RESOURCE LETTER ON GEOMETRICAL RESULTS
FOR EMBODDINGS AND BRANES

Matej Pavšič\textsuperscript{1}

Department of Theoretical Physics
Jožef Stefan Institute
Jamova 39
SI–1000 Ljubljana, Slovenia

Victor Tapia\textsuperscript{2}

Departamento de Física
Universidad Nacional
Bogotá, Colombia

Abstract

Due to the recent renewal in the interest for embedded surfaces we provide a list of commented references of interest.

Introduction

The idea that our space–time can be considered as a four–dimensional space embedded in a higher–dimensional flat space is an old and recursive one. Recently, due to a proposal by Randall and Sundrum this idea has again attracted much attention. Since several related ideas had already been studied and several results had been obtained in the past we have considered convenient to provide a full account of the existing results on the subject.

The choice of the references included in this letter has been dictated by some simple criteria. We have included only articles published on wide

\textsuperscript{1}MATEJ.PAVSIC@IJS.SI
\textsuperscript{2}TAPIENS@CIENCIAS.CIENCIAS.UNAL.EDU.CO
circulation journals and books, etc., except when they are of exceptional relevance or they has a historical character. The main criterion for inclusion in this letter is that the article made use of extrinsic Riemannian geometry.

A. The Geometry of Embeddings

The concept of an abstract Riemannian manifold arises in mathematics as the result of an evolution in mathematical attitudes. In the earlier period, geometers thought of more concretely of curved surfaces embedded in a flat Euclidean space (Gauss, 1827), i.e., they described the geometry of surfaces as embedded in a higher-dimensional space. The concept of an abstract Riemannian manifold, defined intrinsically, was first explicitly formulated by Riemann in his famous, although little read, thesis, in 1850 (published only in 1868). In that work, what it is today known as Riemannian geometry was introduced. The geometrical properties of a manifold were characterised only by intrinsic properties, without any need of a reference to a higher dimensional embedding space.

Almost immediately after this abstract view came into favour, a question naturally arose: the isometric embedding problem: The question of the existence of concrete realisations of abstract Riemannian manifolds as submanifolds of higher dimensional Euclidean spaces. Today we know that the answer is yes: any intrinsically defined Riemannian manifold can be isometrically embedded, locally and globally, in an Euclidean space of appropriate dimension and signature. The works cited in this section are mainly oriented to establish this equivalence.

The embedding problem was first considered by Schläfli (1873) just after the publication of Riemann work. He discussed the local form of the embedding problem and he conjectured that a Riemannian manifold with positive defined and analytic metric can be locally and isometrically embedded as a submanifold of a Euclidean space $E_N$ with $N = n(n+1)/2$.

In 1926 Janet described a method of proof based on a power series development, so it was limited to local results. Furthermore, he required the metric to be analytic. This proof however, as Janet himself noticed, was incomplete. He solved only the local problem for two-dimensional manifolds with analytic metric. In 1927 E. Cartan extended the Janet’s proof to $n$-dimensional manifolds treating it as an application of his theory of Pfaffian forms. The dimensionality requirement was $N = n(n+1)/2$, as conjectured by Schläfli.
In 1931 Burstin completed the Janet’s proof and also extended it to the case in which the embedding space is a given Riemannian manifold $V_N$ with positive defined and analytic metric. The Gauss–Codazzi–Ricci equations are conditions to be satisfied by the embedding, therefore, they can be considered as integrability conditions for the embedding. In 1956 Leichtweiss gave a new proof based more substantially than Burstin in the Gauss–Codazzi–Ricci equations of Riemannian geometry. His proof is more involved than that provided by Burstin. In 1961 Friedman extended the theorem to Riemannian manifolds with indefinite metrics, such as the space–time used in contemporary gravitational theories. For semi–positive definite metrics see (Lense, 1926).

The first global isometric embedding theorem of $V_n$ into $E_N$ were established by Nash (1956). The results depend crucially on the compactness of $V_n$. For $V_n$ compact he obtains $N = n(3n + 11)/2$; for non–compact manifolds $N = n(n + 1)(3n + 11)/2$. The first global results for indefinite metrics were obtained by Clarke (1970) and Greene (1970).

We may therefore conclude that any intrinsically defined Riemannian manifold has a local and a global, isometric embedding in some Euclidean space. Then one can consider the two approaches, intrinsic and extrinsic, to Riemannian geometry as completely equivalent.

This result opened new perspectives in theoretical physics, mainly in the physics of the gravitational field. On the one hand it was considered as a mathematical tool to construct and classify solutions of General Relativity. On the other hand, it allows to introduce new variables, perhaps more amenable for a description of quantum mechanical phenomena.

It is clear that some manifolds can be embedded in a higher dimensional space with a dimension lesser than the requirement fixed by the embedding theorem. Therefore, it is convenient to introduce the concept of class of the embedding, which is the minimal number of extra dimensions necessary to satisfy the Gauss–Codazzi–Ricci equations.

B. Applications to General Relativity

At the beginning of the XX century General Relativity was developed (by Einstein) based on Riemannian geometry. Almost immediately exact solutions were obtained based on the use of extrinsic geometry (Kasner, 1921). Several other results are contained in the references.

C. Results Related to the Class of the Embedding
In this section we list several articles containing results related to the class of the embedding. They are divided in: class 1; class 2; spherically symmetric fields (class 2); embedding of the Schwarzschild solution; and spherically symmetric fields of class 1.

D. Extrinsic Gravity

Since the pioneer work of Sakharov in 1966, several attempts have been done in order to incorporate the concept of bending of space–time in gravitational theories. One way of doing this is by considering space–time as embedded in a higher–dimensional space. In this case the concept of bending has an immediate geometric visualisation. The first attempt in this direction seems to be due to Regge and Teitelboim (1976). The idea was later revived by Pavšič in the eighties, and by Maia and by Tapia in 1989.

E. Strings and Membranes

During the 80’s there was a good lot of work on string theories. Since the string is a (1–dimensional) geometrical object living in a higher–dimensional space the use of embedding should be an immediate tool. However, even when the main developments in strings were in other directions there was also some work dealing with the geometrical, embedding, aspects of strings. Several proposals were done using embeddings and generalisations to other higher–dimensional (membranes) objects were studied.

F. Rigid Particles and Zitterbewegung

The Dirac equation gives rise to the phenomenon of zitterbewegung (Schrödinger, 1930). Soon after that, it was realized that this motion can be explained by a helical motion of the electron (Huang, 1952; Corben, 1961). Later on, it was discovered that a helical motion can be described by an acceleration dependent Lagrangian (Liebowitz, 1969; Riewe, 1971, 1972). The relativistic generalizations making use of embedding geometry are many and are listed in this section.

G. New Brane World
A different approach to considering our universe as embedded in a higher-dimensional flat space was developed mainly by Akama (1978, 1983). Recently this line of approach revived, but this time as a consequence of developments in other areas, mainly in string theory, $M$–theory and the like, with the proposal due to Randam and Sundrum (1999).

Acknowledgements

This work was partially done during a visit of the authors to The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. V.T. has been partially supported by Dirección de Investigación, Universidad Nacional de Colombia.

A. The Geometry of Embeddings


2. B. Riemann, Über die Hypothesen welche der Geometrie zu Grunde liegen (1854). This thesis was presented on June 10th, 1854, in Göttingen and it was first published in Abh. Königl. gesellsch. 13, 1 (1868). It was translated into English by W.K. Clifford and published in Nature 8, 14 (1873). Since the work by Riemann was published only in 1868, this is the date usually used in the references in order to avoid confusion over the order of appearance, and precedence, of ideas in geometry.


10. J. A. Schouten and D. J. Struik, Einführung in die neueren Methoden der Differentialgeometrie, Bd. 2 (Groningen, 1938), S. 142.


B. Applications in General Relativity


   This seminar’s report contains the following individual papers:


C. Results Related to the Class of the Embedding

Embedding class 1:


Embeddings of Class 2:


Spherically Symmetric Embeddings (Class 2):


Embedding of the Schwarzschild metric:


Spherically symmetric embeddings of class 1:


D. Extrinsic Gravity


13. V. Tapia, Gravitation a la String, Clas. Quantum Grav. 6, L49 (1989).


---

**E. Strings and Membranes**


**F. Rigid Particles and Zitterbewegung**


**G. New Brane World**


