordered points and then back from infinity to 0 covering the anti-time-ordered points. The total time-path is in effect closed.

Conversely the correlation functions can be read from $Z_A$

$$G_{A}^{n,m}(a_1, t_1; \ldots; a_n, t_n|b_1, t_1'; \ldots; b_m, t_m') = (-i)^n i^m \delta^n \delta^m \frac{\delta^{a_1}(t_1) \ldots \delta^{a_n}(t_n) \delta^{b_1}(t_1') \ldots \delta^{b_m}(t_m)}{\delta J^a_+(t_1) \ldots \delta J^a_+(t_n) \delta J^b_+(t_1') \ldots \delta J^b_+(t_m)} Z[J_+, J_-]|_{J_+ = J_- = 0}$$ (II. 10)
II.3 Relation between the functionals

Clearly there must be a relation between the decoherence functional and the CTP one. One can see in the correlation functions, if we assume a single operator \( \hat{A} = \sum_i a_i P_i \) and consider a pair of histories \( \alpha(i_1, t_1; \ldots; i_n, t_n) = \{ P_{i_1}, t_1; \ldots; P_{i_n}, t_n \} \) and \( \beta(i'_1, t'_1; \ldots; i'_m, t'_m) = \{ P_{j_1}, t'_1; \ldots; P_{j_m}, t'_m \} \). Then one can easily verify that

\[
G^{n,m}_A(t_1, \ldots, t_n; t'_1, \ldots, t'_m) = \sum_{i_1 \ldots i_n} \sum_{j_1 \ldots j_m} \sum_{a_{i_1} \ldots a_{i_n}} b_{j_1} \ldots b_{j_m} \times d[\alpha(i_1, t_1; \ldots; i_n, t_n), \beta(j_1, t_1; \ldots; j_m, t_m)] \tag{II. 11}
\]

The straightforward relation is nonetheless not possible to show in an elementary fashion. One needs to consider a decoherence functional for continuous-time histories [18, 19, 20, 21] and this requires a significant upgrading of the formalism of quantum mechanical histories. The key idea is to represent histories by projectors on a tensor product of Hilbert spaces \( \otimes_{t \in T} H_t \) [22] in analogy to the construction of the history sample space classically. A suitable Hilbert space (not a genuine tensor product) can be constructed [18] for the case that \( T \) is a continuous set and the decoherence functional can be defined as a bilinear, hermitian functional on this space. It can then be shown that as a functional it is essentially a double "Fourier transform" of the CTP generating functional.

This proof is to be found in [10] and is elementary once one follows the logic of the construction. Here we shall restrict ourselves to a convenient statement of this result. Let us assume that we have a family of commuting self-adjoint operators \( \hat{A}_i \). Their spectrum is then a subset \( \Omega \) of some vector space \( \mathbb{R}^n \). Any function of \( \hat{A}_i \) is in one-to-one correspondence to functions \( f(x) \) with \( x \in \Omega \). Like the classical case we can construct a space of histories \( \Omega^T \) as a suitable subset of \( \times_{t \in T} \Omega_t \). We can then consider subsets of \( \Omega^T \) as corresponding to histories of the quantum mechanical observables \( \hat{A}_i \). This is easily verified if \( T \) is a finite set and in continuous-time case there is a sense in which it is true. The decoherence functional is then a map that to each pair of subsets \( C \) and \( D \) of \( \Omega^T \) it assigns a complex number in such a fashion that the following properties are satisfied [6, 23]

- \( d(D, C) = d^*(C, D) \), hermiticity
- \( d(0, C) = 0 \), null triviality
- \( d(1, 1) = 1 \), normalisation
- \( d(C \cup C', D) = d(C, D) + d(C', D) \) for disjoint \( C \) and \( C' \), additivity.

In fact it can be constructed as a continuum-limit of the discrete-time expressions (2.2). One can write formally the decoherence functional as an integral over \( \Omega^T \times \Omega^T \)

\[
d(C, D) = \int Dx(\cdot)Dx'(\cdot)\Delta[x(\cdot)|x'(\cdot)]\chi_C[x(\cdot)]\chi_D[x'(\cdot)] \tag{II. 12}
\]
Then if $Z_{A}[J_{+}, J_{-}]$ is the CTP generating functional associated to $\hat{A}^{i}$ we have

$$Z_{A}[J_{+}, J_{-}] = \int Dx(\cdot)Dx'\cdot e^{i\int dtJ_{a}^{+}(t)x_{a}^{n}(t)}e^{-i\int dtJ_{a}^{-}(t)x_{a}^{n}(t)}\Delta[x(\cdot)|x'(\cdot)]$$

(II. 13)

In other words viewed as a bi-functional over the functions on $\Omega^{T}$ the decoherence functional is identical to the CTP generating functional. The only difference is on the type of functions upon which they take values - the first on characteristic functions, the second on complex valued functions of unit norm. In fact, equation (2.12) amounts to

$$G_{n,m}(a_{1}, t_{1}; \ldots; a_{n}, t_{n}|b_{1}, t_{1}; \ldots; b_{m}, t_{m}) = \int \int Dx(\cdot)Dx'(\cdot)$$

$$\times x^{a_{1}}(t_{1})\ldots x^{a_{n}}(t_{n})x^{b_{1}}(t')\ldots x^{b_{m}}(t_{m})\Delta[x(\cdot)|x'(\cdot)]$$

(II. 14)

Hence there exists the following correspondence of objects in classical and quantum probability

<table>
<thead>
<tr>
<th>Quantum</th>
<th>Classical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probabilities</td>
<td>$d(C, D)$</td>
</tr>
<tr>
<td>Correlations</td>
<td>$Z[A_{+}, A_{-}]$</td>
</tr>
</tbody>
</table>

The probability measure is a real-valued functional on functions of $\Omega^{T}$, while the decoherence functional is a hermitian bilinear functional on the functions of $\Omega^{T}$. In a given system, one goes from $d$ to $p$, when the decoherence condition (2.5) is satisfied, while in both cases one goes from probabilities to correlations through a Fourier transform.

What we will next show is how to effect the transition from the CTP generating functional to a stochastic process for the coarse-grained variables. Working at the level of the correlation functions makes the construction of stochastic differential equations easier than working at the level of probabilities.

### III  From quantum to classical

#### III.1  The basic choice for coarse graining

In order to study the transition from quantum to classical we need to choose the variables, upon which we shall concentrate. This amounts to a choice of a family $\hat{A}^{i}$ of intercommuting operators. Now a maximal family of intercommuting operators generically contains full information about the evolution of the quantum system (possible exceptions to this rule are trivial cases as for instance when the Hamiltonian and the initial density matrix commute with all $\hat{A}^{i}$).

One can implement a coarse-graining procedure even at this stage. It suffices take for $\hat{A}^{i}$ a non-maximal family of operators. This is the case, for instance, in quantum Brownian motion models. If we assume that the total system consists
of a large number of harmonic oscillators a maximal family of intercommuting operators consists from the position operators of all particles. When we choose to focus on a single one of them, we effectively coarse-grain by treating the rest degrees of freedom as an environment. This is the type of coarse-graining associated with the studies of environment-induced decoherence.

However, this type of coarse-graining does not suffice. One has usually to consider smeared values of the relevant observables. This is effected by considering projectors sufficiently smeared over Ω. We shall take Ω to be a $\mathbb{R}^s$ so its points will be vectors $x^a$. In general it is difficult to work with characteristic functions, so we will work with smeared characteristic functions. If we denote by $|x|$ their Euclidean distance, then a good choice for the projector is the function

$$f_x(x) = \exp(-\frac{1}{2\sigma^2}|x-\bar{x}|^2) \quad \text{(III. 1)}$$

This Gaussian is not a sharp projector; it is strongly peaked in a sphere of length $\sigma$ around the point $\bar{x}$, hence it is a good approximation to a true projector for not very large values of $\sigma$.

We now assume consider a discrete-time history at times $t_i$, consisting each time by projectors $f_{\bar{x}_i}$ centered around $\bar{x}_i$. This can be viewed as a discretised approximation to a coarse-grained history in continuous time centered around a path $t \to \bar{x}(t)$. We now consider two such discretised histories, centered at the same time points $t_i$ each corresponding to a different path $\bar{x}()$ and $\bar{x}'()$. Let us denote them by $\alpha_{\bar{x}}(\cdot)$ and $\alpha_{\bar{x}'}(\cdot)$.

If we expect our system to exhibit classical behaviour then typically the off-diagonal elements of the decoherence functional will fall rapidly whenever $\delta^2 = ||\bar{x}(\cdot) - \bar{x}'(\cdot)||^2 := \sum_i |\bar{x}_i - \bar{x}'(t_i)|^2$ is much larger than $N \times \sigma^2$. (Here $N$ is the number of time-steps). Typically one has

$$d(\alpha_{\bar{x}}(\cdot), \alpha_{\bar{x}'}(\cdot)) = O(e^{-\delta^2/N\sigma^2}), \quad \text{(III. 2)}$$

or some other type of rapid fall-off. For pure initial states this behaviour is typical, when $\sigma^2$ much larger than the uncertainties of the initial state and the Hamiltonian evolution preserves this property [24]. In this case the diagonal elements are close to defining probabilities for coarse-grained histories with width $\sigma$ at each moment of time and centered around $\bar{x}(\cdot)$.

Now, we want to find a probability distribution that approximately would give these values for the probabilities of these histories. A single time projector is centered in a volume of $\Omega$ of size

$$\int dx f_x(x) = (2\pi\sigma^2)^{s/2}. \quad \text{(III. 3)}$$

A probability distribution on the space of discretised) paths that reproduces these expressions for probabilities of these coarse grained sets is

$$p[\bar{x}(\cdot)] = \frac{1}{(2\pi\sigma^2)^{sn/2}}d(\alpha_{\bar{x}}(\cdot), \alpha_{\bar{x}}(\cdot)), \quad \text{(III. 4)}$$

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where \( n \) is the number of time-steps assumed. (Dividing by the volume turns the probabilities of events into a density.)

One can use equation (2.12) to write

\[
d(\bar{\alpha}(\cdot), \bar{\alpha}(\cdot)) = \int D\alpha(\cdot)D\alpha'(\cdot)
\]

\[
\times \exp\left(-\frac{1}{2\sigma^2}\|x(\cdot) - \bar{x}(\cdot)\|^2 - \frac{1}{2\sigma^2}\|x'(\cdot) - \bar{x}(\cdot)\|^2\right)\Delta[x(\cdot)|x'(\cdot)]
\]

(III. 5)

Note that still or expressions are defined with respect to discrete time. From equation (2.13) we see that the kernel \( \Delta \) can be obtained from the inverse Fourier transform of the CTP generating functional. We then obtain

\[
p[\bar{x}(\cdot)] = \int DJ_+ DJ_- e^{-1/4\pi(J_+^2 + J_-^2) - i/J_+ J_-} e^{-\sigma^2/2(J_+^2 + J_-^2)}
\]

(III. 6)

Note, that there is no multiplicative term depending on the number of time steps, and hence from this expression one can safely go to the continuum limit. We have denoted as \( J \cdot x = \sum_i J(t_i) x(t_i) \) and at the continuous limit this expression will correspond to an integral.

To get the generating functional for the classical correlation functions one Fourier transforms the probability measure to get (we now drop the index that refers to our choice of variables for the correlation functions)

\[
\hat{Z}_{cl}[J] = e^{-\frac{\sigma^2}{2}J \cdot J} \int dR(R) e^{-1/2\pi R^2 R} Z[R, \sqrt{2\pi\sigma}, R, \sqrt{2\pi\sigma}]
\]

(III. 7)

with \( R = \frac{1}{2}(J_+ + J_-) \). This \( Z[J] \) needs to be normalised to unity by assuming that \( Z[0] = 1 \). The normalisation condition is not kept, because we have employed approximate characteristic functions. Had we used a sharp characteristic function, the construction would automatically guarantee normalisation \( Z[0] = 1 \). Now there is a deviation from unity of the order of \( \sigma^2 \).

This expression can be simplified. Assume that we have a classical stochastic process for the variables \( x(\cdot) \), with a generating functional \( Z_0[J] \). Let us follow the same procedure for coarse-graining as before, using the approximate projectors (3.1). The coarse-grained generating functional would be

\[
\hat{Z}_{cl}^{cg}[J] = e^{-\frac{\sigma^2}{2}J \cdot J} Z_0[J]
\]

(III. 8)

This means, that we can consider \( Z[J] \) in equation (3.7) as coming from coarse-graining a classical stochastic process, with twice the degree of coarse graining as the one from quantum theory. One can then drop the term outside the integral in (3.7) as coming from coarse-graining of an underlying stochastic process given by
This equation gives the stochastic correlation functions in the classical limit of the quantum system described by the CTP generating functional $Z[J_+, Z_-]$. It should be always kept in mind, that this process gives reliable results only on scales much larger than $\sigma^2$.

We should now pause for a minute and examine the assumptions we used in order to arrive here.

First, one should ask what is the meaning of the parameter $\sigma$. Is it arbitrary or not? In principle it is not. It is the necessary degree of coarse-graining before the fall-off (2.5) of the off-diagonal elements of the decoherence functional is manifested. It, therefore, has to be much larger than the natural scale associated with microscopic processes; however compared to macroscopic scales it ought to be considered small. In principle, it can be identified by a full study of the decoherence functional. Usually a measure of coarse-graining is the trace of the corresponding positive operator. However, here, since we have considered only commuting variables with continuous spectrum, the trace of the operators corresponding to (3.7) is infinite. (Essentially it means that if we focus on position, momenta can take arbitrary values). This is related to the fact that there is no default universal scale, by which to judge whether $\sigma$ is large. This problem is remedied by considering phase space coarse-grainings as we shall see shortly. In this case the natural scale is $\hbar = 1$ and one can say that $\sigma^2 >> 1$ in order to have consistency of histories. It should nonetheless be much smaller than a macroscopic time scale by which we observe phenomena.

More precisely, the stochastic approximation (3.9) is accurate within an order of $(l_{mic}/\sigma)^2$, where $l_{mic}$ is the microscopic scale that is determined by the dynamics or the initial state. However, there is also an error proportional to $(\sigma/L_{mac})^2$, where $L_{mac}$ is the macroscopic scale of observation, i.e. the scale of accuracy we are interested in having. This is due to the use of the Gaussian approximation for the projectors. Overall we have an error of the order of

$$c_1(l_{mic}/\sigma)^2 + c_2(\sigma/L_{mac})^2,$$  

(III. 10)

where $c_1$ and $c_2$ are constants of the order of unity. It is, therefore, evident that a separation of scales is necessary if the stochastic description is to make any sense.

Second, the general logic of this construction is to identify a stochastic process that adequately describes the evolution of the classicalised coarse-grained observables. There is a subtle difference from the consistent histories scheme, in that we do not seek to construct consistent sets for the system and hence make statements about individual quantum systems. Our approach is more operational. Given that quantum theory is a model that provides the statistical behaviour of physical systems, we ask to construct a different model based on
probability theory that describes some regime of the same physical system. For this purpose we utilise the consistency condition in order to identify the validity of our approximation and build the probability distribution from the diagonal elements of the decoherence functional.

III.2 Phase space coarse grainings

One does not have to restrict to correlation functions of a family of commuting operators in order to construct the CTP generating functional. By considering correlation functions in both position and momentum it is possible to generalise the definition (2.9). Indeed in this equation it is not necessary to assume that the operators $\hat{A}^a$ are commuting. The reason we assumed commutativity was in order to get a correspondence with the decoherence functional, that can be described in terms of paths on the common spectrum of this operators.

However, in any Hilbert space that there exists a representation of the canonical commutation relations

$$[\hat{q}^i, \hat{p}^j] = i\delta^{ij} \quad \text{(III. 11)}$$

it is possible to assign a function on the phase space $\Gamma = \{(q^i, p^i)\}$ for each operator by means of the Wigner transform

$$\hat{A} \rightarrow F_A(q, p) = \int d\xi d\chi e^{-i\xi q - i\chi p} \text{Tr} \left( \hat{A} e^{i\xi \hat{q} + i\chi \hat{p}} \right) = \text{Tr} \left( \hat{A} \hat{\Delta}(q, p) \right) \quad \text{(III. 12)}$$

An important property of this transform is that it preserves the trace:

$$\text{Tr} \hat{A} = \int dq dp F_A(q, p) \quad \text{(III. 13)}$$

However, the Wigner transform does not preserve multiplication of operators. The defining condition $\hat{P}^2 = \hat{P}$ for projectors is therefore not preserved and a projector is mapped into some general positive function, rather than a characteristic function of a subset of $\Gamma$.

Since any operator can be represented by a function of $\Omega$, histories would be represented by functions on a space $\Gamma^T$, which will be a suitable subspace of $\times_{t \in T} \Gamma_t$. In reference [10] it was shown that a decoherence functional can be constructed as an hermitian bilinear functional on the space of functions of $\Gamma^T$. And it is related to the CTP generating functional by means of a Fourier transform.

All the formulas in the previous paragraph can then be reinterpreted to fit the phase space context, by allowing the variables $x$ to denote both $q$ and $p$. (The dimension of $\Gamma$ is clearly even). The main difference is that the Gaussian function $f_x$ corresponds to an operator $\hat{F}$ with a finite trace. By virtue of (2.1) and (2.13)

$$\text{Tr} \hat{F} = (2\pi \sigma^2)^{r/2} \quad \text{(III. 14)}$$
Hence $\sigma$ has units of action is an absolute measure of the degree of coarse-graining of the description. Consistency occurs whenever $\sigma^2 >> \bar{\hbar}$, where $\bar{\hbar}$ provides the natural length scale on phase space. In fact, in the study, of a large class of closed quantum systems Omnés has showed [25] that the off diagonal elements of the decoherence functional fall in the order of $(\hbar/\sigma)^{r/4}$, where $r = 2k$ is the dimension of $\Gamma$. Hence, even if $\hbar/\sigma \sim 10^{-8}$ there is a substantial degree of decoherence to justify the use of classical probability and $\sigma$ is still sufficiently small compared to some external macroscopic scales to justify the use of the Gaussian approximation for the projector. From a macroscopic perspective it would be sufficient to consider the leading order in $\sigma^2$ of the correlation functions.

The study of phase space histories is more intricate, though, because one has to choose proper units for position and momentum, by which to write Euclidean norm in the coarse-grained projector (2.1). For classicality it is not only necessary to have a large value of $\sigma$, but the choice of units has to be preserved by the dynamical evolution [25, 26]. This is a non-trivial condition that largely depends on the system’s Hamiltonian. For this we shall prefer to employ configuration space coarse-grainings.

Whenever we have a representation of the canonical commutation relations we can define the coherent states

$$|z\rangle = |\chi\xi\rangle = e^{i\hat{q}\xi + i\hat{p}\chi}|0\rangle,$$

where $|0\rangle$ is a fiducial vector, often taken to be the lowest energy eigenstate. The important point is that one can assign to a large class of density matrices $\hat{\rho}$ a function $f_\rho(\chi, \xi)$ (its P-symbol) defined by

$$\hat{\rho} = \int d\chi d\xi f(\chi, \xi)|\chi\xi\rangle\langle \chi\xi|.$$

If one then denotes by $Z_{\chi_0\xi_0}[J_+, J_-]$ the CTP generating functional corresponding to an initial state given by $|\chi_0\xi_0\rangle$ then the CTP generating functional for the same system but a different initial state $\hat{\rho}$

$$Z[J_+, Z_-] = \int d\chi_0 d\xi_0 f_\rho(\chi_0, \xi_0)Z_{\chi_0\xi_0}[J_+, J_-]$$

and a similar equation would hold for the classical limit, provided that the degree of coarse-graining necessary for decoherence is determined by the study of the state $\hat{\rho}$ rather than the coherent states.

### III.3 Gaussian processes

Let us now consider a quantum system described by a Gaussian CTP generating functional. Its most general form would be

$$Z[J_+, J_-] = \exp \left( -\frac{i}{2} J_+ \cdot \Delta \cdot J_+ + \frac{i}{2} J_- \cdot \bar{\Delta} \cdot J_- \right)$$
Here we have denoted by $\Delta$ kernels of the form $\Delta^{ab}(t, t')$, by $J \cdot \Delta \cdot J' = \int dt dt' J^a(t) \Delta^{ab}(t, t') J^b(t')$ and the bar denotes complex conjugation. $X$ denotes the one-point function $G^{10} = G^{01}$ and

$$i \Delta^{ab}(t, t') = G^{2,0}(a, t; b, t') - X(a, t)X(b, t') \quad \text{(III. 19)}$$

$$i K^{ab}(t, t') = G^{1,1}(a, t|b, t') + G^{1,1}(b, t'|a, t) - 2X(a, t)X(b, t') \quad \text{(III. 20)}$$

We can write $\Delta = \Delta_1 - i \Delta_2$ and $K = K_1 - i K_2$, in terms of the real-valued kernels $\Delta_1, \Delta_2, K_1, K_2$. The hermiticity condition on the CTP generating functional would then entail

$$\Delta_1^T = \Delta_1 \quad \Delta_2^T = \Delta_2 \quad \text{(III. 21)}$$

$$K_1 = 0 \quad K_2 = 2 \Delta_2 \quad \text{(III. 22)}$$

Evaluating the integral (3.7) yields

$$Z^{cl}[J] = e^{-J \cdot \Xi \cdot J + iJ \cdot X} \quad \text{(III. 23)}$$

where

$$\Xi = \Delta_2 + \frac{1}{4 \sigma^2} \Delta_1 \cdot \Delta_1 \quad \text{(III. 24)}$$

It is worth noticing, that whenever the term $\Delta_2$ is dominant, the classical two-point function is independent of the coarse-graining scale and equal to the real part of the quantum two-point function. However, this simplification can occur only in Gaussian systems.

**IV Examples**

**IV.1 Harmonic oscillators**

For a single harmonic oscillator with frequency $\omega$ and mass $m$ in a thermal state, we have for the configuration space correlation functions

$$\Delta_1(t, t') = -\frac{1}{2m\omega} \sin(\omega|t - t'|) \quad \text{(IV. 1)}$$

$$\Delta_2(t, t') = \frac{1}{2m\omega} \coth(\beta\omega/2) \cos(\omega(t - t')) \quad \text{(IV. 2)}$$

and equation (3.24) gives $\Xi(t, t')$.

One should recall that the smearing scale $\sigma$ is determined by the condition (2.5) on the fall-off of the diagonal elements of the decoherence functional. Here $\sigma^2$ should be much larger than $(2m\omega)^{-1}$, the position uncertainty of the *ground
state. This can be verified by direct evaluation, but it is made plausible by the following observation: a thermal state has a positive P-symbol, and hence its quantum behaviour is identical to the one of the coherent states, which in a Gaussian system is identical with that of the vacuum.

The term $\Delta \Delta$ is proportional to $(\sigma^2 m \omega)^{-1}$, hence comparatively small. In particular, at high temperature $\beta \omega \ll$ the $\Delta \Delta$ term is clearly dominant, the correlation function is $\sigma$ - independent and one recovers the classical result.

Let us recall that this system does not describe a harmonic oscillator in contact with a heat bath; it describes a closed system, evolving unitarily that is prepared in a thermal state (whatever that might mean). Physically more relevant is the case of an oscillator undergoing quantum Brownian motion, to be taken up later.

But we shall first examine the case, where the system is initially prepared in a squeezed state. A squeezed state $|r, \phi\rangle$ is the zero eigenstate of the operator

$$\hat{b} = \cosh r/2\hat{a} + \sinh r/2e^{i\phi}\hat{a}^\dagger,$$

where $r \geq 0$. The correlation function $\Delta_1$ is identical to the one for the vacuum case while

$$\Delta_2(t, t') = \frac{1}{2m\omega}(\cosh r \cos \omega(t - t') + \sinh r \cos \omega(t + t' - \phi))$$

Clearly it is necessary that $\sigma^2 > \frac{\cosh r}{2m\omega}$ in order to have decoherence. In that case the $\Delta_2$ term is again negligible. For values of $r$ at the order of unity, it is not different from the vacuum case, but for large $r$ the degree of coarse-graining necessary for classicality might become too large to allow us to obtain any useful information. This is what is meant, when we say that squeezed states are highly non-classical states.

### IV.2 Quantum Brownian motion

We shall study here the Caldeira-Leggett model [27, 28], i.e. a single harmonic oscillator of mass $M$ and frequency $\omega$ coupled linearly to a bath of harmonic oscillators in a thermal state. More precisely the system is defined by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2M} + \frac{1}{2} M \omega^2 \hat{x}^2 + \hat{x} \sum_i c_i \hat{q}_i + \sum_i \left( \frac{\hat{p}_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 \hat{q}_i^2 \right)$$

From the Heisenberg equations of motion we get

$$M \frac{d^2}{dt^2} \hat{x} + M \omega^2 \hat{x}^2 = -\sum_i c_i \hat{q}_i$$

$$\frac{d^2}{dt^2} \hat{q}_i + \omega_i^2 \hat{q}_i^2 = -\frac{c_i}{m_i} \hat{x}$$
The second equation has a solution
\[ \dot{q}_i(t) = \dot{q}_0 i \cos \omega_i t + \frac{\dot{p}_{0i}}{m_i \omega_i} \sin \omega_i t - \frac{c_i}{m_i \omega_i} \int_0^t ds \sin \omega_i (s - s') \hat{x}(s), \] (IV. 9)
which when substituted into (4.7) yields
\[ \frac{d^2}{dt^2} \hat{x} + \omega^2 \hat{x} - \frac{2}{M} \int_0^t ds \eta(t - s) \hat{x}(s) = -\frac{1}{M} \sum_i c_i \dot{q}_i(t) \] (IV. 10)
Here
\[ \eta(s) = \sum_i \frac{c_i^2}{2m_i \omega_i} \sin \omega_i s, \] (IV. 11)
is known as the dissipation kernel. Let us denote by \( u(t) \) the solution of the homogeneous equation corresponding to (4.10) with the initial conditions \( u(0) = 1 \) and \( \dot{u}(0) = 0 \). It can be identified as the inverse Laplace transform of the function
\[ \tilde{u}(s) = \frac{1}{s^2 + \omega^2 - 2/M \eta(s)}, \] (IV. 12)
where \( \tilde{\eta} \) is the Laplace transform of the dissipation kernel. We can then write the solution of (4.10) as
\[ \hat{x}(t) = \hat{x}_0 u(t) + \frac{\dot{p}_0}{M} \tilde{u}(t) - \frac{1}{M} \sum_i c_i \int_0^t ds u(t - s) \cos \omega_i s + \frac{\dot{p}_{0i}}{m_i \omega_i} \int_0^t ds u(t - s) \sin \omega_i s \] (IV. 13)
Now we assume that the initial state of the system is factorisable to a thermal state at temperature \( T = \beta^{-1} \) for the environment and a density matrix \( \hat{\rho}_0 \) for the distinguished oscillator. In this case, we can easily see that the expectation value
\[ x(t) = \text{Tr}[\hat{\rho}_0 \hat{x}(t)] = x_0 u(t) + \frac{p_0}{M} \tilde{u}(t), \] (IV. 14)
is a solution of the dissipative equations of motion, while the two-point function reads
\[ \text{Tr}[\hat{\rho}_0 \hat{x}(t) \hat{x}(t')] = (\Delta x_0)^2 u(t) u(t') + \frac{(\Delta p_0)^2}{M^2} \tilde{u}(t) \tilde{u}(t') + \frac{C_{pq}}{M} \left[ u(t) \dot{u}(t') + u(t') \dot{u}(t) \right] + \frac{i}{2M} \left[ u(t) \ddot{u}(t') - u(t') \ddot{u}(t) \right] + \frac{1}{M^2} \int_0^t ds \int_0^t ds' u(s) \nu(s - s') u(s') + i \int_0^t ds \int_0^t ds' u(s) \eta(s - s') u(s') \] (IV. 15)
Here $\Delta x_0$, $\Delta p_0$ and $C_{pq}$ refers to the uncertainties and correlation between position and momenta at $t = 0$ and

$$\nu(s) = \sum_i \frac{c_i^2}{2m_i\omega_i} \coth \frac{\beta \omega_i}{2} \cos \omega_i s \quad \text{(IV. 16)}$$

is known as the noise kernel.

From this equation, it is easy to determine the kernels $\Delta$ and $K$. If we write the last line in equation (4.15) as $rac{1}{M^2}[N(t, t') + iH(t, t')]$ we have

$$\Delta_1(t, t') = \theta(t - t') \left[ \frac{1}{M}[u(t)\dot{u}(t') - u(t')\dot{u}(t)] + \frac{1}{M^2}H(t, t') \right]$$

$$+ \theta(t' - t) \left[ \frac{1}{M}[u(t')\dot{u}(t) - u(t)\dot{u}(t') + \frac{1}{M^2}H(t', t)] \right] \quad \text{(IV. 17)}$$

$$\Delta_2(t, t') = (\Delta x_0)^2u(t)u(t') + \frac{(\Delta p_0)^2}{M^2}\dot{u}(t)\dot{u}(t')$$

$$+ \frac{C_{pq}}{M}[u(t)\dot{u}(t') + u(t')\dot{u}(t)] + \frac{1}{M^2}N(t, t') \quad \text{(IV. 18)}$$

Now we want to derive the stochastic limit, to which these quantum correlation functions correspond. We shall use equation (3.24). Note however, that the necessary degree of coarse-graining in order to achieve decoherence must be in a phase space region much larger than the one occupied by the initial state. Hence we take the coarse-graining scale to be much larger than the uncertainties and correlations of the initial state. This allows us to drop all terms but $N(t, t')$ in equation (4.15). The semiclassical equations are the largely independent of the details of the initial state.

To see this in more detail we need to specify a given distribution of modes and couplings in the environment. This information is encoded in the spectral density

$$\tilde{I}(k) = \sum_i \frac{c_i^2}{2m_i\omega_i} \delta(k - \omega_i) \quad \text{(IV. 19)}$$

If the spectral density is specified, then we can fully determine the noise and dissipation kernels as

$$\eta(s) = \int dk \tilde{I}(k) \sin ks \quad \text{(IV. 20)}$$

$$\nu(s) = \int dk \tilde{I}(k) \coth \frac{\beta k}{2} \cos ks. \quad \text{(IV. 21)}$$

A large class of physically interesting choices for spectral density are

$$I(k) = \frac{2M\gamma}{\pi} k \left( \frac{k}{\Lambda} \right)^{s-1}, \quad k \leq \Lambda$$

$$0, \quad k > \Lambda, \quad \text{(IV. 22)}$$
where $\Lambda$ is a high-frequency cut-off. The exponent $s$ determines the infra-red behaviour of the bath. For $s = 1$ the environment is called ohmic, for $s < 1$ subohmic and for $s > 1$ supraohmic.

**High temperature** For high temperature $\beta \Lambda \ll 1$, the $\Delta_2$ term is proportional to $\beta^{-1}$ and dominates the classical correlation function. The classical stochastic process will be then independent of the precise choice of coarse-graining. The two point function for $x$ will then be

$$\Xi(t, t') = \frac{1}{M^2} \int_0^t ds u(t-s) \int_0^{t'} ds' u(t'-s') \nu(s-s') \quad \text{ (IV. 23)}$$

But this is the correlation function for the solution of the classical stochastic differential equation

$$M \frac{d^2}{dt^2} x(t) + M \omega^2 x(t) - \int_0^t ds \eta(t-s) x(s) = f(t), \quad \text{ (IV. 24)}$$

where $f(t)$ is a Gaussian process with two-point function

$$\langle f(t) f(t') \rangle = \eta(t-t'). \quad \text{ (IV. 25)}$$

The noise kernel then gives the correlation function for an external noise perturbing the classical dissipative equations of motion. This justifies its name and recover results suggested by path integral techniques [28], or explicitly proved only in particular regimes [6]. *Note, however, that this is true only when the $\Delta_2$ term in (3.24) dominates, as is the case of high temperature.* In the previous paragraph it is by necessity that the $\Delta_1^2$ term is small. Here it is not the case, because $\Delta_1$ contains also a contribution from the environment degrees of freedom that might give a substantial contribution in certain regimes. In the general case the $\Delta_2^2$ term might be of importance, something that implies that the stochastic limit will not be given by such a simple expression in terms of an external force guided by the noise kernel.

### IV.3 Scalar field

Consider now the case of a free, massive scalar field in vacuum. The correlation functions read

$$\Delta_1(x, t; x', t') = - \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-ik\cdot x} \sin \omega_k |t - t'| \quad \text{ (IV. 26)}$$

$$\Delta_2(x, t; x', t') = - \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-ik\cdot x} \cos \omega_k (t - t') \quad \text{ (IV. 27)}$$

Since $\omega_k \geq m$, coarse-grainings with $\sigma^2 m^2 >> 1$ for each mode will manifest a suppression of the interferences. As in the harmonic oscillator the $\Delta_1^2$ term, will be negligible and the $\Delta_2$ will provide the classical correlation function.
For a massless field, there is no natural coarse-graining scale for all modes. One needs to take larger values of \( \sigma \) to adequately deal with the infra-red modes. Configuration space coarse-graining is clearly bad in such a case. The analysis would have to be performed in phase space and would provide the same results as the \( m \neq 0 \) case [29].

Consider now the case of a free scalar field in a general globally hyperbolic spacetime. The reader is referred to the standard treatments of Birrell and Davies [30] and Wald [31]. Let us by \( t \) denote the time coordinate in a space-like foliation, by \( x \) the spatial coordinates and by \( \alpha \) label the modes. The Heisenberg-picture field will read then

\[ \hat{\phi}(x, t) = \sum_{\alpha} \hat{a}_\alpha u_\alpha(x, t) + \hat{a}_\alpha^\dagger \bar{u}_\alpha(x, t), \]  

(IV. 28)

where \( u_\alpha(x, t) \) are some complex-valued solutions to the Klein-Gordon equation and

\[ [\hat{a}_\alpha, \hat{a}_\beta^\dagger] = \delta_{\alpha\beta}. \]  

(IV. 29)

Let us assume that the system is found in a Gaussian state \( |\Omega\rangle \), which is annihilated by the operator \( \hat{a}_\alpha \). The action of \( \hat{a}_\alpha^\dagger \) on \( |\Omega\rangle \) produces all states of the Fock space. It is easy to compute

\[ \langle \Omega | \hat{\phi}(x, t) \hat{\phi}(x', t') | \Omega \rangle = \sum_{\alpha} \bar{u}_\alpha(x', t') u_\alpha(x, t) \]  

(IV. 30)

Let us now consider a particular instant \( t = 0 \) as reference time, in which the mode functions \( u_\alpha^0(x) = u_\alpha(x, t) \) form an orthonormal basis

\[ \int dx u_{\alpha}^0(x) u_{\beta}^0(x) = \delta_{\alpha\beta} \]  

(IV. 31)

\[ \int dx \bar{u}_{\alpha}^0(x) u_{\beta}^0(x) = 0. \]  

(IV. 32)

Here, we wrote as \( dx \) the volume element of the push-backed spacetime metric in the spacelike surface \( t = 0 \). Any scalar function on a instant of time can be decomposed in modes

\[ u_\alpha(x, t) = A_{\alpha\beta}(t) u_{\beta}^0(x) + B_{\alpha\beta}(t) \bar{u}_{\beta}^0(x) \]  

(IV. 33)

The matrices \( A \) and \( B \) are the Bogolubov coefficients and satisfy the matrix identity

\[ A^\dagger A - B^\dagger B = 1, \]  

(IV. 34)

which essentially means that time evolution is given (classically) by a symplectic transformation. The correlation function (4.30) reads then

\[ \langle \Omega | \hat{\phi}(x, t) \hat{\phi}(x', t') | \Omega \rangle = \bar{u}_{\alpha}^0(x') A^\dagger(t') A(t) u_{\beta}^0(x) + u_{\alpha}^0(x') B^\dagger(t') B(t) \bar{u}_{\beta}^0(x) + \bar{u}_{\alpha}^0(x') A^\dagger(t') B(t) \bar{u}_{\beta}^0(x) + u(x') B^\dagger(t') A(t) u_{\beta}^0(x), \]  

(IV. 35)
where a matrix notation has been employed. This gives for the kernels
\[
\Delta_1(t, x; t', x') = -2\theta(t - t') \text{Im} \tilde{u}^0(x') [A^\dagger(t') A(t) - B^\dagger(t') B(t)] u^0(x)
\]
\[
-2\theta(t' - t) \text{Im} \tilde{u}^0(x) [A^\dagger(t) A(t') - B^\dagger(t) B(t')] u^0(x) \quad (\text{IV. 36})
\]
\[
\Delta_2((t, x; t', x') = 2 \text{Re} \tilde{u}^0(x') [A^\dagger(t') A(t) + B^\dagger(t') B(t)] u^0(x)
\]
\[
+2 \text{Re} \tilde{u}^0(x') A^\dagger(t') B(t) \tilde{u}^0(x) \quad (\text{IV. 37})
\]

The Bogolubov transformation is a generalisation of the squeezing transformation we studied earlier. For the case of a single mode we had to coarse-grain in regions of configuration space much larger than the uncertainty. It is similar in this case. We can coarse-grain each mode separately at a scale \( \sigma^2 \). From equation (4.35) we can read the uncertainty for each mode (defined using the mode functions \( u^0_\alpha \)). This will be equal to
\[
(\Delta \phi_\alpha)^2 = [A^\dagger A + B^\dagger B]_{\alpha\alpha}(t) + [A^\dagger B + B^\dagger A]_{\alpha\alpha}(t) \quad (\text{IV. 38})
\]
The first term is always larger in norm than the second due to Schwarz inequality, and therefore coarse graining for each mode must be much larger than \( (A^\dagger A + B^\dagger B)_{\alpha\alpha}(t) \) for all \( t \). To have a uniform coarse-graining scale for all modes it is necessary that \( \sigma^2 \) has to be much larger than the norm of the matrix
\[
||A^\dagger A + B^\dagger B||(t) = ||2A^\dagger A - 1||(t) < 2||A||^2(t) \quad (\text{IV. 39})
\]
The matrix \( A \) has a finite norm be virtue of the Bogolubov identity (4.34). Now, in order to have a meaningful coarse-graining procedure it is necessary that the norms of the Bogolubov matrices must be bounded in time. Hence one can write a sufficient condition for the possibility of a coarse-graining scale \( \sigma^2 \) valid for all modes is that
\[
\sigma^2 >> \sup_t ||A(t)||^2 \quad (\text{IV. 40})
\]
We can then employ equation (3.24) to identify the correlation function for the classicalised field. Due to (4.40) the \( \Delta_2^1 \) term will in general be smaller, as in the case of the squeezed system hence the distribution (4.37) gives the correlation function of the classical stochastic process.

### IV.4 Perturbative expansion

So far we have considered only the case of Gaussian processes. We shall now study the case of non-quadratic systems through the use of perturbation theory. Let us by \( Z_0[J_+, J_-] \) denote a CTP generating functional that can be exactly evaluated, e.g. a Gaussian. In general, a perturbative expansion around \( Z_0 \) will be of the form
\[
Z[J_+, J_-] = \exp \left( i F[i, \frac{\delta}{\delta J_+}, -i \frac{\delta}{\delta J_-}] \right) Z[J_+, J_-] \quad (\text{IV. 41})
\]
in terms of a functional $F[x(\cdot), x'(\cdot)]$, that depends on some coupling constant. In the case that the CTP generating functional depends on configuration space variables, the Hamiltonian is of the form $\hat{H}_0 + V(\hat{x})$ and the initial state is the vacuum, then we have

$$Z[J_+, J_-] = \exp \left( i \int dt V[\frac{\delta}{\delta J_+(t)}] - iV[\frac{\delta}{\delta J_-(t)}] \right) Z_0[J_+, J_-] \quad (IV. 42)$$

Substituting $J_\pm = R/\sqrt{2\pi\sigma} \pm J/2$ in equations (4.42) yields for the classical generating functional

$$Z_{cl}[J] = \int dR(\cdot) e^{-\frac{1}{2\pi} R^2} \exp \left( i \int dt V[\frac{i}{2} \frac{\delta}{\delta J(t)} + i\sqrt{\frac{\pi}{2\sigma}} \frac{\delta}{\delta R(t)}] \right) Z_0\left[\frac{R}{\sqrt{2\pi\sigma}} + \frac{J}{2}, \frac{R}{\sqrt{2\pi\sigma}} - \frac{J}{2}\right] \quad (IV. 43)$$

If we assume that the length scale by which the potential varies is much larger than $\sigma$ we can keep the lower order term in $\sigma$ in the exponential to get

$$Z_{cl}[J] = \int dR(\cdot) e^{-\frac{1}{2\pi} R^2} \times \exp \left( - \int dt \sqrt{\frac{\pi}{2\sigma}} V'[\frac{i}{2} \frac{\delta}{\delta J(t)} + \frac{\delta}{\delta R(t)}] \right) Z_0[R, J] \approx Z_0^cl[J] - \int dt \frac{1}{2\sqrt{2\pi}} \sigma V'[\frac{i}{2} \frac{\delta}{\delta J(t)}] \int dR(t) R(t) e^{-\frac{1}{2\pi} R^2} Z_0[R, J], \quad (IV. 44)$$

where $Z_0[J]$ is defined from $Z_0[J_+, J_-]$ via (3.9) and we wrote for brevity $Z_0[R, J] = Z_0[R/\sqrt{2\pi\sigma} + J/2, R/\sqrt{2\pi\sigma} - J/2]$.

The above expression is the leading order in $\sigma$ of the generating functional and is valid only when the potential $V$ is assumed to vary in macroscopic scales.

Note, that as we see from (4.44), a perturbation expansion of the quantum theory does not generically amount to a perturbative expansion in the corresponding classical limit.

## V Concluding remarks

Let us now summarise our results. We first showed that the quantum mechanical correlation functions do not correspond to the statistical properties of a physical system and hence do not correspond to a classical stochastic process. Then we explained the relation of Schwinger and Keldysh’s CTP generating functional to the decoherence functional of the consistent histories approach to quantum theory. This enabled us to use the decoherence condition for histories in order
to develop a procedure of going from a quantum process to a stochastic process that corresponds to a given degree of coarse-graining for a class of observables. The end result was equation (3.9) that gives the relation between the classical generating functional and the CTP generating functional. But we should keep in mind that any results of the stochastic description that give some detailed structure in scales smaller than $\sigma$ are clearly unreliable.

We then proceeded to study examples. We showed that in Gaussian processes the classical limit is a Gaussian process, with a correlation function given by the real part of the quantum two-point function plus a term that depends on the coarse-graining scale. The second term is negligible in general, but this is not necessary in the case of quantum Brownian motion at low temperatures, because then it largely depends on properties of the bath, rather than the initial condition of the distinguished system.

Now, what is the use of these results? The answer is that a stochastic description might be more amenable to our intuition than a quantum one. For instance, given a stochastic process one can simulate the evolution of individual systems and identify typical behaviours. This is something that cannot be done in quantum theory. We have no (uncontroversial) way to describe the random evolution of an individual system's observables, while in classical probability we can write equations of the Langevin type. In this sense, the stochastic approximation captures an aspect of the quantum mechanical randomness that might be manifested on macroscopic or even intermediate scales. Hence for Brownian particle, the knowledge of the noise due to the environment might be deemed sufficient knowledge in certain applications.

Substituting the full quantum behaviour for an approximate description in terms of Langevin equation is necessary in cases where the quantum system acts as a driving force term for another classical one. This is the case, for instance, of any detector coupled to a quantum field. While one often uses a quantum description for a detector, this is clearly inappropriate for realistic systems: a detector is a macroscopic system and the description of its effective behaviour in terms of a simple Schrödinger equation is an idealisation that is hardly justified from first principles. On the other hand a detector can be treated as a classical system - e.g. with respect to its center of mass - that is coupled to a stochastic driving force due to the quantum system to which it is measured. In this case the stochastic description of a quantum system is not only convenient, but necessary.

The situation is similar in the case of quantum field theory in curved spacetime as far as the issue of backreaction is concerned. The quantum field acts as a source for the classical gravitational field via its stress-energy tensor. One writes then the semiclassical gravity equation

$$G_{\mu\nu} = \kappa \langle T_{\mu\nu} \rangle$$  \hspace{1cm} (V. 1)

There is an underlying assumption in any use of this equation. As it stands it is nonsensical: on the left hand side is an observable for an individual system
and on the right hand side an ensemble average. What is implied is that i) the
behaviour of the quantum field is in some regime approximately deterministic
and ii) the corresponding classical value of the stress energy tensor is equal to
the expectation value over the field’s state.

Of course these two assumptions need to be verified. In order for the quan-
tum field to behave classically, coarse-graining is necessary. Even with coarse-
graining it is not necessary that the system will classicalise. For instance, in
the one-dimensional case, time evolution might cause a continuous increase of
the squeeze parameter in a given mode. In this case no fixed degree of coarse-
graining is good at all times and it is very little we can do with a classical
description. Let us assume, however, that this is not the case. According to
our earlier analysis, this would mean that we can choose a coarse-graining scale
satisfying equation (4.40). Then, one might expect that the system will exhibit
stochastic behaviour for at least some of its properties.

In order for assumption i) to be valid, the stochastic process for the stress
energy tensor should have small deviations from its mean. In fact this is rarely
true, as in simple spacetimes one can show that the quantum fluctuations of the
stress-energy tensor are of the same order of magnitude as its mean value [32].
This leads to the point, advocated by Hu and collaborators [33, 34, 35, 36], that
the semiclassical description of backreaction of quantum fields onto geometry
ought to have a stochastic component.

This is where our results in section 4.3 are of relevance. First, we argued
that the condition (4.40) is sufficient to obtain classicality; then we showed that
it is essentially the real part of the two-point function of the fields that gives
the two-point function at the classical limit. Now the stress energy tensor is a
quadratic functional of the fields

\[ T^{\mu\nu}(x) = \frac{1}{2} \nabla^\mu \phi(x) \nabla^\nu \phi(x) - \frac{1}{2} g^{\mu\nu} (\nabla^\rho \nabla^\rho + m^2 \phi^2(x)) \tag{V. 2} \]

Its expectation value can be read from the two-point function via point
splitting and renormalisation. This means that we can define

\[
\langle T^{\mu\nu}(x) \rangle = \lim_{x' \to x} \frac{1}{2} \left[ \frac{1}{2} g^{\mu\nu}(x) \nabla^\rho \nabla_x^\rho \nabla^\nu \phi(x) + \Delta_2(x, x') \right] - \frac{1}{2} \left[ g^{\mu\nu}(x) + g^{\mu\nu}(x') \right] \left[ \nabla^\rho \nabla_x^\rho + m^2 \phi^2(x) \right] \tag{V. 3} \]

Now, the imaginary point of the two point function vanishes as \( t \to t' \) as
can be easily checked by equation (4.35). Since this part is antisymmetric the
symmetrisation of the derivatives in the definition (5.3) implies that it vanishes.
Hence, the quantum mechanical expectation of the stress energy tensor equals the
classical stochastic expectation. However, as we said before one needs to take the
higher order correlations into account in order to have a consistent backreaction.
In this case, the naive prescription of using the quantum mechanical correlation
functions of the stress energy tensor breaks down. The correct higher order
correlation functions for the stress energy tensor ought to be constructed from the classical correlations (4.37). And clearly the stochastic process for the stress energy tensor is not Gaussian, even though it is obtained from the Gaussian process for the field \( \phi \).

The general conclusion in this description is that the backreaction of quantum fields to geometry can be described by a stochastic differential equation of the type

\[ G_{\mu \nu} = \kappa T_{\mu \nu}[\phi] \]  

(V. 4)

where \( T_{\mu \nu} \) is a random variable, a functional of the classicalised field \( \phi(\cdot) \) that is defined by a Gaussian stochastic process with generating functional given by (4.37). This result is conditional upon (4.40) holding that defines the possibility of having a robust coarse-graining.

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**References**


