New Construction of an Universal Quantum Network and Its Applications *

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We proposed a new candidate for the construction of an universal quantum network, which are built from the elementary quantum gates to the elements of quantum circuit, again to quantum subnetworks and finally to a whole quantum network. The main advantages of our scheme are its simplicity and standardization in construction, facility and directness in assembly and scale, reusage and update in engineering, realizability and procedurization in an effective construction. Moreover our universal quantum networks are potentially programmable. It is proved that all the elementary quantum gates can be written as our quantum subnetwork. This implies that our scheme is compatible with the known quantum gate-assembly schemes, and so the known results from a quantum computation to a set of the elementary quantum gates can be directly inherited and used in our scheme. As the applications of our construction scheme, we obtain the whole quantum networks for Shor’s algorithm, Grover’s algorithm and simulating Schrödinger equation in general. This means that our universal quantum network can generally describe the known main results in quantum computation.

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The combination of information science and quantum mechanics has created a series of amazing results. One of them is just the idea of quantum computer [1,2] which can speed up a computation greatly and even exponentially than a classical computer can do, for example, factorization of a large number [3] and search for the unstructured data. [4] The rapid developments both in theory and experiments seem to indicate that quantum computer can be implemented in future.

A central part of quantum computer is the quantum network. Here, quantum network means an array of quantum gates, in which the quantum gates are assembled and arranged according to some principles and rules. Just is well known, an universal quantum network can be constructed by a set of universal elementary gates in principle [5]. However, it is still extremely important to know how to construct a whole quantum network in terms of the quantum gates so that it can carry out a general quantum computing task. To be able to do and how to do are two completely different problems, because the latter needs a determined procedure and a feasible method. To solve the problem how to do, we proposed a new construction for an universal quantum network based on some physical ideas and engineering considerations. Our construction scheme has three steps, that is from the elementary gates to the elements of quantum circuit, again from the elements of quantum circuit to the quantum subnetworks, finally from the quantum subnetworks to a whole quantum network. Our construction scheme is not to put the known quantum gate-assembly schemes into a nicety and concretization because our ideas have been beyond the known schemes. Of course, on the construction for an universal quantum network, the next two important problems are how to effectively construct it and how to implement it in experiments. In this paper, we will consider how to obtain a construction with high efficiency in principle by using of our method. However, we do not intend to directly touch at the great difficulties on the experimental implement.

By the word quantum subnetwork, we mean a quantum network corresponding to a
quantum computing step or a quantum computing part in a quantum computation including quantum algorithm and quantum simulating. Suppose a quantum computation $U$ is expressed by a product of a series of quantum computing steps, that is $U = U_1 U_2 \cdots U_n$, or a summation of a set of quantum computing parts, that is $U = U_1 + U_2 + \cdots + U_n$. Thus, a quantum network corresponding to $U_i (i = 1, 2, \cdots, n)$ is a quantum subnetwork with respect to the whole (or total) quantum network for $U$. Moreover, a quantum subnetwork can have its hierarchy more than one level. The number of its hierarchy depends on the requirement how to decompose a whole quantum network so that the constructions for the lowest hierarchy of the quantum subnetworks are effective and facile enough as possible. A quantum sub-subnetwork is always called with respect to its upper hierarchy quantum subnetwork. As to the proposed elements of quantum circuit, they have two kinds of basic elements: "Rotator" and "Transistor", and two kinds of the auxiliary elements "Jointer" and "Connector", which will be introduced in the section III.

In fact, a quantum computation including quantum algorithm and quantum simulating usually consists of a series of quantum computing steps or a set of quantum computing parts. It is very natural to think that a whole (or total) quantum network is made of some quantum subnetworks in which each of them corresponds a quantum computing step or a quantum computing part. So, we first successfully find a method through introducing an auxiliary qubit (or a quantum system), which can easily build a whole quantum network in terms of its quantum subnetworks. Then, a quantum subnetwork can be constructed by directly connecting a series of the elements of quantum circuit together. And a element of quantum circuit, such as the rotator and transistor, is clearly defined by the matrix elements in a transformation matrix corresponding to a quantum computing step or a quantum computing part. On the other hand, an element of quantum circuit also can be made of the elementary quantum gates. Consequently, the procedure in our construction scheme is fully determined. In this sense, our scheme make a big progress. Indeed, in terms of the ideas to decompose the whole to the parts and divide one step to several steps, we can simplify and standardize the scheme to construct an universal quantum network and provide a design principle to
decompose a whole quantum network into such some quantum subnetworks that they can effectively be constructed with the biggest possibility. It will be seen that our construction is very easy to assemble and scale up a whole quantum network in terms of its quantum subnetworks for the summation and product of simultaneous and successive transformations. However, this is difficult in the known quantum gate-assembly schemes. In this aspect, our construction scheme is significantly different from the known ones and then advances the art of construction for a whole quantum network.

Furthermore, it is worth discussing how to construct effectively the quantum (sub-)networks. This is usually too complicated to do [6] because the mathematical symmetries and physical features of a total quantum computation often can not be directly found out or utilized. But, in our scheme, an effective construction for a quantum subnetwork is simpler since itself is just simpler and it can reduce into more simpler hiberarchy of quantum subnetworks. Moreover, our decomposition principle is to let the lowest hierarchy quantum subnetworks has enough symmetries and physical features which can be directly used for an effective construction as possible. In addition, we can use some known and knowing in future effective constructions for some quantum subnetworks, for example, quantum network for quantum Fourier transformation, which will be seen in the following. This is the reasons why our construction scheme have more means do it than the known ones.

At present, although there are several excellent quantum algorithms, but the knowledge about the constructions of their whole quantum networks are still not complete. Moreover, a whole quantum network for simulating Schrödinger equation in general has not been found. One of reasons is the fact mentioned above, that is, it is often difficult to try to directly decompose a quantum computation to the elementary gates. Sometimes even this is not feasible. For example, a quantum simulating for Schrödinger equation in general can be based on a discretization of the time evolution in a series of short enough interval [7,8]. If we want to construct a quantum network directly for a total transformation, then the result will back to a finite time and lose the advantage of discretization so that we can not do it in physics. But in our construction scheme, this conflict with this physical idea for simulating
Schrödinger equation is overcome because we can easily construct a whole quantum network in terms of its quantum subnetworks.

Recently, there is a series of new results in experiments for implementing quantum computation. One of reasons is that ones are trying every possible thing. For example, Ahn et al. did not use $n$ qubits as a recourse, but rather a single atom prepared in a superposition of $n$ Rydberg state [9]. In contrast, we do not directly start with the elementary quantum gates, but introduce the elements of quantum circuit as the essential components of quantum (sub-)network. Although our elements of quantum circuit can be constructed by the elementary quantum gates, perhaps, their experimental implement is not finished through combining the implements for the elementary quantum gates, but is directly obtained by the suitable quantum systems. If this guess comes true, then the elements of quantum circuit will be more fundamental in a quantum network.

It is worth mentioning, in mathematical, our elements of quantum circuit are explicitly and directly related with the natural basis of the matrix and matrix elements. The natural basis leads them to become some standard accumulating units as a transformation matrix. Resetting their parameters corresponds to rewriting the matrix elements and so their combination will form a new transformation matrix. Usually, these elements of quantum circuit have been combined to build a quantum (sub-)network in a way with the effective construction. Resetting their parameters is just to program this quantum (sub-)network [10]. Therefore, our construction provides a possible way to program a quantum (sub-)network.

From a view in engineering, a whole quantum network in our construction scheme is built in a “motherboard” with many “slots”. The behavior of a quantum subnetwork corresponding to a quantum computing step appears a plug-in board in a “slot” which is an interspace between two connectors. Building a whole quantum network for a product of a series of quantum computing steps is simply to insert such quantum subnetworks in their slots. The quantum (sub-)network for a summation of a set of quantum computing parts is even simply to put the quantum (sub-)subnetworks together by virtue of the so-called jointers, in which a quantum (sub-)subnetwork corresponding to a quantum computing part. This leads that
a quantum network is updatable. Moreover, a quantum subnetwork and its any hierarchy in one quantum network can be reused in another quantum network. All we do is to draw it out from the slot or move it out from a quantum network and insert it into a given slot or put it together with the other subnetworks in the other quantum network. Updatable, reusable and potentially programmable are very important advantages in our construction scheme. However, the known quantum gate-assembly schemes are not so.

Our universal quantum network can overcome some difficulties stated above (their details see the following) and can easily be applied to the main known quantum algorithms and quantum simulating for Schrödinger equation in general. This implies that we find a standard way to construct a whole quantum network and universally describe a general quantum computation. It must be emphasized that we do not try to find a physical system to implement a concrete quantum network here, but our aim is only to study how to construct an universal quantum network from a theoretical view.

It is interesting and important whether our construction can be used to the known decomposition from a quantum computation to a set of the universal quantum gates. The answer is positive, because we have proved that all the elementary quantum gates can be written as our quantum subnetworks. As soon as we know how to combine the elementary quantum gates to form a quantum computation, the construction of the quantum network in our scheme is just very easy. All that we need to do is to assemble the quantum subnetworks for these elementary quantum gates into a whole quantum network like inserting the “plugging board” and playing the “building blocks”. Therefore, we can say that our construction scheme is compatible with the known quantum gate-assembly schemes.

It is true that we are filled with admiration for the beauty of the known achievements, but our believe is “let’s try every thing” and this believe must be also suitable for the study on theory. It seems to us the present theory is not really so mature that we do not need to propose some new candidates for theory and research some new possible methods. Consequently, in this paper, we try to propose a new candidate for the construction of an universal quantum network. Our construction for an universal quantum network is sim-
plified, standardized, scalable and potentially programmable. In special, our method more easily obtains the effective construction for an universal quantum network and its quantum subnetworks is reusable and updatable. Moreover, our construction scheme is compatible with the known quantum gate-assembly schemes. Obviously, now is still a coexistence period of all the possible candidates for the theory. Let’s do better and better in future.

In the next section we first list two main difficulties in the known quantum gate-assembly schemes. Our construction scheme for an universal quantum network from a theoretical view is presented in Section III. Then, we give out the whole quantum network for Shor’s and Grover’s algorithms. Moreover, we obtain a whole quantum network for generally simulating Schrödinger equation in Section IV. The last section is used for discussion and comparison with the known schemes, and for concluding remarks.

II. DIFFICULTIES IN KNOWN QUANTUM GATE-ASSEMBLY SCHEMES

The first example is quantum Fourier transformation. It plays an important role in quantum computation including factorization, search algorithms and quantum simulating. Its matrix $F$ reads

$$F = \frac{1}{\sqrt{N}} \sum_{m,n=0}^{N-1} e^{2\pi imn/N} |x_m\rangle\langle p_n|, \quad F^{-1} = \frac{1}{\sqrt{N}} \sum_{m,n=0}^{N-1} e^{-2\pi imn/N} |p_m\rangle\langle x_n|. \quad (1)$$

Quantum Fourier transformation can be rewritten [11] as

$$F = \sum_{n=0}^{2^k-1} \prod_{j=1}^{k} \left( |0\rangle + e^{2i\pi n/(2^k-1)} |1\rangle\right) \langle p_n| = \sum_{n=0}^{2^k-1} \prod_{j=1}^{k} B_j (2^j \pi n/(2^k - 1) H_j |0\rangle\langle p_n|. \quad (2)$$

where $B_j$ is a rotator gate and $H_j$ is a Hadamard gate acting on the $j$–qubit. Every term in the above summation can be regarded as the construction for a quantum (sub-)network with high efficiency in the known schemes. However, The total transformation is not an ideal form of an universal quantum network since it is not a product of its quantum subnetworks. In other words, one only constructed the effective quantum subnetwork for a part of quantum Fourier transformation acting on a given state $|p_n\rangle$, but has not obtained its whole quantum
network. Obviously, this is not enough, for example, Shor’s algorithm needs a whole quantum network for quantum Fourier transformation. Therefore, it is necessary to find such a method so that a whole quantum network can easily be built by its quantum subnetworks. If this is done, then we not only can utilize the known some effective constructions, but also we can have a powerful mean to seek an effective construction, that is, to decompose a quantum network into such a set of the quantum subnetworks that they can be effectively constructed as possible.

The second example is about the quantum simulating for Schrödinger equation. In quantum mechanics, the evolution of Schrödinger wavefunction with time can be written as:

\[
\psi(x, t) = T \exp \left\{ -i \int_0^t d\tau H \right\} \psi(x, 0)
\]

where \( T \) is a time-order operator and the natural unit system is taken. When \( H \) is independent on time, it simplifies as

\[
\psi(x, t) = e^{-iHt} \psi(x, 0)
\]

Choosing a short time step \( \Delta t \), the time evolution operator \( e^{-iHt} \) reads

\[
\Omega(\Delta t) = (1 - iH\Delta t)
\]

If we wish to advance the system by time \( T \), we can repeat the whole process \( T/\Delta t \) times, that is

\[
\Omega(T) = (1 - iH\Delta t)^{T/\Delta t}
\]

Now we would like to construct a quantum network for the time evolution operator according to the known quantum gate-assembly schemes. But, there exists an obvious difficulty. If we start directly from the total time evolution operator, we will lose the advantage of the discretization of the time. Actually it seems not to be feasible since departing from the physical idea. A very natural idea is to construct a quantum subnetwork for \( (1 - iH\Delta t) \) and then assemble all \( T/\Delta t \) quantum subnetworks together. However, the known quantum gate-assembly scheme has not provided how to do it. This, again, call us to find a method which can easily build a whole quantum network in terms of its quantum subnetworks.
A quantum computation, in mathematical, can be described by a (unitary) transformation matrix $U$ acting on an input state of a quantum system, that is

$$|\Psi\rangle' = U|\Psi\rangle = \sum_{m,n} U_{mn}|m\rangle\langle n|\sum_s \psi_s|s\rangle = \sum_{m,n} U_{mn}\psi_n|m\rangle$$

and each component of this quantum state is transformed as

$$\psi'_m = \sum_{n=0}^{2^k-1} U_{mn}\psi_n$$

The output state $|\Psi\rangle'$ contains the information that we need. Based on our ideas stated above, we first introduce an auxiliary qubit $A$ prepared in $|0\rangle_A$ and define two kinds of basic elements of quantum circuit $R_m, T_{mn}$

$$R_m(U_{mm}) = \exp\{(U_{mm}|m\rangle\langle m| \otimes I_A) \cdot C^\dagger\} = I_R \otimes I_A + (U_{mm}|m\rangle\langle m| \otimes I_A) \cdot C^\dagger$$

$$T_{mn}(U_{mn})(m \neq n) = \exp\{(U_{mn}|m\rangle\langle n| \otimes I_A) \cdot C^\dagger\} = I_R \otimes I_A + (U_{mn}|m\rangle\langle n| \otimes I_A) \cdot C^\dagger$$

where $I_R$ and $I_A$ are unit matrices in the register space and the auxiliary qubit space respectively, $N_A$ is a NOT gate acting on the auxiliary qubit. While $C^\dagger$ is an auxiliary element of quantum circuit

$$C^\dagger = I_R \otimes |1\rangle_{AA}\langle 0| = I_R \otimes c_A^\dagger$$

which satisfies that $C^{\dagger 2} = 0$. This leads to an exponential form to reduce to only two terms. It is clear that the action of $C^\dagger$ is able to joint the transformations for various basic vectors, that is the different elements of quantum circuit together. For example, $R_m(\alpha)R_n(\beta) = I_R \otimes I_A + [(\alpha|m\rangle\langle m| + \beta|n\rangle\langle n|) \otimes I_A] \cdot C^\dagger = \exp\{(\alpha|m\rangle\langle m| + \beta|n\rangle\langle n|) \otimes I_A\} \cdot C^\dagger$. So, $C^\dagger$ can be called as a “Jointer”. Furthermore, since $c_A = |0\rangle_{AA}\langle 1|$ and $c_A^\dagger = |1\rangle_{AA}\langle 0|$, it is easy to verify that $c_A^2 = c_A^{\dagger 2} = 0; c_A c_A^\dagger + c_A^\dagger c_A = I_A$. Thus $c_A$ and $c_A^\dagger$ can be thought as the fermionic annihilate and create operator respectively in the auxiliary qubit. From this, the auxiliary system might be able to extended to a larger system beyond a qubit if one can implement easily a nilpotent transiting operator.
$R_m(U_{mn})$ can be called the “Rotator” for its action makes $|m⟩\otimes|0⟩_A$ to rotate to $|m⟩\otimes|0⟩_A + U_{mn}|m⟩\otimes|1⟩_A$. $T_{mn}$ can be called the “Transistor” for its action makes $|n⟩\otimes|0⟩_A$ to map to $|n⟩\otimes|0⟩_A + U_{mn}|m⟩\otimes|1⟩_A$.

It is worth emphasizing that there is an essential difference between a classical gate and a quantum gate. It is just that a classical gate is always to carry out a determined operation, but a quantum gate can carry out a kind of operations. For example, a quantum rotation gate can rotate the state to any angle and then it needs a parameter $\phi$ or $e^{i\phi}$ to determine its operation. So do the rotator and transistor. Each rotator $R_n$ or transistor $T_{mn}$ depends on one parameter $U_{mn}$ or $U_{nm}$.

Since the action of the joiners, the rotator and transistor can be connected together and form a $(k+1)$-qubit quantum network,

$$Q(U) = \prod_{m,n=0}^{k-1} \exp\{U_{mn}|m⟩⟨n| \otimes I_A\}$$

$$= \prod_{m,n=0}^{k-1} \exp\{|(U_{mn}|m⟩⟨n| + U_{nm}|n⟩⟨m|) \otimes I_A/2\} \cdot C^\dagger \}$$

where $k$ qubits form the register space and a qubit belongs to the auxiliary space. It is easy to obtain its reversible operation

$$Q^{-1}(U) = \prod_{m,n=0}^{k-1} \exp\{-U_{mn}|m⟩⟨n| \otimes I_A\}$$

$$= \prod_{m,n=0}^{k-1} \exp\{-|(U_{mn}|m⟩⟨n| + U_{nm}|n⟩⟨m|) \otimes I_A/2\} \cdot C^\dagger \}$$

$Q(U)$ acting on $|Ψ⟩\otimes|0⟩_A$ just carries out a general quantum computation:

$$Q(U)|Ψ⟩\otimes|0⟩_A = |Ψ⟩\otimes|0⟩_A + |Ψ⟩'\otimes|1⟩_A$$

Thus, two project measurements

$$D = C^\dagger C = I_R \otimes |1⟩_A⟨1|, \quad P = CC^\dagger = I_R \otimes |0⟩_A⟨0|$$

result in an output state $|Ψ⟩'\otimes|1⟩_A$ and an input state $|Ψ⟩\otimes|0⟩_A$ respectively in a quantum computation (8).
It is worth emphasizing, in our construction scheme, that every element of quantum circuit is like a “building block”. They are simply putting together even without the order limitation, and all of them form a quantum (sub-)network. Obviously, the action of the joiner plays an important role for it. Likewise, because there are the jointers in our construction for an universal quantum network, we can obtain a quantum network for a summation of a set of quantum computing parts

\[ Q(U) = Q(U_1 + U_2 + \cdots + U_r) = Q(U_1)Q(U_2)\cdots Q(U_r), \]  

where \( Q(U_i) \) is a quantum subnetwork for a quantum computing part \( U_i \). This is one of the most important and new features of our construction. Again introduce a so-called “Connector” defined as

\[ C = I_R \otimes |0\rangle_{AA} \langle 1| = I_R \otimes c_A \]  

which is used to prepare a middle state so that this prepared state can be used in a successive transformation. Thus, we can obtain a quantum network for a product of a series of quantum computing steps

\[ Q(U) = Q(U_1U_2\cdots U_r) = I_R \otimes I_A + C^\dagger \left( \prod_{j=1}^{r} CQ(U_j) \right) CC^\dagger \]  

\[ = I_R \otimes I_A + \tilde{Q}(U) = \exp\{\tilde{Q}(U)\}, \]  

where \( Q(U_i) \) is a quantum subnetwork for a quantum computing step \( U_i \). This is another the most important and new feature of our construction. Note that in Eq.(21) \( I_R \otimes I_A \) is added so that the transformation \( U_1U_2\cdots U_r \) is reversible. The another way is to use two registers respectively to input state and out state and the latter includes one auxiliary qubit. Thus, the quantum network for product of a series of quantum computing steps becomes a form of full multiplication:

\[ \bar{Q}(U) = \bar{Q}(U_1U_2\cdots U_r) = (I_R)_{\text{input}} \otimes \left[ C^\dagger \left( \prod_{j=1}^{r} CQ(U_j) \right) CC^\dagger \right]_{\text{out}} = (I_R)_{\text{input}} \otimes \bar{Q}(U), \]  

while the initial state is now prepared as \( (|\Psi(t)\rangle)_{\text{input}} \otimes (|\Psi(t)\rangle \otimes |0\rangle_A)_{\text{out}} \). In special, if \( U = U_1 \otimes U_2 \otimes \cdots \otimes U_r \), we have
\[
\tilde{Q}(U) = \tilde{Q}(U_1 \otimes U_2 \otimes \cdots \otimes U_r)
\]
\[
= (I_R)_{\text{input}} \otimes \left[ C^\dagger \left( \prod_{j=1}^{r} CQ(I_1 \otimes \cdots \otimes I_{j-1} \otimes U_j \otimes I_{j+1} \otimes \cdots \otimes I_r) \right) CC^\dagger \right]_{\text{out}}
\]
\[
= (I_R)_{\text{input}} \otimes \tilde{Q}(U),
\]

Furthermore, we can obtain
\[
\tilde{Q}((U_1 \otimes U_2 \otimes \cdots \otimes U_r)V) = (I_R)_{\text{input}} \otimes C^\dagger C \tilde{Q}(U_1 \otimes U_2 \otimes \cdots \otimes U_r)CQ(V)CC^\dagger,
\]

This means that the role of \(\tilde{Q}\) is the same as one of \(Q\) when we connect them by the connectors. But, the role of \(\tilde{Q}\) is different from one of \(Q\) when they are the quantum networks for a summation of various quantum computing parts. Actually, this is a reason why we take a quantum subnetwork for a set of quantum computing parts to be, in general, a lower hierarchy than a quantum subnetworks for a series of quantum computing steps.

Therefore, a whole quantum network is constructed via a so-called “motherboard” with many “slots”. A slot is an interspace between a pair of connectors and then a motherboard is made of some connectors. The quantum subnetworks for a series of quantum computing steps, just like some “plug-in boards”, are inserted into these slots, and the quantum subnetworks for a set of quantum computing parts, even like some “building blocks”, are put together directly. All of them are so easily assembled and scaled as a whole quantum network which can carry out a general quantum computation.

Our construction provides not only a procedure to simplify a whole quantum network, but also a way to obtain its effective construction. That is, we can decompose a total quantum computation into a summation of a set of quantum computing parts and/or a product of a series of quantum computing steps so that quantum subnetworks for them can be constructed effectively as possible. Note that a quantum subnetwork for a set of quantum computing parts is usually a lower hierarchy than a quantum subnetworks for a series of quantum computing steps. As soon we obtain the effective constructions for the quantum subnetworks, an effective construction for a whole quantum network is just direct and facile.
Eq. (18), it is easy to get a whole quantum network for a quantum Fourier transformation in terms of its quantum subnetworks [11]

$$Q(F) = \prod_{n=0}^{2^{k-1}} Q[B(n)HM_{0n}] = \prod_{m=0}^{2^{k-1}} \prod_{n=0}^{2^{k-1}} \exp\{[(B(n)H)_{m0}|m\rangle \otimes I_A] \cdot C\dagger\},$$

(25)

where $B(n)H = \prod_{j=0}^{2^{k-1}} B_j[2^{j\pi n}/(2^k - 1)]H_j$ and $M_{0n} = |0\rangle\langle x_n|$. Because, each quantum subnetwork for a part of Fourier transformation acting on a given state is an effective construction, the whole quantum network $Q(F)$ is also so. The difficulty in the known construction scheme of quantum network is then overcome.

It is very important and interesting how to program a quantum network [10]. In our construction, the transformation parameters in a quantum computation are directly and explicitly related with the elements of quantum circuit. Resetting them is just program a quantum network. Therefore, our construction for an universal quantum network is potentially programmable.

Now, let’s build the relation between the elements of quantum circuit and the elementary quantum gate. In mathematics, the elements of quantum circuit are the natural basis of the operator in Hilbert space and each of them directly relates with a matrix element of the transformation matrix for a quantum computation. Of course, they can be constructed by the elementary quantum gates. [12] It is easy to see

$$|m\rangle\langle n| = \frac{1}{2^k} \prod_{i=0}^{k-1} [\delta_{\alpha_i0}\delta_{\beta_i0}(I + Z) + \delta_{\alpha_i1}\delta_{\beta_i1}(I - Z)]$$

$$+ \delta_{\alpha_i0}\delta_{\beta_i1}(X + Y) + \delta_{\alpha_i1}\delta_{\beta_i0}(X - Y)];$$

$$|m\rangle = \prod_{i=0}^{k-1} |\alpha_i\rangle, \quad \langle n| = \prod_{i=0}^{k-1} \langle \beta_i|,$$

(26)

(27)

where $X = \sigma_x; iY = \sigma_y; Z = \sigma_z$ and $\sigma_{x,y,z}$ are usual Pauli spin matrix. In fact, $|m\rangle\langle n|$ can be written as a product of a general exchange transformation between two adjacent states and a measurement $|n\rangle\langle n|$ from left or a measurement $|m\rangle\langle m|$ from right. To do this, we introduce the generalized exchange gate $E(m, m + 1)$ for two adjacent basic states $|m\rangle$ and $|m + 1\rangle$ defined by
\[ E(m, m + 1) = E(m + 1, m) = \sum_{j=0; j \neq m, m+1}^{2^k-1} |j\rangle\langle j| + |m\rangle\langle m+1| + |m+1\rangle\langle m|. \] (28)

Obviously, it is Hermian and unitary. It is a two state gate but not a qubit gate in general. It is easy to see that it acts \(|m\rangle\) or \(|m+1\rangle\) leads to their exchange and keeps the other basic states invariant. Note that for two qubits, \(E(2, 3)\) is a CNOT gate and \(E(1, 2)\) is a swapping gate. [13] The exchange transformation of arbitrary two states \(|m\rangle\) and \(|n\rangle\) can be constructed by

\[
E(m, n) = \begin{cases} 
\prod_{j=n}^{m-1} E(j + 1, j) & (n < m), \\
\prod_{j=0}^{n-m-1} E(n - j - 1, n - j) & (n > m), \\
I_R & (n = m).
\end{cases}
\] (29)

The order of product is arranged from the left side with index \(j\) increasing. Therefore, \(E(m, n)(m \neq n)\) can be expressed by a product of \(|m-n|\) successive the generalized exchange gates in which each of the generalized exchange gate is unitary and only involves two adjacent states. In special, \(E(m, m)\) is just an identity gate. Obviously, \(E(m, n)\langle n| = |m\rangle\) and \(\langle m|E(m, n) = \langle n|\). This implies that \(E(m, n)\) is a transiting unit from \(|n\rangle\) to \(|m\rangle\). So an arbitrary transformation \(U\) for \(k\) qubits can be written as

\[
U = \sum_{m,n=0}^{2^k-1} U_{mn}E(m, n)|n\rangle\langle n| = \sum_{m,n=0}^{2^k-1} U_{mn}|m\rangle\langle m|E(m, n). \] (30)

Thus the universal quantum network for \(U\) is just defined by

\[
Q(U) = \prod_{m,n=0}^{2^k-1} \exp\{(U_{mn}E(m, n)|n\rangle\langle n| \otimes I_A) \cdot C^\dagger\}. \] (31)

If the graphic rules for the elements of quantum circuit with form \(\exp\{(U_{mn}E(m, n)|n\rangle\langle n| \otimes I_A) \cdot C^\dagger\}\) are given out, the picture of the quantum network \(Q(U)\) can be drawn easily, because the construction of a quantum network is constructed by directly connecting the elements of quantum circuit together. In fact, the picture of the elements of quantum circuit such as rotators and transitors also can be drawn if we introduce the graphic rules for the general exchange gate for two adjacent states and the measurement gate.
It is worth noting that an element of quantum circuit is constructed by a transiting unit and a measurement and can be written as an exponential form. Perhaps, it is helpful to implement an element of quantum circuit in experiment and understand why the elements of quantum circuit may be fundamental in our construction scheme. Obviously, \( Q(U) \) is universal and the Eq.(31) is an alternative of Reck et.al's formula [6] since it keeps the advantages such as the product form, only involving two states (not qubit) and the closed relation with the elementary gates et.al. Even our construction can be applied to an irreversible and/or non-unitary transformation. This means we can have more ways to seek an effective construction and even a new quantum algorithm.

The key point is the efficiency of an universal quantum network. It seems that an universal quantum network consisting of \( 2^k \times 2^k \) basic elements is the same as a classical computer in use of computing resources. In fact, this is a price to reach at universality. Because the basic elements of quantum circuit only act on a branch (path) of the quantum data flow from its definition, \( Q(U) \) needs \( 2^k \times 2^k \) basic elements in general. However, in practice, our construction is obtained in terms of the symmetries and physical features in a quantum computation. This leads that many elements of quantum circuit do not really appear. For example, the simplest is a quantum network for a transformation of a single quantum state: \( |n\rangle \rightarrow e^{i\alpha_n}|n\rangle, |m\rangle (m \neq n) \rightarrow |m\rangle \)

\[
Q(S(e^{i\alpha_n})) = \exp\{C^\dagger\} \exp\{[(e^{i\alpha_n} - 1)|n\rangle\langle n| \otimes I_A] \cdot C^\dagger\}
\] (32)

where \( \exp\{C^\dagger\} = Q(I_R) \) is a quantum subnetwork for an identity gate and we construct \( Q(S(e^{i\alpha_n})) \) by decomposing this transformation into \( I + (e^{i\alpha_n} - 1)|n\rangle\langle n| \). In special, when \( \alpha_n = \pi \), this transformation is a reflection of \( |n\rangle \), which can be used in Grover’s algorithm.

Another example is a quantum subnetwork

\[
Q(U(i)) = \exp\{[I_1 \otimes I_2 \otimes \cdots \otimes U(i) \otimes \cdots \otimes I_k] \otimes I_A] \cdot C^\dagger\}
\] (33)

for a transformation only acting on the \( i \)-th qubit which leads \( |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \cdots \otimes |\alpha_i\rangle \otimes \cdots \otimes |\alpha_k\rangle \) to \( |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \cdots \otimes (U(i)|\alpha_i\rangle) \otimes \cdots \otimes |\alpha_k\rangle \). In special, for a controlled gate \( U_{\text{control}} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U \), we have

\[ \]
\[
Q(U_{\text{control}}) = \exp\{(|0\rangle\langle 0| \otimes I_2 \otimes I_A)C^\dagger\} \exp\{(|1\rangle\langle 1| \otimes U \otimes I_A)C^\dagger\}
\]

(34)

When \(U\) is a NOT gate, it is a quantum network for a controlled NOT. Likewise, we can write down a quantum network for Toffoli gate (controlled-controlled-NOT), in which the third qubit experiences NOT if and only if the others are in the state \(|1\rangle\):
\[
Q(\text{Toffoli}) = \exp\{C^\dagger\} \exp\{-(|1\rangle\langle 1| \otimes I \otimes I_A)C^\dagger\} \exp\{(|1\rangle\langle 1| \otimes N \otimes I_A)C^\dagger\}
\]

(35)

since we can decompose \(U_{\text{Toffoli}} = I - |1\rangle\langle 1| \otimes I + |1\rangle\langle 1| \otimes N\). Therefore, all the elementary quantum gates can be written down by our quantum networks. This implies that our construction scheme is compatible with the known quantum gate-assembly schemes because as soon as we have obtained an expression of a quantum computation in terms of a set of the elementary quantum gates, we can first construct the quantum subnetworks for these elementary quantum gates and then assemble easily them into a whole quantum network by virtue of the properties of our quantum network.

It must be emphasized that our construction has further simplification and more ways to build an universal quantum network. One of the simplest examples is a quantum network for a transformation of a single state stated above. Actually, for a diagonal transformation, its quantum network can be defined by \(Q(U_d) = \prod_{m=0}^{k-1} \exp\{(U_m|m\rangle\langle m| \otimes I_A)C^\dagger\}\). Moreover, Rotator, Transitor, even their combination with many branches can be introduced. For example, the conditional rotation gate \(R_2\) only acting on the second qubit in 3-qubit register is made of two Rotators with \(2^3/2\) branches. \[11\] Another typical case is the computing step \(V\) can be written as the direct product \(V(1) \otimes V(2)\). Suppose \(V(1)\) is \(2^{k_1} \times 2^{k_1}\) and \(V(2)\) is \(2^{k_2} \times 2^{k_2}\) \((k_1 + k_2 = k)\). Then, if in \(V(1)\) we can decrease a parameter or an elements, the result leads that \(2^{k_2} \times 2^{k_2}\) parameters or the number of the corresponding elements are decreased. The use of computing resource is then at high efficiency. So, to simplify \(U\) into the direct product of subspaces as possible is a better method to advance the efficiency of the use of computing resources. Quantum Fourier transformation only acting on a given state is just an example. Its quantum network can be effectively constructed since the
above reason [11]. Generally speaking, the number of the elements of quantum circuit in a quantum network for the transformation $U$ is at least equal to the number of the different matrix elements in $U$ except for zero. In general, we decompose it into a summation of a set of quantum computing parts and/or a product of a series of quantum computing steps so that we can find the symmetries and physical features in these quantum computing parts or steps as possible. The symmetries and physical features in a quantum computing part, step or a whole computation will largely decreased the number of the element of quantum circuit. Thus, in principle, we can finally obtain an effective construction for an universal quantum network.

It necessary to point out that there are three important features of the connector introduced by our construction. First, a connector is a standard interface unit between quantum subnetworks. In a classical algorithm, one usually does not need to consider how to connect two computing steps because the classical data flow is generally single branch (unless in parallel), but, in a quantum algorithm, one has to think over this problem because the quantum data flow is generally many branches. If with respect to two quantum computing steps or parts, that is two unitary transformations, one designs their quantum networks independently, then the arrangement ways of quantum data of input and output for two quantum subnetworks are different in general. This means that an interface unit is needed. How to design this interface unit just becomes a problem. Here, in our construction for an universal quantum network, connector as a standard interface unit is introduced and one does not worry about this problem again. Second, the interspace between two connectors looks like a “slot” for a quantum subnetwork and a quantum subnetwork appears as a “pinboard”. For example, the quantum network for quantum Fourier transformation can be reused in a whole quantum network for Shor’s algorithm or Grover’s algorithm by plugging in the corresponding slot. At present, one still does not know how to program a quantum network. This means that a quantum network is only able to carry out a given quantum computation. Therefore, it is important and interesting that the quantum subnetworks is made as plug-in and reusable one, because many resources can be significantly saved. Finally, the connector
provides a tool to assemble and scale some quantum (sub-)networks in order to carry out a larger quantum computation.

From this universal quantum network proposed by this paper, it follows a general design principle for a quantum algorithm and quantum simulating. That is, we ought to find such a suitable and optimized decomposition that the quantum subnetworks for its components can be effectively and easily constructed as possible. The process of decomposition can continue until our aim is arrived at. In principle, this is possible if a quantum computation can be effectively carried out. In order to do this, we need to use the fundamental laws of physics, specially the principles and features of quantum mechanics, for example, coordinate system choice, representation transformation, picture scheme and quantum measurement theory, if we have thought that a quantum computing task is a physical process. Moreover, we have to use the symmetry properties of every computing parts and/or step $U^i$ as possible, such as the direct product decomposition, the transposing invariance $U^i_{nm} = U^i_{mn}$, or the row equality $U^i_{nm} = U^i_{mn}$ for all $n$ or the line equality $U^i_{mn} = U^i_{nm}$ for all $m$ as well as make $U_i$ with zero elements and equal elements as many as possible. In general, we always can find such some symmetries through decomposing $U$ into a summation $U_1 + U_2 + \cdots + U_n$ and/or a product $U_1U_2\cdots U_n$. In the following, we will give some examples.

In the above sense, say it has been designed an universal quantum network which can carry out a general quantum computing. Obviously, our construction scheme can keep the advantages in the known schemes. More importantly, our method simplifies the construction for an universal quantum network in terms of the elementary quantum gates and standardize the process of construction as a procedure, that is, from the elementary quantum gates to the elements of quantum circuit, again to the quantum subnetwork and finally to a whole quantum network. Moreover, each step in this procedure is facile and determined. The significances of simplification and standardization is to provide not only the facility in engineering, but also the realizability in an effective construction. Furthermore, our construction provides a way to reuse and update a quantum subnetwork as well as potential possibility to program a quantum network.
In this section, we would like to show that the whole quantum networks of the known main quantum algorithms can be described by our universal quantum network.

Let’s start with Shor’s algorithm which is used to the factorization of a large number \( N \). It can speed up exponentially computing in a quantum computer than doing this in a classical computer. Shor’s algorithm can be described by the following five steps. First, start with two \( k \)--qubit registers in \(|0\rangle|0\rangle\), then prepare the first register into a superposition with the equal weight in terms of Fourier transformation or \( k \)--qubit Hadamard gate denoted by \( H \):

\[
H|0\rangle|0\rangle = \sum_{n=0}^{2^k-1} |n\rangle|0\rangle. \tag{36}
\]

Second, select randomly a factor \( a \) and make a mapping

\[
\sum_{n=0}^{2^k-1} |n\rangle|0\rangle \xrightarrow{G} \sum_{n=0}^{2^k-1} |n\rangle|a^n \text{mod} N\rangle. \tag{37}
\]

Obviously

\[
G = \sum_{n=0}^{2^k-1} |n\rangle|a^n \text{mod} N\rangle\langle n|\langle 0|. \tag{38}
\]

Third, measure the second register by \( I_1 \otimes |a^m \text{mod} N\rangle\langle a^m \text{mod} N| \) and obtain the result:

\[
\sum_{j=0}^{[2^k/r]-1} |jr + l\rangle|u\rangle. \]

Fourthly, do Fourier transformation \( F \) to the first register so that

\[
U_{\text{DFT}} |jr + l\rangle = \frac{1}{\sqrt{2^k}} \sum_{y=0}^{2^k-1} \exp\{2\pi i (jr + l)y/2^k\} |y\rangle. \tag{39}
\]

and obtain the final state \( \frac{1}{\sqrt{r}} \sum_{m=0}^{r-1} \exp(2\pi im/r) |m2^k/r\rangle \). The last, measure the first register in the basis \( y = m2^k/r \). If one obtains one values \( y \), then solve equation \( y/2^k = m/r \) to find the period. Once \( r \) is known the factors of \( N \) are obtained by calculating the greatest common divisor of \( N \) and \( a^{r/2} \pm 1 \).

Thus, the main steps in Shor’s algorithm can be represented by one total transformation matrix:
The product of the several matrices is an easy problem. After we know the form of $U(\text{Shor})$, we are able to obtain all of the parameters, that is the elements of $U$, to determine the construction of a quantum network for Shor’s algorithm in our method. In principle, this quantum networks also can be constructed by Barenco’s method. However, one can not guarantee that two kinds of constructions are effective in general. In addition, the latter may be a difficult work. In fact, the key skill in our method is to find an suitable and optimal decomposition of $U$ and seek the effective constructions of quantum subnetworks for every components in the decomposition. Since we have had an effective construction of the quantum network for quantum Fourier transformation, we only need to find the quantum subnetworks for $H$ and $G$ according to Shor’s decomposition. Obviously, starting from the initial state $|0\rangle_1 \otimes |0\rangle_2 |0\rangle_A$, it follows that $Q(H)$ is

$$\tilde{Q}(H) = I_{\text{input}} \otimes C\dagger \left( \prod_{j=1}^{k} (CQ(H_j \otimes I_2)) \right) CC\dagger = I_{\text{input}} \otimes \tilde{Q}(H).$$

(41)

Note that $H_j$ is a Hadamard transformation only acting on the $j$–qubit in the first register and the second register has kept the original state. While the realization of our quantum network for mapping $G$ can read

$$Q(G) = \prod_{n=0}^{2^k-1} \exp\{(|n\rangle|a^n \text{mod} N\rangle \langle n| \otimes |0\rangle \otimes I_A) \cdot C\dagger\}. \quad (42)$$

Furthermore, in terms of the connectors, the whole quantum network for Shor’s factorization can be obtained as the following:

$$\tilde{Q}(\text{Shor}) = (I_R)_{\text{input}} \otimes \left( C\dagger CQ(F \otimes I_2)M(a^m \text{mod} N) \otimes I_A CQ(G)C\tilde{Q}(H)CC\dagger \right)_{\text{out}}. \quad (43)$$

Obviously, our whole quantum network for Shor’s algorithm is a effective construction because that its main quantum subnetworks for Fourier transformation and Hadamard transformation are effective.

Another famous quantum algorithm is Grover’s algorithm which is used to search the expected term in an unstructured data. It can be described by the following four steps. First,
start with a $k$-qubit registers in $|0\rangle$, then prepare it into a superposition with the equal weight in terms of Fourier transformation or $k$-Hadamard gate, that is $H|0\rangle = \sum_{n=0}^{2^k-1} |n\rangle$.

Second, do a reflection:

$$R_2 = I - 2|j\rangle\langle j| = \sum_{n=0}^{2^k-1} (-1)^{\delta_{jm}} |x_m\rangle\langle x_m|,$$  \hspace{1cm} (44)

where $j$ corresponds to the expected data. Third, make the following operation:

$$R_1 = F^{-1}R_0 F = F^{-1}[2|0\rangle \langle 0|-I]F = -F^{-1}\sum_{m=0}^{2^k-1} (-)^{\delta_{0m}} |x_m\rangle\langle x_m|F,$$  \hspace{1cm} (45)

where $I$ is an identity matrix, $F$ is a quantum Fourier transformation, $F^{-1}$ is its inverse and $R_0 = 2|0\rangle \langle 0|-I$. The last, repeat $R_1R_2 \sqrt{N\pi/4}$ times and then do all measurements.

Since the quantum network for quantum Fourier transformation has been obtained and its inverse has the similar realization but its parameters with a negative sign. While $R_0$ and $R_2$ is diagonal, it is very easy to get from our construction for a quantum network

$$Q(R_0) = \exp\{(2|0\rangle \langle 0| \otimes I_A) \cdot C^\dagger\} \exp\{-C^\dagger\}$$  \hspace{1cm} (46)

$$Q(R_2) = \exp\{C^\dagger\} \exp\{(-2|x_j\rangle \langle x_j| \otimes I_A) \cdot C^\dagger\}.$$  \hspace{1cm} (47)

Thus the quantum network for Grover’s algorithm is just obtained

$$Q(\text{Grover}) = (I_R)_{\text{input}} \otimes \left(C^\dagger CQ(F^{-1})CQ(R_0)CQ(F)CQ(R_2)CQ(H)CC^\dagger\right)_{\text{out}}.$$  \hspace{1cm} (48)

Again, because we have used an effective construction of quantum subnetwork for quantum Fourier transformation, our whole quantum network for Grover’s algorithm is an effective construction.

In fact, the known main quantum algorithms have realized such a suitable decomposition so that their every computing steps and/or parts have some symmetries. Thus, we can find the effective constructions of the quantum subnetworks for the quantum computing steps and/or parts and connected these subnetwork together in terms our method and finally obtain a whole quantum network with the effective construction.
Simulating quantum systems has such a meaning that using a specially designed quantum system, for example quantum computer, which is called a simulated system, to simulate another so-called physical quantum system. In 1982, Richard Feynman first proposed that a quantum system would be more efficiently simulated by a computer based on the principles of quantum mechanics rather than by one based on the principles of classical mechanics [1]. This is because that the size of the Hilbert space grows exponentially with increase of the number of particles. A full quantum simulating demands the exponential resources on a classical computer so that it is in general intractable. Since the discovery by Shor of a quantum algorithm for factoring in polynomial time [3], there has been tremendous activity in the field of quantum computation including quantum simulating. For example, Lloyd has shown how a quantum computer is in fact an efficient quantum simulator [14]. In addition, some the general ideas and schemes of quantum simulating to several special quantum systems were proposed and discussed [7,8,15–21]. At present, quantum simulating mainly performs a simulation of the dynamics.

If one would like to simulate a quantum system by a quantum computer, the first task is how to “write” quantum state into the quantum computer, in other words, how to store the information of physical system – quantum state in the quantum computer. Although quantum computer has an ability to store information increasing exponentially than classical computer, it deals with the information as digital one just like classical computer. Consequently, a basic skill is to discretize the wavefunction which describes the quantum state in a finite space.

\[ \psi(x = a, t) \rightarrow \psi(x_m, t) \quad (m = \left\lfloor \frac{a}{L/N} \right\rfloor). \]  

(49)

where \( L/N \) is the length of \( |x_m - x_{m-1}| \). In addition, when the quantum system is limited within a box, it ought to impose the periodic boundary conditions, that is

\[ \psi(x_{m+N}, t) = \psi(x_m, t). \]  

(50)
Therefore, the wavefunction in time $t$ can be written as a vector

$$|\Psi(t)\rangle = \sum_{m=0}^{N-1} \psi(x_m, t) |x_m\rangle,$$

(51)

where $|x_m\rangle$ form a set of basis with the properties of orthogonality and completeness
\[\langle x_m | x_n \rangle = \delta_{mn}, \quad \sum_{m=0}^{N-1} |x_m\rangle\langle x_m| = 1\] in $N$-dimensional Hilbert space. When this is extended to two particles, it follows that

$$|\Psi(t)\rangle = \sum_{m_1=0}^{N_1-1} \sum_{m_2=0}^{N_2-1} \psi(x_{m_1}, x_{m_2}, t) |x_{m_1}, x_{m_2}\rangle.$$

(52)

Of course, the extension to high dimensional is similar. In fact, the above equation also can describe two dimensional case. (Usually taking $N_1 = N_2$, this means that the particle moves in a square box). A quantum register with $k$ qubits can express a state in Hilbert space at most with $N = 2^k$ dimensional. For two particles in one dimensional, its Hilbert space should be $N_1 \times N_2 = 2^{k_1+k_2}$. Thus, in a classical computer, we need exponentially increasing bits to deal with the quantum state. But in a quantum computer, we see that $k_1 + k_2$ qubits, or two quantum registers with $k_1$ and $k_2$ qubits respectively, are suitable to this task. In fact, this is just one of reasons why simulating a quantum system can be more rapidly done by use of quantum computer than by use of classical computer. Generally speaking, for 3-dimensional and $n$ particles, we can use $3n$ quantum registers to store the discretizing the wavefunction. For the system of identical particles the initial state of the quantum computer has to be chosen symmetrically or anti-symmetrically. For quantum field theory, its discretized method can be similar to one in lattice gauge theory.

It is worth emphasizing how to realize the fundamental operators such as coordinates and momentum is a key point. In coordinate representation, coordinates are directly written as a diagonal matrix whose diagonal elements are $x_m$. But the momentum has a little complication because it is a derivative action as the following

$$\hat{p}\psi(x, t) = -i \frac{\partial \psi(x, t)}{\partial x} = -i \frac{\psi(x + \Delta x, t) - \psi(x)}{\Delta x}.$$

(53)

So the momentum operator can be defined by:
\[
\hat{p} = -i \frac{1}{2} \left( \frac{N}{L} \right) \sum_{m=0}^{N-1} \left( |x_m\rangle \langle x_{m+1}| - |x_m\rangle \langle x_{m-1}| \right),
\]

where, in order to make the momentum operator is Hermian, an average of the left and right derivative has been taken and the periodic boundary conditions (50) has been used. It is easy to verify

\[
\hat{p}|\Psi(t)\rangle = \sum_{m=0}^{N-1} \left[ -i \frac{1}{2} \frac{\psi(x_{m+1},t) - \psi(x_{m-1},t)}{(L/N)} \right] |x_m\rangle.
\]

Further, the kinetic energy operator (natural unit system \(\hbar = c = 1\)) can be obtained:

\[
\hat{T} = -\frac{1}{8\mu} \left( \frac{N}{L} \right)^2 \left[ \sum_{m=0}^{N-1} \left( |x_m\rangle \langle x_{m+2}| + |x_m\rangle \langle x_{m-2}| \right) - 2I_N \right].
\]

It is clear that the potential operator is a diagonal transformation if one only consider the local potential \(U(x)\) or the external field \(V_e(x)\), that is

\[
\hat{U} = \sum_{m=0}^{N-1} U(x_m) |x_m\rangle \langle x_m|,
\]

\[
\hat{V}_e = \sum_{m=0}^{N-1} V_e(x_m) |x_m\rangle \langle x_m|,
\]

while two-body local interaction \(U(x_1, x_2)\) can be written as

\[
\hat{U} = \sum_{m_1 = 0}^{N_1-1} \sum_{m_2 = 0}^{N_2-1} U(x_{m_1}, x_{m_2}) |x_{m_1}, x_{m_2}\rangle \langle x_{m_1}, x_{m_2}|.
\]

It is easy to extend to high dimensional and many particles formally. For example, in two particles case, the fundamental momentum operators are \(\hat{p}_1 = \hat{p}^{(1)} \otimes I_2, \hat{p}_2 = I_1 \otimes \hat{p}^{(2)}\). Then, the kinetic energy and the potential energy operators can be constructed in a similar way.

Hamiltonian has been here. Now it appears, in principle, that one can use the known quantum gate-assembly schemes to design a quantum network for the time evolution operator. However, obviously if one start directly from the total time evolution operator, he/her will lose the advantage of the discretization of the time. Actually it seems not to be feasible since departing from the physical idea. Even in mathematics, the quantum network for a
time evolution operator $\Omega(t)$ in a finite time is very difficult to construct since $\Omega(t)$ with infinite terms. A very natural idea is to construct a quantum subnetwork for $(1 - iH\Delta t)$ and then assemble all $T/\Delta t$ quantum subnetworks together. However, the known quantum gate-assembly scheme has not provided how to do it. Moreover, if one tries to write a small enough evolution operator within a given precision in a small time interval, then maybe one does not know clearly how to treat with an approximately unitary transformation in terms of the known schemes. In fact, the known scheme was proved only for an unitary transformation. In order to overcome the above difficult, we use our construction scheme for an universal quantum network. First, we define the momentum quantum subnetwork as:

$$Q(\hat{p}) = I_R \otimes I_A + \hat{p}_R \cdot C^\dagger = \exp\{\hat{p}_R \cdot C^\dagger\}$$

$$= \prod_{m=1}^{2^k-1} \exp\left\{ -i \frac{L}{2N} (|x_m\rangle\langle x_m| [E(m, m + 1) - E(m, m - 1)] \otimes I_A) \cdot C^\dagger \right\} ,$$

where

$$\hat{p}_R = -i \frac{L}{2N} \sum_{m=0}^{2^k-1} |x_m\rangle\langle x_m| (E(m, m + 1) - E(m, m - 1)) \otimes I_A$$

In terms of the above method the quantum subnetworks for the kinetic energy and the potential energy with the factor $-i\Delta t$ can be constructed as:

$$Q(-i\Delta t T) = \exp\left\{ -i \frac{\Delta t}{4\mu} \frac{L^2}{N^2} C^\dagger \right\} \prod_{m=1}^{2^k-1} \exp\left\{ i \frac{\Delta t}{8\mu} \left( \frac{L}{N} \right)^2 (|x_m\rangle\langle x_m| E(m, m + 2) \otimes I_A) \cdot C^\dagger \right\} \exp\left\{ i \frac{\Delta t}{8\mu} \left( \frac{L}{N} \right)^2 (|x_m\rangle\langle x_m| E(m, m - 2) \otimes I_A) \cdot C^\dagger \right\} ,$$

$$Q(-i\Delta t V) = \prod_{m=0}^{2^k-1} \exp\{ -i\Delta t V(x_m)|x_m\rangle\langle x_m| \otimes I_A \cdot C^\dagger \} .$$

According to the method stated above, the quantum subnetwork for the time evolution operator in a short time interval reads

$$Q(\Omega(\Delta t)) = Q(I_R)Q(-i\Delta t H) = \exp\{C^\dagger\}Q(-i\Delta t T)Q(-i\Delta t V) .$$

For a finite time, take the product of all the time evolution operators at small enough time steps and then obtain finally the whole quantum network for the time evolution operator:
\[ \hat{Q}(e^{-iHt}) = (I_R)_{\text{input}} \otimes \left[ C^\dagger \left( \prod_{i=1}^{[T/\Delta t]} CQ(\Omega(\Delta t)) \right) CC^\dagger \right]_{\text{out}}. \quad (66) \]

By using of it, Schrödinger equation can be simulated in general. In above procedure, the whole quantum network for the time evolution operator is divided into \([T/\Delta t]\) quantum subnetworks for the time evolution operator in a short time interval, and each such quantum subnetwork is decomposed to two quantum sub-subnetworks \(Q(T)\) and \(Q(U)\) for the kinetic energy and potential energy respectively. Then, \(Q(T)\) and \(Q(U)\) are effectively constructed by using of the elements of quantum circuit. In our scheme, these quantum subnetworks are easily assembled as a whole quantum network in order to simulate Schrödinger equation in general.

For two particles, the extension of this method is direct, but it is not efficient enough in the use of computing resources if one does it directly. Although the quantum network for the kinetic energy operator is obtained by the same method, but the quantum network for two body potential needs all \(2^k \times 2^k\) basic elements. Thus, a better method is first to reduce Schrödinger equation to the mass center and the relative coordinate systems. In the mass center system, we need to simulate a free practice, and in the relative coordinate system, we need to simulate Schrödinger equation in a single-body potential action. For a free particle, its Hamiltonian is diagonal in the momentum representation. Thus, we can simulate it by a quantum subnetwork for Fourier transformation and a quantum subnetwork for a diagonal Hamiltonian. For the relative motion with a single-body potential, its quantum network is able to be obtained in our method stated above. This is just a example to build an effective construction by virtue of the physical principles. Of course, the efficiency problem here is also said with respect to the comparison among the different quantum algorithms. Because, the quantum network is built in quantum parallelism, that is, it acts on all states at the same time. Therefore, with respect to classical computing, it must be efficient. In this sense, the above method to simulate Schrödinger equation can be extended to higher dimensional and more particles.
VI. DISCUSSIONS

At present, it seems that known construction schemes for a quantum network are not really “universal”. One of the reasons is that one has not a standard procedure to decompose a quantum computation to the elementary quantum gates. Another reason is a quantum network can not be reapplied to the other quantum computation task in general. However, in our view, a perfect physics theory in a field should be able to describe everything within its field, and a powerful science method in a field should be able to deal with every interesting problem within its field. In other words, a useful and competitive candidate for a theory will be able to describe the known results and had better to predict some new results. It is clear that our new construction scheme has such some features because it can give out the whole quantum networks for the main known quantum algorithms, can obtain the quantum network for quantum simulating of Schrödinger equation in general, and is helpful to find new quantum algorithms. Moreover, our scheme can use the known results in an effective construction of the quantum network. In special, our scheme is compatible with the known quantum gate-assembly schemes because we have prove that all the elementary quantum gates can be written as our quantum subnetworks. Thus, the known results from a quantum computation to a set of the elementary quantum gates can be directly inherited and used in our scheme. On other hand, our construction scheme also keeps some advantages in the known schemes such as in a product form, every factor only involves two states (not qubit) et. al. Furthermore, our construction scheme of quantum (sub-)network can be extended to a non-unitary transformation and every element of quantum circuit is put into an exponential form. This implies that we have more means to get an effective construction of a quantum network and implement it in experiment.

In engineering, a whole “machine” should be assembled in terms of the its components by plugging and putting together, every component should be able to be replaced or moved out in order to reuse, update as well as repair, and a set of “machine” had better to be able to be scaled up a larger “machine” so that its power gets stronger and satisfy more
purposes. Moreover, the assembling and scaling scheme should be clearly given out and had better to be so facile like playing the building blocks and plugging boards. In order to pursue these goals, we introduce the elements of quantum circuit as the in-betweens from the elementary quantum gates to the quantum subnetworks, and the elements of circuit are like some building blocks for the quantum subnetworks. Furthermore, a quantum subnetwork corresponding to a quantum computing step is made just like a “plug-in board” and a quantum subnetwork corresponding to a quantum computing part is made just like a “building block”. Usually, a quantum subnetwork corresponding to a quantum computing part is a lower hierarchy than a quantum subnetwork corresponding to a quantum computing step. In special, we design a powerful connector. A “motherboard” for a whole quantum network is made of some connectors and a interspace between a pair of connectors forms a “slot”. The boards of quantum subnetworks are inserted into these slots and they together play a whole quantum network. And the blocks of quantum sub-subnetworks are just simply put together to play a quantum subnetwork. Thus, the construction of an universal quantum network for a quantum computation is simplified and standardized. So we can easily assemble them and generally deal with a quantum computing step or quantum computing part, even a total quantum computation in our way. Obviously, in our construction the motherboard of a quantum network, quantum subnetworks and its any hierarchy becomes reusable and updatable. This is very important in engineering. In addition, since the direct connection with the matrix elements in the elements of quantum circuit and quantum subnetworks, a quantum network is potentially programmable [10], that is resetting its parameters—matrix elements is possible.

Note that only an effective construction is really useful, standardization, universality and facility are then limited to some content. We have to put those elements with the same or similar physical features and mathematical symmetries together in order to form some effective quantum subnetworks. This belongs to the art of design, which is still “individuation”. In other words, the key point is now how to decompose a quantum computation into its computing steps and/or parts so that the quantum subnetworks for those steps and
parts can be effectively constructed with the biggest possibility. Our scheme has made the improvements in two aspects. One is that if one wants to find an effective construction for a quantum network, in general, his/her needs to use the symmetries in the corresponding transformation matrix and physical features in a given quantum system, while these symmetries and features are usually connected with the elements of this transformation matrix. Based on our scheme related with the matrix elements directly, to find an effective construction should be easier than the known schemes without obvious and direct relation with the matrix elements. Another is that in our construction we can continuously to decompose the quantum subnetwork to its lower hierarchy only in a finite steps so that the lowest hierarchy of subnetwork can be constructed effectively as possible. This implies that our construction scheme have more means than the known ones. This is not trivial. An effective construction of a whole quantum network is often too complicated to do [6] because its symmetries can not be directly found out or utilized. But an effective construction for a quantum subnetwork is simpler since itself is just simpler and it can reduce into more simpler hierarchy of quantum subnetworks for the effective constructions. Moreover, we can use some known and knowing in future effective constructions for some quantum subnetworks, for example, quantum network for quantum Fourier transformation.

In conclusion, the main advantages of our scheme are its simplicity and standardization in construction, facility and directness in assembly and scale, reusage and update in engineering, realizability and procedurization in an effective construction, compatibility with and heritage from the known schemes. Moreover our universal quantum networks are potentially programmable. In special, we have shown our scheme can be applied to the known main quantum algorithms and simulating Schrödinger equation. Therefore, we are sure that our scheme is a possible candidate for an universal quantum network.

Of course, our construction scheme leaves some open questions. For example, how to program it and how to implement it in experiment.

This research is on progressing.
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