Flux tube solutions in noncommutative gauge theories

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Abstract

We derive nonperturbative classical solutions of noncommutative $U(1)$ gauge theory, with or without a Higgs field, representing static magnetic flux tubes with arbitrary cross-section. The fields are nonperturbatively different from the vacuum in at least some region of space. The flux of these tubes is quantized in natural units.
Noncommutative gauge theories are the topic of much recent interest [1]-[9]. Early examples of such theories arose in nonperturbative regularizations of membranes [10, 11] and D-branes [12, 13], and they have also recently emerged at various limits of M theory. They are, therefore, relevant to aspects of string and D-brane dynamics. They exhibit an interesting space uncertainty relation [14] and have many similarities with large-\(N\) gauge theories and lattice gauge theories [15, 16].

The above theories are field theories on a noncommutative space [18] (for a concise review see also [19]). For a flat such space the coordinates \(X_i\) obey the commutation relations

\[
[X_i, X_j] = i\theta_{ij}
\]

with \(\theta_{ij}\) a set of commuting parameters. Specializing to 3+1 dimensions, and assuming a commutative time dimension, we can always bring the above to the form

\[
[X, Y] = i\theta, \quad [X, Z] = [Y, Z] = 0
\]

with \(\theta > 0\). So only the \((X, Y)\) plane is noncommutative and resembles a quantum phase space (with \(\theta\) playing the role of \(\hbar\)) on which \(X\) and \(Y\) act as coordinate and momentum operators.

One expects that at the limit \(\theta \to 0\) such theories go over to ordinary commutative ones. There can be, however, interesting nonperturbative effects that arise for nonzero \(\theta\) and do not have a smooth limit. These effects could be present already at the classical level, in the form of nonperturbative solutions. Several such solutions have recently appeared [20]-[25]. In [25], in particular, a solution representing a magnetic monopole has been derived. In this letter we will demonstrate the existence of nonperturbative solutions corresponding to magnetic flux tubes of arbitrary cross-section.

One way to describe field theories on noncommutative spaces is through star products of ordinary functions. In this approach, a one-to-one correspondence between functions of operators \(\hat{f}\) and ordinary functions \(f\) is established by Weyl ordering:

\[
\hat{f}(X, Y) = \frac{1}{2\pi} \int d^2k \tilde{f}(k_1, k_2) e^{i(k_1 X + k_2 Y)}
\]

where \(\tilde{f}\) is the Fourier transform of \(f\). The product of two Weyl ordered operators \(\hat{f} \hat{g}\), Weyl reordered, corresponds to a function denoted \(f \ast g\)

\[
f \ast g (x, y) = e^{\frac{i}{2\hbar}(\partial_{x_1} \partial_{y_2} - \partial_{x_2} \partial_{y_1})} f(x_1, y_1) g(x_2, y_2) \bigg|_{x_1=\bar{x}_2, y_1=\bar{y}_2}
\]

which defines the noncommutative star product. One then writes noncommutative field theory lagrangians in terms of commutative fields but with products replaced by star products.

We will specialize to the case of noncommutative \(U(1)\) gauge theories, with or without Higgs fields. The action involves the gauge fields \(A_\mu\) and, possibly, a Higgs field \(\Phi\):

\[
S = \frac{1}{4g^2} \int d^4 x \left\{ F_{\mu\nu}^2 + (D_\mu \Phi)^2 \right\}
\]
where
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i(A_\mu * A_\nu - A_\nu * A_\mu) \] (6)
and
\[ D_\mu \Phi = \partial_\mu \Phi + i(A_\mu * \Phi - \Phi * A_\mu) \] (7)
Indices (0, 1, 2, 3) = (t, x, y, z) are summed with the flat Minkowski metric. (We used the fact \( \int dx dy f * g = \int dx dy fg \) to eliminate star products in \( S \).) In the standard commutative limit the above is a free theory, \( g \) is irrelevant and the Higgs field \( \Phi \) decouples from the Maxwell field. Here, however, we have a coupled interacting system.

An alternative approach to noncommutative theories is to acknowledge that they refer to operator-valued fields and work directly with operators on the quantum phase space \((X, Y)\) which are functions of \(z\) and \(t\) [17, 25, 26]. In this approach, star products become operator products and integration over the \((X, Y)\) plane becomes trace:
\[ \int dx dy f(x, y) = 2\pi \theta \text{Tr} \hat{f}(X, Y) \] (8)
Derivatives in the \(X\) and \(Y\) direction become the standard quantum generators of translations in these variables and they act on operators in the adjoint way, via commutation. That is, for any operator \(F\)
\[ \partial_x F = \frac{i}{\theta} [Y, F], \quad \partial_y F = -\frac{i}{\theta} [X, F], \] (9)
Understanding, now, \(A_\mu\) and \(\Phi\) as anti-hermitian operator-valued functions of \(z\) and \(t\), the action becomes
\[ S = -\frac{\pi \theta}{4g^2} \int dz dt \text{Tr} \left\{ F_{\mu\nu}^2 + (D_\mu \Phi)^2 \right\} \] (10)
with
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \] (11)
\[ D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi] \] (12)
We can also write the field strength, in analogy to the commutative case, as the commutator of affine derivative operators:
\[ \hat{F}_{\mu\nu} = [\hat{A}_\mu, \hat{A}_\nu] \] (13)
where we defined
\[ \hat{A}_\mu = \partial_\mu + A_\mu \] (14)
\( \hat{F} \) differs from \(F\) by the commutator of the two derivatives on the noncommutative plane:
\[ \hat{F}_{\mu\nu} = F_{\mu\nu} + \omega_{\mu\nu} \] (15)
ω is a constant magnetic field in the z-direction (proportional to the symplectic 2-form on the (X,Y) plane) with one ‘quantum’ of flux per unit cell of area $2\pi \theta$:

$$\omega_{\mu\nu} = -\frac{i}{\theta} (\delta_{\mu 1} \delta_{\nu 2} - \delta_{\nu 1} \delta_{\mu 2})$$  \hspace{1cm} (16)$$

Covariant derivatives also become

$$D_\mu \Phi = [\hat{A}_\mu, \Phi]$$  \hspace{1cm} (17)$$

This rewriting may be of little interest for the z- (and t-) components, since $\partial_z$ and $A_z$ are operators on different spaces, but it is quite relevant for the X and Y components since both $\partial_i$ and $A_i$ are operators on the same space; $\hat{A}_i$ are the only operators appearing in the theory.

The action in terms of the new field strength becomes

$$\hat{S} = -\frac{\pi \theta}{2 g^2} \int dz dt \text{Tr} \left\{ [\hat{A}_\mu, \hat{A}_\nu]^2 + [\hat{A}_\mu, \Phi]^2 \right\}$$

$$= S + \frac{\pi}{g^2 \theta} \int dz dt \left[ \text{Tr} 1 - i \theta F_{12} \right]$$  \hspace{1cm} (18)$$

The first additional term is an irrelevant (infinite) constant while the second is the trace of a commutator which contributes only “boundary” terms. The two actions lead to the same equations of motion:

$$[\hat{A}_\nu, [\hat{A}_\nu, \hat{A}_\mu]] + [\Phi, [\Phi, \hat{A}_\mu]] = 0$$

$$[\hat{A}_\nu, [\hat{A}_\nu, \Phi]] = 0$$  \hspace{1cm} (19)$$

For static, purely magnetic configurations ($A_0 = 0, \partial_0 = 0$) we can also write the Bogomolny equations

$$[\hat{A}_i, \hat{A}_j] = \pm \epsilon_{ijk} [\hat{A}_k, \Phi]$$  \hspace{1cm} (20)$$

which minimize the energy of such configurations and automatically satisfy the equations of motion (19) by virtue of the Jacobi identity.

We will look for solutions of (19,20) that represent static magnetic flux tubes in the z-direction. We therefore take $\partial_z A_i = 0$ and also make the gauge choice $A_z = 0$. Defining the new fields

$$A = \frac{1}{\sqrt{2}} (\hat{A}_1 + i \hat{A}_2)$$

$$\bar{A} = \frac{1}{\sqrt{2}} (\hat{A}_1 - i \hat{A}_2)$$  \hspace{1cm} (21)$$

the equations of motion for pure gauge theory simply become

$$[A, [\bar{A}, A]] = 0$$  \hspace{1cm} (22)$$

while the Bogomolny equations for the Higgs theory become

$$0 = \partial_z A = \pm [A, \Phi]$$

$$\partial_z \Phi = \pm [A, \bar{A}]$$  \hspace{1cm} (23)$$

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The second equation in (23) is solved as
\[ \Phi = \pm z[A, \bar{A}] + \Phi_0 \]
with \( \Phi_0 \) a constant operator. The first equation, then, becomes
\[ [A, [\bar{A}, A]] = 0, \quad [A, \Phi_0] = [\bar{A}, \Phi_0] = 0 \]
Thus the equation for \( A \) is the same as in the theory without Higgs, and we have an extra \( \Phi_0 \) field (which can be proportional to the unit operator and does not contribute to the action or energy).

Instead of working with \( X \) and \( Y \) it is more convenient to work with oscillator operators
\[ a = \frac{X + iY}{\sqrt{2\theta}}, \quad a^\dagger = \frac{X - iY}{\sqrt{2\theta}}; \quad [a, a^\dagger] = 1 \]
acting on oscillator states \( |n> \), \( n = 0, 1, \ldots \) diagonalizing the number operator \( N = a^\dagger a \). In terms of these, \( A \) and \( \bar{A} \) become
\[ A = \frac{a}{\sqrt{\theta}} + A_1 + iA_2, \quad \bar{A} = \frac{a^\dagger}{\sqrt{\theta}} + A_1 - iA_2 \]
Note that \( N \) plays the role of the square of the distance from the origin \( R \), \( N \sim \frac{R^2}{2\theta} \).

The ‘vacuum’ configuration \( A_i = 0 \) corresponds to \( A = a/\sqrt{\theta} \). We shall look for rotationally invariant solutions in the form of a deformed oscillator \[ A = f(N) a \]
with \( f(\cdot) \) a scalar function. The above is a shorthand for the explicit expression
\[ A = \sum_{n=0}^{\infty} g(n)|n><n+1|, \quad g(n) = \sqrt{n+1} f(n) \]
Substituting the above ansatz in (22) we obtain for \( g(n) \):
\[ g(n) \left\{ |g(n+1)|^2 - 2|g(n)|^2 + |g(n-1)|^2 \right\} = 0 \]
valid for all \( n \) and subject to the boundary condition \( g(-1) = 0 \). Thus, either \( g(n) \) or the expression in brackets (which is the discrete laplacian on \( g \)) should vanish.

Assume that two successive zeros of \( g(n) \) are \( n_1 < n_2 \). Then the above equation for \( n_1 < n < n_2 \) gives a system of \( n_2 - n_1 - 1 \) linear homogeneous independent equations for the quantities \( |g(n_1 + 1)|^2, \ldots |g(n_2 - 1)|^2 \), which implies that all these \( g(n) \) have to vanish as well. We conclude that the general situation is \( g(-1) = \ldots = g(n_0 - 1) = 0 \) with \( n_0 \) some nonnegative integer. The rest of the \( g(n) \) are essentially the same as those for the operator \( a \), only shifted up by \( n_0 \). Choosing vanishing phases for \( g(n) \) and an overall normalization \( 1/\sqrt{\theta} \) we obtain
\[ g(n) = \sqrt{n-n_0 + \frac{1}{\theta}}, \quad n \geq n_0; \quad g(n) = 0, \quad 0 \leq n < n_0 \]
\[ A = \frac{a}{\sqrt{\theta}} \sqrt{\frac{N}{N - n_0}} \hat{P}_{n_0} \]  

(32)

where

\[ \hat{P}_{n_0} = 1 - P_{n_0} = \sum_{n=0}^{\infty} |n> <n| \]  

(33)

is the operator that projects out the first \( n_0 \) levels. Clearly \( n_0 = 0 \) reproduces the vacuum solution \( A = a/\sqrt{\theta} \). For large distances from the origin, that is, for \( N \gg 1 \), \( A \) becomes

\[ A \approx \frac{a}{\sqrt{\theta}} \left( 1 + \frac{n_0}{2N} \right) \sim \frac{a}{\sqrt{\theta}} + \frac{n_0}{\sqrt{2}} \frac{X + iY}{R^2} \]  

(34)

so it reproduces the gauge field of a Dirac string of strength \( n_0 \). (Remember that \( A_i \) are anti-hermitian, so (34) implies that \( iA_j \sim \epsilon_{jk}X_k/r^2 \), a vortex configuration.)

The field strength \( \hat{F} \) is

\[ \hat{F}_{\mu\nu} = \omega_{\mu\nu} \hat{P}_{n_0} \]  

(35)

and so

\[ F_{\mu\nu} = -\omega_{\mu\nu} P_{n_0} \]  

(36)

The above represents a circular static magnetic flux tube centered at the origin of the \((X,Y)\) plane with radius \( \sim \sqrt{\theta n_0} \). It resembles a circular quantum Hall ‘droplet’ with \( n_0 \) particles. The total magnetic flux is

\[ \phi = 2\pi \theta \text{Tr}(iF_{12}) = -2\pi n_0 \]  

(37)

and is obviously quantized. We stress that the obtained solution is nonperturbative since, in the area around the origin, it cannot be written as \( A = a/\sqrt{\theta} + O(\theta^0) \).

This solution is by no means unique. Firstly, we can perform any unitary transformation and still have a solution. Such deformations are considered to be generalized gauge transformations, but they lead to configurations with varying profiles on the \((X,Y)\) plane; for instance, the coherent-squeezed state operator

\[ U = e^{\mu a^\dagger - \mu a} e^{\lambda a^2 - \lambda a^\dagger^2} \]  

(38)

deforms the cross-section of the tube into an ellipsoidal one and also moves it to a different position on the plane. In general, conjugation with all possible \( U \) will change the position and shape of the cross-section into any closed curve with the same area (the set of all \( U \) generates all area-preserving diffeomorphisms on the noncommutative space). Thus all quantum Hall droplets correspond to flux tube solutions.

(Note that \( U \) in (38) above alters the asymptotic behavior of \( A \) for \( N \gg 1 \) and thus would lead to a field nonperturbatively different from the vacuum everywhere.)
We can, however, choose appropriate operators $U'$ which perform the required reshaping of the tube while leaving the asymptotics of $A$ unchanged. Such a $U'$ would be, e.g.,

$$U' = U f(N)$$

(39)

with $U$ as above and $f(N)$ a real function such that $f(N) \to 0$ for $N \to \infty$ while $f(N) = 1$ for $N \sim n_0$.

Secondly, we can start with similar higher-moment ansatze, such as

$$A = f(N) a^k = \sum_n g(n) |n > < n + k|$$

(40)

The analysis is similar, but now the equations group into $k$ uncoupled subsystems, each involving the coefficients $g(kn + q)$ for a single $q = 0, 1, \ldots k - 1$. We can choose a solution as above for each $q$, that is,

$$g(kn + q) = C_q \sqrt{n - n_q + 1}, \quad n \geq n_q$$

(41)

with different normalizations $C_q$ and shifts $n_q$ for each $q$. The field strength $\hat{F}$ of such configurations becomes

$$\hat{F}_{\mu\nu} = \theta \omega_{\mu\nu} \sum_{q=0}^{k-1} |C_q|^2 \sum_{n=n_q}^{\infty} |kn + q > < kn + q|$$

(42)

So by choosing all $|C_q|^2 = 1/\theta$ we obtain a localized $F$

$$F_{\mu\nu} = -\omega_{\mu\nu} \sum_{q=0}^{k-1} \sum_{n=0}^{n_q-1} |kn + q > < kn + q|$$

(43)

This is a flux tube with spatial extent $\sim \sqrt{\theta kn_{\text{max}}}$, where $n_{\text{max}}$ is the maximum of $n_q$, and a varying radial profile. (We can, e.g., obtain an annular profile between the values $n_1$ and $n_2$ by choosing $k = n_2$ and $n_q = 0$ for $q < n_1$, $n_q = 1$ for $n_1 < q < n_2$.) We note, however, that the asymptotics of $A$ are always different than the vacuum, so this seems to be an entirely nonperturbative solution (although we have not excluded the possibility that an appropriate unitary transformation will make it asymptotically perturbative).

Finally, we can take advantage of the scaling symmetry of the equations of motion (19) when expressed in terms of $\hat{A}_i$ and simply multiply the previous solutions by any number. Taking the original circular-profile solution (32) and multiplying by $\lambda$

$$F_\lambda = (\lambda^2 - 1)\omega - \lambda^2 \omega P_{n_q} = -\omega P_{n_q} + (\lambda^2 - 1)\omega \hat{P}_{n_q}$$

(44)

It represents a flux tube identical to the original one plus a constant magnetic field outside it. By choosing $\lambda^2 = 1 + \mathcal{O}(\theta)$ we can make the constant field perturbative.
if we wish. We point out that this scaling transformation exists for any general solution of the equations of motion, as

\[ \hat{A}_{x,y}(z,t) \rightarrow \lambda \hat{A}_{x,y}(\lambda z, \lambda t) \]

\[ A_{z,t}(z,t) \rightarrow \lambda A_{z,t}(\lambda z, \lambda t) \]

\[ \Phi(z,t) \rightarrow \lambda \Phi(\lambda z, \lambda t) \]

(45)

which leads to a transformation of the field strength

\[ F_{\mu\nu}(z,t) \rightarrow \lambda^2 F_{\mu\nu}(\lambda z, \lambda t) + (\lambda^2 - 1) \omega_{\mu\nu} \]

(46)

We stress that this is a nonperturbative transformation which does not go over to the usual scaling transformation of the commutative theory at the limit \( \theta \to 0 \). Applied, for example, to the monopole solution found in [25], it produces a monopole with a fractional strength \( \lambda^2 \) dilated in the \( z \)-direction by \( \lambda \), in the presence of a constant magnetic field.

In conclusion, we have explored several nonperturbative solutions of the static equations of motion of noncommutative \( U(1) \) gauge theory with or without a Higgs field, in the shape of flux tube configurations. The flux of these tubes seems to obey a quantization condition similar to the one in compact electrodynamics, for which we have no topological explanation. This is presumably related to the integrality of the index of some operator. The situation is certainly rather different from the commutative case, since, for instance, we can evade the quantization of the magnetic monopole charge by embedding it in a constant magnetic field, as demonstrated in the previous paragraph.

The analysis of this paper may also shed some light to the question of the stability of the monopole solution of [25]. The monopole was tied to a string in the \( z \)-direction, asymptotically similar to our flux tube solution for \( n_0 = 1 \). This string carries energy per unit length, and would naturally want to contract and pull the monopole in the \( z \)-direction. As we stressed, however, noncommutative gauge theory is most naturally formulated in terms of \( \hat{A}_i \) rather than \( A_i \) (there is no natural separation of \( \hat{A}_i \) into a \( \partial_i \) and a \( A_i \) piece in the action). The field strength \( \hat{F} \) in terms of \( \hat{A} \) has an additional constant magnetic field \( \omega \). In the presence of this field the string carries no field inside it and represents a depletion (rather than addition) of energy. It therefore has a negative tension and wants to expand, pushing the monopole in the negative \( z \)-direction. On the other hand, due to the magnetic field \( \omega \), the monopole feels a magnetostatic force pulling it upwards. This exactly balances the push of the string and stabilizes it.

Clearly the above heuristic discussion touches upon the question of field self-interaction and charges in noncommutative gauge theories. These and similar issues are left for future investigation.

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References


[22] K. Furuuchi, hep-th/9912047


