Connecting Green’s Functions in an Arbitrary Pair of Gauges and an Application to Planar Gauges

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Abstract

We establish a finite field-dependent BRS transformation that connects the Yang-Mills path-integrals with Faddeev-Popov effective actions for an arbitrary pair of gauges F and F’. We establish a result that relates an arbitrary Green’s function [either a primary one or one that of an operator] in an arbitrary gauge F’ to those in gauge F that are compatible to the ones in gauge F by its construction [in that the construction preserves expectation values of gauge-invariant observables]. We establish parallel results also for the planar gauge-Lorentz gauge connection.

1 INTRODUCTION

The importance of Standard Model [SM] calculations in particle Physics cannot be overestimated. The Standard Model calculations require the choice of a gauge. There are a variety of these which have been used in various different context. Some of these are the Lorentz-type gauges, axial-type gauges, light-cone gauge, planar gauges, radial gauges, nonlinear gauges, the R_\xi-gauges, Coulomb
gauge etc. Different gauges have been found useful and convenient in different calculational context [1]. A priori, we expect from gauge invariance, that the values for physical observables calculated in different gauges are identical. Formal proofs of such equivalence for S-matrix elements has been given in a given class of gauges, say the Lorentz-type gauges with a variable gauge parameter $\lambda$ [2,3]. Some isolated attempts to connect $S$-matrix elements in singular [rather than a class of them] gauges also have been done. For example, formal equivalence of $S$-matrix elements in the Coulomb and the Landau gauges [both singular gauges] has also been established [2,4]. Similar formal attempts to connect the [singular] temporal gauge with Feynman gauge in the canonical formalism has also been done [5,6]. It is important to note that, however, the Green’s functions in the gauges such as the Coulomb [7], the axial, the planar and the light-cone [8,9] in the path integral formulation are ambiguous on account of the unphysical singularities in their propagators. Hence, it becomes important to know how to define the Green’s functions in such gauges in such a manner that they are compatible to those in a well-defined covariant gauge such as the Lorentz gauge. A general procedure that connects Green’s functions in the path integral formulation in two classes of gauges, say the Lorentz and the axial, has been lacking until recently. Such comparisons are important not just in a formal sense but also in practice. Precisely because of this, the proper treatment of the $1/\eta \cdot q$ type poles in axial and light-cone gauges (and also similar questions in the Coulomb gauge [7]) has occupied a lot of attention [8,9] and the criterion used for their validation has, in fact, been the comparison with the calculational results in the Lorentz gauges. Such comparative calculations, where possible [8,9, 10] have to be done by brute force and have been done to $O(g^4)$ generally; thus limiting the scope of their confirmation. At a time, a physically observable anomalous dimension was reported to differ in Lorentz and axial gauges [1]. Such questions motivate us to develop a general path integral formalism that can address all these questions in
a wide class of gauges in a single framework. In a purely Feynman diagrammatic approach, we, of course, have the attempt of Cheng and Tsai [7].

Recently, we developed a general path integral formalism [11] for connecting pairs of Yang-Mills effective actions and applied it, in particular, to connecting the Lorentz and the axial type gauges [12,1]. In reference 11, we have also applied the procedure to connecting to the general BRS-anti-BRS invariant effective action of Baulieu and Thierry-Mieg. Our formalism is based on the Finite Field-dependent BRS [FFBRS] transformations [11] that connect the two path-integrals. These transformations are of a [field-dependent] BRS-type [11] and are evaluated in a closed form and leave the vacuum expectation value of a gauge-invariant observable explicitly invariant [11,13] as one transforms from say, a Lorentz-type to an axial-type gauge. Our procedure, in fact, gives a way of defining carefully Green’s functions in the axial-type gauges by a path-integral that explicitly takes care of the ill-defined nature of the propagator [14] and in a manner compatible with those in Lorentz gauges. We found an effective treatment of the axial propagator using this procedure [14,15] and applied this formalism also to show the preservation of the Wilson loop in axial gauges [16]. To summarize, the output of the works [11-14] has been (i) an explicit closed field transformation in \([A,c,\tau]\) space to connect the path-integrals in the two gauges that preserves gauge-invariant observables, (ii) A relation that allows one to calculate the Green’s functions in axial gauges, compatible with those in Lorentz gauges by the very construction, (iii) way of dealing with axial poles that is compatible with the Lorentz gauges. A simplified proof of (ii) above was also given using the BRS WT-identity [17].

Now that the above techniques have matured, we propose in this work to generalize our previous works in several directions in this one. In Section 3, we prove
the existence of the FFBRS transformation the connects Faddeev-Popov effective actions \([FPEA]\) with any two arbitrary gauge functions \(F\) and \(F'\). This applies to all the gauges mentioned above in the first a paragraph [expect the Planar gauge for which the action in not in the manifest FPEA form]. In section 4, we generalize the identity that connects the Green’s functions in gauge \(F'\) to the Green’s functions in the gauge \(F\) to the case of an arbitrary pair. In section 5, we develop the results for connecting Green’s functions in planar gauges to those in Landau gauge. In section 6, we give a heuristic treatment that uses the results of section 3 and which connects the vacuum expectation of gauge-invariant observable in planar gauges to those in Lorentz gauges. This result can also be verified by explicit calculations that run parallel to those in the section 3; and is not limited, however, by the heuristic treatment we have given for compactness.

2 PRELIMINARY

In this section, we shall review the results in earlier works \([11-13]\) and introduce notations. We consider two arbitrary gauge fixing functions \(F[A]\) and \(F'[A]\) which could be nonlinear or non-covariant and an interpolating gauge function \(F^M[A] = \kappa F'[A] + (1-\kappa)F[A] \); \(0 \leq \kappa \leq 1\). The Faddeev-Popov effective action\([FPEA]\) in each case is given by

\[
S_{eff} = S_0 + S_{gf} + S_{gh}
\]

with

\[
S_{gf} = \frac{1}{2\pi} \int d^4x \, \hat{F}^\gamma [A]^2
\]

\[
S_{gh} = -\int d^4x \, \overline{c}^\alpha \hat{M}^{\alpha\beta} c^\beta
\]

with

\[
\hat{M}^{\alpha\beta} = \frac{\delta}{\delta} F^\gamma [A] \hat{D}^\gamma_{\mu} [A]
\]

\[
S_{gh} = -\int d^4x \, \overline{c}^\alpha \hat{M}^{\alpha\beta} c^\beta
\]

\[
\delta = \frac{\hat{D}^\gamma_{\mu} [A]}{\delta} \hat{D}^\gamma_{\mu} [A]
\]

\[
\hat{M}^{\alpha\beta} = \frac{\delta}{\delta} F^\gamma [A] \hat{D}^\gamma_{\mu} [A]
\]

\[
\frac{\delta}{\delta} F^\gamma [A] \hat{D}^\gamma_{\mu} [A]
\]
\[ D^{\alpha\beta}_\mu [A] = \delta^{\alpha\beta}_\mu \partial_\mu + g_0 f^{\alpha\beta\gamma}_\mu A^\gamma \quad (2.3a) \]

We denote the FPEA for the three cases \( \hat{F} = F, F', F^M \) by \( S_{eff}, S'_{eff} \) and \( S^M_{eff} \) respectively. BRS transformations for the three effective actions are:

\[ \phi_i' = \phi_i + \delta_{iBRS}[\phi] \delta \Lambda \quad (2.4) \]

with \( \delta_{iBRS}[\phi] \equiv D^{\alpha\beta}_\mu c^\alpha \gamma \partial_\mu c^\beta, \) \( g_0 f^{\alpha\beta\gamma}_\mu c^\alpha c^\beta c^\gamma \) and \( \tilde{F}/\lambda \) respectively for \( A, c \) and \( \bar{c} \). In the case of the mixed gauge condition, \( \delta_{iBRS}[\phi] \) for \( \bar{c} \) is \( \kappa \)-dependent and in this case we show this explicitly by expressing (2.4) as

\[ \phi_i' = \phi_i + \bar{\delta}_{iBRS}[\phi, \kappa] \delta \Lambda \equiv \phi_i + \{ \bar{\delta}_{iBRS}[\phi] + \kappa \bar{\delta}_{2iBRS}[\phi] \} \delta \Lambda \quad (2.4a) \]

Following observations in [11] and [12], we guess and later prove the field transformations that take one from the gauge \( F \) to gauge \( F' \). It is given by the finite field-dependent BRS transformation [FFBRS]

\[ \phi_i' = \phi_i + \delta_{iBRS}[\phi] \Theta[\phi] \quad (2.5) \]

where \( \Theta[\phi] \) has been constructed by the integration [11] of the infinitesimal field-dependent BRS [IFBRS] transformation

\[ \frac{d\phi_i}{dt} = \delta_{iBRS}[\phi(\kappa)] \Theta'[\phi(\kappa)] \quad (2.6) \]

(where \( \delta_{iBRS}[\phi(\kappa)] \) refers to the BRS variations of the gauge \( F \)) with the boundary condition \( \phi[\kappa=1]=\phi' \) and \( \phi[\kappa=0]=\phi \) and is given in a closed form by [11]

\[ \Theta[\phi] = \Theta[\phi][\exp[f[\phi]]-1]/f[\phi] \quad (2.6a) \]

and \( f \) is given by

\[ f = \sum_i \frac{\delta \Theta[\phi]}{\delta \phi_i} \delta_{iBRS}[\phi] \quad (2.6b) \]

We wish to develop, in this work, an FFBRS for connecting two arbitrary gauges \( F \) and \( F' \). We shall show that in this case it is given by an FFBRS of the form (2.5) with \( \Theta'[\phi(\kappa)] \)
given by

$$\Theta[\phi(\kappa)] = \int d^4y \, \psi(y) \, (F^\gamma A(\kappa)) - F^\gamma A(\kappa))$$  \hspace{1cm} (2.7)

The Jacobian for the IFBRS transformation is defined as

$$D\phi[\kappa = 0] = D\phi[\kappa] J(\kappa) = D\phi[\kappa + d\kappa] J(\kappa + d\kappa)$$  \hspace{1cm} (2.8)

The change in the Jacobian for the IFBRS of (2.6) is given by

$$-\frac{i}{d\kappa} d\kappa = \int d^4x \, \sum_i \left[ \pm \frac{\delta^2 \phi(x, \kappa)}{\delta \psi_i(y, \kappa)} \right]_{x=y}$$  \hspace{1cm} (2.9)

and is evaluated easily as

$$-\frac{i}{d\kappa} d\kappa = i \int d^4x \{ \tau(M-M') c + \frac{1}{\lambda} F[F - F'] \}$$  \hspace{1cm} (2.10)

Further we define

$$S_{\text{eff}}^M[\phi(\kappa), \kappa] = S_{\text{eff}}[\phi(\kappa)] + S_1[\phi(\kappa), \kappa]$$  \hspace{1cm} (2.11)

We then have,

$$S_1[\phi(\kappa), \kappa] = \int d^4x \left[ \frac{1}{4\pi} \kappa^2 F^\gamma A(\kappa) F^\gamma A(\kappa) + 2\kappa (1 - \kappa) F^\gamma A(\kappa) F' A(\kappa) + \kappa (\kappa - 2) F^\gamma A(\kappa) F^\gamma A(\kappa) F^\gamma A(\kappa) \right]$$

$$+ \kappa (\kappa - 2) F^\gamma A(\kappa) F^\gamma A(\kappa) F^\gamma A(\kappa) \right] c(\kappa) \right]$$  \hspace{1cm} (2.12)

We introduce the following notation

$$< f[\phi(\kappa)] > \kappa = \int D\phi(\kappa) f[\phi(\kappa)] \exp \left\{ i \, S_{\text{eff}}^M[\phi(\kappa), \kappa] \right\}$$  \hspace{1cm} (2.13)

In references [11] and [13] it was established that the expectation value of a gauge-invariant observable $$< O[A(\kappa)] > \kappa$$ is independent of $$\kappa$$ iff the Jacobian $$J(\kappa)$$ and the effective action $$S_{\text{eff}}^M[\phi(\kappa), \kappa]$$ satisfy

$$< \frac{1}{d\kappa} dJ \frac{dS_1[\phi(\kappa), \kappa]}{d\kappa} > \kappa = 0$$  \hspace{1cm} (2.14)

In Section 3, we shall verify (2.14) for the IFBRS of (2.6).

### 3 FFBRS Transformations Connecting Any Pair \([F,F']\) of Gauges

In the references [11,12] explicit field transformations of the FFBRS type that connected various pairs of equivalent effective actions for the Yang-Mills theory were constructed. In this work, we wish to give a result that generalizes it for
FPEA in a pair of arbitrary gauges $F$ and $F'$ that includes for example those mentioned at the beginning of the Introduction [except the planar one]. These field transformations are such that they preserve, by an explicit construction, the expectation values of gauge-invariant observables.

Consider the expectation value of a gauge invariant observable $O[A]$ in the mixed gauge:\footnote{It is understood that like the Lorentz gauges, an appropriate $O(\epsilon)$ term \cite{14,15} is necessary in (3.1) to make it well-defined.}

\[
<<O[A]>>_\kappa \equiv \int D\phi(\kappa) O[A(\kappa)] \exp\{ i S_{eff}^M[\phi(\kappa),\kappa]\} \quad (3.1)
\]

where $\phi(\kappa)$ represent fields as defined in (2.6) with the specific $\Theta'$ given by (2.7).

We show that

\[
\frac{d}{dc}<<O[A]>>_\kappa \equiv 0 \quad (3.2)
\]

As shown in \cite{13}, (3.2) is valid iff

\[
0 \equiv \int D\phi(\kappa) \left\{ \frac{1}{d}\frac{dI}{dc} - \frac{i}{d} \frac{dS_i[\phi(\kappa),\kappa]}{dc} \right\} O[A(\kappa)] \exp\{ i S_{eff}^M[\phi(\kappa),\kappa]\} \quad (3.3)
\]

where we recall from (2.11) [noting the definitions below (2.3a)]

\[
S_i[\phi(\kappa),\kappa] = S_{eff}^M[\phi(\kappa),\kappa] - S_{eff}[\phi(\kappa)] \quad (3.4)
\]

We verify the result (3.3) explicitly below. The proof proceeds much as in \cite{11} and \cite{12} and hence we shall give it briefly. The change in the Jacobian under the IFBRS of (2.6)

\[
\delta \phi_i(\kappa) = \delta_{IBRS}[\phi(\kappa)] \Theta'[\phi(\kappa)] d\kappa
\]

\[
= \delta_{IBRS}[\phi(\kappa)] \int_d d^4y \bar{\tau}(y) \left( F^\gamma[A(\kappa)] - F'\gamma[A(\kappa)] \right) d\kappa \quad (3.5)
\]

viz. $\frac{d}{dc} \frac{d}{d\kappa}$ is given by (2.10); viz

\[
\frac{d}{dc} d\kappa = i \int d^4x \{ \bar{\tau}(M-M')c + \frac{1}{\lambda} F[F - F'] \} \quad (3.6)
\]

Further, using the expression for $S_i[\phi(\kappa),\kappa]$ of (2.12), we find

\[
\frac{dS_i[\phi(\kappa),\kappa]}{dc} = -\frac{d}{dc} \Theta' + \frac{B}{A} \quad (3.7)
\]

with

\[
\Theta' = \int d^4x \left( \kappa^2 F^\gamma(M - M')^\gamma + \kappa(1 - \kappa) F^\gamma(Mc)^\gamma + \kappa(1 - \kappa) F^\gamma(M'c)^\gamma + \kappa(\kappa - 1) F^\gamma(Mc)^\gamma 
\]

\[
+ \kappa F^\gamma[(M - M')c]^\gamma + \kappa \int d^4yd^4x \bar{\tau}\gamma(x) \left( \frac{\delta F^\gamma(x) - \delta F'(x)}{\delta A_\mu(y)\delta A_{\beta}(z)} \right)(Dc)^\gamma(z)(Dc)^\beta(y) \right\} \quad (3.7a)
\]
and
\[
\hat{B} = \int d^4x \left\{ \frac{1}{\lambda} [\kappa F'^2 + (1 - 2\kappa) F F' + (\kappa - 1) F^2] + \tau [(M - M')c] \right\}
\]  
(3.7b)

We note that the last term in \(\hat{A}\) vanishes by Bose symmetry; and \(\hat{A}\) can be reorganized further as,

\[
\hat{A} = \int d^4x \left\{ \frac{1}{\lambda} [\kappa [F' - F] [(1 - \kappa) Mc + \kappa M'c] \right\}
\]  
(3.7c)

The [generalized] antighost equation of motion is given by

\[
0 \equiv \int \text{D}\phi(\kappa) \text{O}[\Lambda(\kappa)] \exp \left\{ i S_{\text{eff}} M[\phi(\kappa),\kappa] \right\} f(\Lambda(\kappa),c(\kappa),\kappa)
\]

\[
\cdot \left\{ \text{M}[\Lambda(\kappa)] (1 - \kappa) + \kappa \text{M}'[\Lambda(\kappa)] \right\} c(\kappa)
\]

(3.8)

where \(\text{f}[\Lambda,c,\kappa]\) is an arbitrary functional of \(\Lambda\) and \(c\) but not of \(\tau\).

Using (3.8) above, we can convert \(\hat{A} \Theta'\) term as

\[
\int \text{D}\phi(\kappa) O[\Lambda(\kappa)] \hat{A} \Theta' \exp \left\{ i S_{\text{eff}} M[\phi(\kappa),\kappa] \right\}
\]

\[=-\int \text{D}\phi(\kappa) O[\Lambda(\kappa)] \Theta' \int d^4x \frac{\xi}{\lambda} [F' - F] \gamma_\lambda \frac{\delta}{\delta \phi(x)} \exp \left\{ i S_{\text{eff}} M[\phi(\kappa),\kappa] \right\}
\]

\[=\int d^4x \frac{\xi}{\lambda} \Theta' [F - F'] \text{O}[\Lambda] \rangle \rangle_{\kappa}
\]

(3.9)

Using (3.9), (3.7b) and (3.6), we find

\[
\langle \left\langle \frac{d}{d\kappa} \text{S}_{\text{eff}}[\phi(\kappa),\kappa] \right\rangle \rangle_{\kappa} \equiv 0
\]

(3.10)

This implies \([11,13]\) that \(\langle \left\langle \text{O}[\phi] \right\rangle \rangle_{\kappa}\) is independent of \(\kappa\). Hence,

\[
\int \text{D}\phi \text{O}[\Lambda] \exp \left\{ i S_{\text{eff}}[\phi] \right\} = \int \text{D}\phi' \text{O}[\Lambda'] \exp \left\{ i S_{\text{eff}}'[\phi'] \right\}
\]

(3.11)

Thus the FFBRS transformation of (2.5) takes one from gauge F to gauge F’ in the sense already pointed out at the beginning of the section. Such transformations can be used in establishing a relation between Green’s functions in the two gauges along the lines of [13,14] that correctly tackle the inherent problems in many of the gauges.

### 4 A RELATION BETWEEN ARBITRARY GREEN’S FUNCTIONS IN TWO GAUGES

In the work of reference [13], we had obtained a relation between arbitrary Green’s functions in axial gauges and Green’s functions in the Lorentz gauges and
the former were “compatible” with those in the well-defined Lorentz gauges. This arbitrary Green’s function in axial gauges could be expressed either as [i] a series of Green’s functions in Lorentz gauges that also involve insertions of the BRS variations in the Lorentz gauges OR [ii] as an integral over parameter $\kappa$ involving the Green’s functions evaluated in the Mixed gauges. It was found from a practical view-point that the latter result is indeed more amenable to calculations. An axial Green’s function, to a given order, can be evaluated by an integral over $\kappa$ of a sum of a finite number of Feynman diagrams in the mixed gauge. This form has been employed in obtaining an Axial gauge prescription compatible with the Lorentz gauges [15,14]. A simpler derivation, based on the BRS, was also given[17] for this latter result (only). In this section, we generalize, the procedure of the reference [17] to connecting arbitrary Green’s functions in an arbitrary pair of gauges. We emphasize that our treatment in [17,13] applies equally well to operator Green’s functions [needed say in perturbative QCD applications] as to the usual Green’s functions.

In this section, we shall show that the method of Ref.[17], based on BRS WT-identity, can be generalized to a any pair of two gauges $F$ and $F'$.

We define$^2$, for any operator $O[\phi]$, not necessarily local,

$$
<<O[\phi]>>_\kappa = \int D\phi O[\phi] \exp\{ i S_{eff}^{M}[\phi,\kappa]\} \tag{4.1}
$$

[Note: Unlike in section 3 [see (3.1)], we do not now allow the integration variable $\phi$ to depend on $\kappa$. This suits us here.] Then

$$
\frac{d}{d\kappa} <<O[\phi]>>_\kappa = i<<\int d^4x \ O[\phi] \left[ -\frac{1}{\kappa}(x) \right] [F + \kappa(F' - F) + [F' - F] + \tau(M-M')c] >>_\kappa \tag{4.2}
$$

The WT-identities for $S_{eff}^{M}[\phi,\kappa]$ are

$$
<<\int d^4x \left( J^\alpha \mu (x) D^\alpha_\mu e^{\beta} + \xi^\alpha (x) [-1/2 g_0 f^{\alpha \beta \gamma} e^{\beta} c^\gamma] (x) \right)(x)
$$

$^2$As pointed out in section 3, a proper $\varepsilon-term$ is required to be added to $S_{eff}^{M}[\phi,\kappa]$ to make $<<O[\phi]>>_\kappa$ well defined. This term depends on $F,F'$ and has to be done separately in each specific context. See e.g. the applications in section 5 to planar gauges and also in [14,15].
\[ -\xi^\alpha(x)\left[\frac{F + \kappa(F' - F)}{A}\right]^\alpha(x) \gg = 0 \quad (4.3) \]

Recalling the definition (2.7) of \( \Theta \); we operate by
\[
\Theta \left[ i\frac{\delta}{\delta J(y)} \right] - i\frac{\delta}{\delta \xi(y)} - i\frac{\delta}{\delta \xi(y)} = -f d^4y \{ F^\gamma - F^\gamma \} \left[ \frac{F + \kappa(F' - F)}{A}\right]^\alpha(x) \left( -i\Theta' \right) \quad (4.4)
\]

We then obtain,
\[
-i f d^4x \{ J_\mu(x)D^\beta \mu \epsilon_\beta + \xi^\alpha(x) [ -1/2g_0 f^{\alpha\beta\gamma} \epsilon_\beta c^\gamma ] (x) - \xi^\alpha(x) \left[ \frac{F + \kappa(F' - F)}{A} \right]^\alpha(x) \} (-i\Theta')
\]
\[
- i f d^4x (M-M') \epsilon(x) c(x) + if d^4x \left[ \frac{F + \kappa(F' - F)}{A} \right]^\alpha[F - F'] \gg = 0 \quad (4.5)
\]

Finally, we operate on both sides by \(-i\Theta\left[ i\frac{\delta}{\delta J(y)} \right] - i\frac{\delta}{\delta \xi(y)} - i\frac{\delta}{\delta \xi(y)} \) and set \( J = \xi = \bar{\xi} = 0 \) in the end to obtain
\[
0 = \int d\phi \exp \left( i S_{eff} [\phi, \kappa] \right) \{-f d^4x \{ J_\mu(x)D^\beta \mu \epsilon_\beta + \xi^\alpha(x) [ -1/2g_0 f^{\alpha\beta\gamma} \epsilon_\beta c^\gamma ] (x) - \xi^\alpha(x) \left[ \frac{F + \kappa(F' - F)}{A} \right]^\alpha(x) \} (-i\Theta')
\]
\[
+ O[A, c, \tau] \{ f d^4x \{ \tau(M-M') \epsilon + \frac{F + \kappa(F' - F)}{A} [F - F'] \} \} \quad (4.6)
\]

Using (4.2) and (2.4a); we obtain
\[
\frac{d}{d\kappa} \langle \langle O[\phi] \rangle \rangle = i f d\phi \exp \left( i S_{eff} [\phi, \kappa] \right) \quad (4.7)
\]

Integrating from \( \kappa = 0 \) to 1 we obtain
\[
\langle \langle O[\phi] \rangle \rangle = \langle \langle O[\phi] \rangle \rangle + i f_0^1 d\kappa D\phi \exp \left( i S_{eff} [\phi, \kappa] \right) \quad (4.8)
\]

This gives us the expression for the Green’s function [which depending on the choice of \( O[\phi] \) could be a primary one or of an operator] in one gauge \( F' \) related to that in \( F \), with finite number of additional terms that can be evaluated by Feynman diagram techniques as mentioned in the beginning of this section.

## 5 Planer Gauges

Planar gauges have been found very useful in the perturbative QCD calculations on account of a simpler propagator structure [that avoids the double pole in \( (\eta, k) \)] and other attractive features [18]. Gauges similar to the planer gauges have also been used in the renormalization [19] of higher derivative gravity theories.
Planar gauges are defined by\(^3\)

\[
S^P_{gf} = -\frac{1}{2\lambda^2} \int d^4x \eta.A \partial^2 \eta.A ; \quad \eta^2 \neq 0, \lambda = 1. \tag{5.1}
\]

and the accompanying ghost term

\[
S^P_{gh} = -\int d^4x \bar{c}^\alpha \partial^2 \eta.Dc^\alpha \tag{5.2}
\]

or equivalently

\[
S^P_{gh} \equiv -\int d^4x \bar{c}^\alpha \eta.Dc^\alpha \tag{5.3}
\]

with \(\bar{c}^\alpha\) defined as \(\partial^2 \bar{c}^\alpha\). The net FPEA for the planar gauges, \(S^P_{\text{eff}}\), has a BRS invariance under

\[
\delta \bar{c} = \frac{\eta A}{\eta^2 A} \delta \Lambda \tag{5.4}
\]

and \(\delta A\) and \(\delta c\) as in (2.4)

We note the the gauge fixing term is not manifestly of the form of (2.2) viz. \(~ \int d^4x \bar{F}^\gamma [A]^2\). Hence, we cannot apply the results established in the last two sections directly to this case\(^4\). We can establish a route to connect this gauge to the Lorentz gauge as follows:

\[
S^P_{gf} \rightarrow S_{\tilde{L}gf} \equiv -\frac{1}{2}\int d^4x \partial.A \partial^2 \partial.A \rightarrow S^L_{gf} \tag{5.5a}
\]

or as,

\[
S^P_{gf} \rightarrow S^A_{gf} \rightarrow S^L_{gf} \tag{5.5b}
\]

We shall call, for the sake of nomenclature, the gauge \(\tilde{L}\) the “pseudo-Lorentz” gauge. The formal connection between the pseudo-Lorentz and the Lorentz gauges as well as that between planar and the axial gauge can be established easily from the work of Lee and Zinn-Justin [2] itself. We note from [2],

\(^3\)For mathematical rigor, we may replace \(\partial^2\) by \(\partial^2 - i\epsilon\) in the equations (5.1), (5.2) and (5.5) and in the definition of \(\bar{c}^\alpha\) in (5.3).

\(^4\)See, however, an alternate way presented in Section 6.
\[
<<O[A]\>>_f \equiv \int D\phi \exp \left\{ i S_0 + i S^A_G \right\} \prod_{\alpha, x} \delta \left( \eta^a A^a - f^a \right) O[A] \quad (5.6)
\]

is independent of 'f' for any gauge-invariant observable O[A]. Now, the vacuum expectation values of gauge-invariant observable O[A] in the planar and the axial gauges are related to the above quantity \(<<O[A]\>>_f\) by the relations:

\[
<<O[A]\>>_A = \int Df \exp \left\{ -\frac{i}{2\lambda} \int d^4xf^2 \right\} <<O[A]\>>_f \quad (5.7)
\]

and\(^5\)

\[
<<O[A]\>>_P = \frac{1}{N_P} \int Df \exp \left\{ -\frac{i}{2M^2} \int d^4xf \partial^2f \right\} <<O[A]\>>_f \quad (5.8)
\]

[Equation (5.8) is compensated by appropriate normalization factor \(\frac{1}{N_P}\) relative to (5.7)]. Therefore, in view of the f-independence of \(<<O[A]\>>_f\), the two quantities above are equal:

\[
<<O[A]\>>_A = <<O[A]\>>_P \quad (5.9)
\]

Thus, as for as the gauge-invariant observables are concerned, once the Lorentz-axial route is established, Lorentz-planar gauge connection also becomes available. We shall establish this explicit connection later in section 6 using the results of Sections 3. But first, we would like to derive a result similar to that in Section 4 for the the planar–Lorentz Green’s functions connection applicable to the arbitrary Green’s functions. We define

\[
W^P[J, \xi, \overline{\xi}] = \frac{1}{N_P} \int D\phi \exp \left\{ i S_0 + i S^A_G - \frac{i}{2M^2} \int d^4x \eta.A \partial^2\eta.A + \text{Source Terms} \right\} \quad (5.10)
\]

As for the ghost term, we can use an identical one both for the planar and the axial gauges, obtained from (5.2) by a field redefinition as in (5.3).

\[
S^A_G = -\int d^4x \overline{c}\eta.Dc^a \quad (5.11)
\]

\(^5\)For mathematical rigor, we may replace \(\partial^2\) by \(\partial^2 - i\epsilon\) as earlier, in the equation below and everywhere else necessary.
We now define, for the singular axial gauge \( (\eta A^\alpha - f^\alpha )=0 \),

\[
    W_f [J, \xi, \vec{\xi}] = \int D\phi \exp\left\{ i S_0 + i S_G^A + \text{Source Terms} \right\} \prod_{\alpha, x} \delta (\eta A^\alpha - f^\alpha ) \quad (5.12)
\]

We then have

\[
    W^P[J, \xi, \vec{\xi}] = \frac{1}{N_p} \int Df \exp \left\{ -i \frac{1}{2\lambda f} \int d^4x \, \partial^2 f \right\} W_f [J, \xi, \vec{\xi}] \quad (5.13)
\]

\[
    W^A[J, \xi, \vec{\xi}] = \int Df \exp \left\{ -i \frac{1}{2\lambda} \int d^4x f^2 \right\} W_f [J, \xi, \vec{\xi}] \quad (5.14)
\]

We already know how the Green’s functions in Axial gauges are linked to those in Lorentz gauges[13,17]. To relate the Green’s functions in the planar gauge to those in Lorentz gauges, we proceed as follows:

Consider the expectation value of an arbitrary operator \( O[\phi] \), possibly multi-local, in the Lorentz gauge, the singular axial gauge \( (\eta A^\alpha - f^\alpha )=0 \) and in the planar gauge:

\[
    \langle \langle O[\phi] \rangle \rangle_L = \int D\phi O[\phi] \exp \left\{ i S_{eff}^L[\phi] + \varepsilon O_L \right\} \quad (5.15)
\]

where \( \varepsilon O_L \) are the \( \varepsilon \)-terms in the Lorentz gauges:

\[
    \varepsilon O_L = \int d^4x \left[ \frac{1}{2} A^\mu A^\mu - \mathfrak{c}^L \right] \quad (5.16)
\]

\[
    \langle \langle O[\phi] \rangle \rangle_f = \int D\phi O[\phi] \exp \left\{ i S_0[\phi] + i S_G^A[\phi] + \varepsilon O_A(f) \right\} \prod_{\alpha, x} \delta (\eta A^\alpha - f^\alpha ) \quad (5.17)
\]

where \( \varepsilon O_A(f) \) terms are the appropriate \( O[\varepsilon] \) terms arrived at as in [14] that can depend on \( f \); and

\[
    \langle \langle O[\phi] \rangle \rangle_f = \int D\phi O[\phi] \exp \left\{ i S_{eff}^P[\phi] + \varepsilon O_P \right\} \quad (5.18)
\]

From (5.18) and (5.17),

\[
    \langle \langle O[\phi] \rangle \rangle_f = \frac{1}{N_p} \int Df \exp \left\{ -i \frac{1}{2\lambda f} \int d^4x \, \partial^2 f \right\} \langle \langle O[\phi] \rangle \rangle_f \quad (5.19)
\]

[and the above relation in fact determines what \( \varepsilon O_P \) should be]. We note⁶,

\[
    \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma^2 \mathfrak{c}^L}} \exp \left\{ \frac{-ix^2[1-i\epsilon]}{\sigma^2} \right\} = \delta (x).
\]

Now, \( \langle \langle O[\phi] \rangle \rangle_{f,\sigma} \) can be related to \( \langle \langle O[\phi] \rangle \rangle_{L,\sigma} \) by the result (4.8). We thus have.

\[
    \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma^2 \mathfrak{c}^L}} \exp \left\{ \frac{-ix^2[1-i\epsilon']}{\sigma^2} \right\} = \delta (x).
\]

\[
    \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma^2 \mathfrak{c}^L}} \exp \left\{ \frac{-ix^2[1-i\epsilon']}{\sigma^2} \right\} = \delta (x).
\]
\[<\langle 0|\phi|0\rangle\rangle_f,\sigma = \langle\langle 0|\phi|0\rangle\rangle_{L,\sigma} + \frac{iNp}{\sigma} \int d\kappa fD\phi \exp\{iS_0 + \frac{i}{2\sigma} \int d^4x \ F_1^M[A]^2 + \varepsilon O_L\} \]
\[ \cdot \sum_i (\bar{\delta}_{1i}[\phi] + \kappa(\bar{\delta}_{2i}[\phi]) (\text{i} \Theta'_i) \frac{\delta L}{\delta \phi_i} \]

(5.21)

Here, on account of the results in Section 4, we have

\[ F_1^M[A] = \partial.A(1 - \kappa) + \kappa(\eta.A - f) \]

(5.22)

and

\[ \Theta'_i = i \int d^4y \bar{\psi}(y) (\partial.A^\gamma - \eta.A^\gamma + f^\gamma) \]

(5.23)

\[ S_{G^M}[\phi, \kappa] = -i \int d^4x c [M(1 - \kappa) + \kappa M'] \]

(5.24)

We substitute (5.21) in (5.19) to obtain,

\[ <\langle 0|\phi|0\rangle\rangle_p = \langle\langle 0|\phi|0\rangle\rangle_{L,\sigma} + \lim_{\sigma \to 0} \frac{1}{\sigma} \int d\kappa fD\phi \exp\{iS_0 + \frac{i}{2\sigma} \int d^4x \ F_1^M[A]^2 + \varepsilon O_L\} \sum_i (\bar{\delta}_{1i}[\phi] + \kappa(\bar{\delta}_{2i}[\phi]) (\text{i} \Theta'_i) \frac{\delta L}{\delta \phi_i} \]

(5.25)

As \[<\langle 0|\phi|0\rangle\rangle_{L,\sigma}\] is independent of 'f' and \[\int d\kappa fD\phi \exp\{-\frac{i}{2\sigma} \int d^4x \partial^2f\}\] is absorbed in normalization, we arrive at

\[ <\langle 0|\phi|0\rangle\rangle_p = \langle\langle 0|\phi|0\rangle\rangle_{L,\sigma} + \lim_{\sigma \to 0} \int d\kappa I(\sigma, \kappa) \]

(5.26)

where

\[ I(\sigma, \kappa) = i \frac{1}{Np} \int d\kappa fD\phi \ exp\{-\frac{i}{2\sigma} \int d^4x \partial^2f\} \int d\kappa fD\phi \ exp\{iS_0 + \frac{1}{2\sigma} \int d^4x \ [\partial.A(1 - \kappa) + \kappa(\eta.A - f)]^2 + \varepsilon O_L\} \sum_i (\bar{\delta}_{1i}[\phi] + \kappa(\bar{\delta}_{2i}[\phi]) (\text{i} \Theta'_i) \frac{\delta L}{\delta \phi_i} \]

(5.27)

Now, we take the limit \[\sigma \to 0\] inside to obtain

\[ \exp\{-\frac{i}{2\sigma} \int d^4x \ F_1^M[A]^2\} \sim \prod_{\alpha, x} \delta\{\partial.A(1 - \kappa) + \kappa(\eta.A - f)\} \]

(5.28)

We further note that

\[ D_{\bar{\tau}} \prod_{\alpha, x} \delta\{\partial.A(1 - \kappa) + \kappa(\eta.A - f)\} \equiv \prod_{\alpha, x} d^{\bar{\tau}}(x) \delta\{\partial.A(1 - \kappa) + \kappa(\eta.A - f)\} \]

\[ = \prod_{\alpha, x} d^{\bar{\tau}}(x) \delta\{\partial.A(1 + \kappa - 1) + (\eta.A - f)\} \]

\[ \sim D_{\bar{\tau}} \prod_{\alpha, x} \delta\{\partial.A(1 + \kappa - 1) + (\eta.A - f)\} \]

(5.29)
with \(\mathcal{P} = \mathcal{T}\kappa\). We use this \(\delta\)-function to simplify
\[
\Theta' = i \int d^4 y \mathcal{P}^*(y) \left( \partial \cdot A^\gamma \right) / \kappa = i \int d^4 y \mathcal{P}^*(y) \left( \partial \cdot A^\gamma \right) / \kappa^2
\]  
(5.30)
and
\[
\exp \left\{ -\frac{i}{2\lambda^2} \int d^4 x \partial^2 f \right\} \rightarrow \exp \left\{ -\frac{i}{2\lambda^2} \int d^4 x \left[ \partial \cdot A(1/\kappa - 1) + \eta \cdot A \right] \partial^2 \left[ \partial \cdot A(1/\kappa - 1) + \eta \cdot A \right]
\]
\[
\equiv \exp \left\{ i \int S_{\partial f}(\kappa) \right\}
\]  
(5.31)
We further re-express \(S_G^M\) as
\[
S_G^M = \int d^4 x \mathcal{P} \left[ M(1/\kappa - 1) + M' \right] c \equiv S_G^M
\]  
(5.32)
Thus, we obtain the result that connects arbitrary Green’s functions in planar gauges to those in Landau gauge:
\[
\langle \langle O[\phi] \rangle \rangle_p = \langle \langle O[\phi] \rangle \rangle_{\text{landau}} + i \frac{1}{N_p} \int_0^1 \frac{d\kappa}{\kappa^2} \int \mathcal{D} \phi \exp \{ i \int S_{G}^\gamma \mathcal{P}^*(y) \left( \partial \cdot A^\gamma \right) / \kappa \} \cdot \sum_i \left[ \delta_{1i}^2 \left[ \phi^2 + \kappa \left( \delta_{2i} \left[ \phi^2 \right] \right) \right] \right] \left( \frac{\delta O[A,c,\kappa]}{\delta \phi_i} \right)
\]
(5.33)
We may change the notation for the integration variable \(\mathcal{P} \rightarrow \mathfrak{P}\). Noting that
\[
\sum_i \left[ \delta_{1i}^2 \left[ \phi^2 + \kappa \left( \delta_{2i} \left[ \phi^2 \right] \right) \right] \right] \left( \frac{\delta O[A,c,\kappa]}{\delta \phi_i} \right)
\]  
(5.34)
with \(\phi_i = A, c, \kappa, c\) and here
\[
S_G^M_{\text{eff}} = S_0 - \frac{1}{2\lambda^2} \int d^4 x \left[ \partial \cdot A(1/\kappa - 1) + \eta \cdot A \right] \partial^2 \left[ \partial \cdot A(1/\kappa - 1) + \eta \cdot A \right] \int d^4 x \mathcal{P} \left[ M(1/\kappa - 1) + M' \right] c
\]  
(5.35)
is the effective action for a mixed planar gauge but with a mixed gauge fixing function:
\[
\left[ \partial \cdot A(1/\kappa - 1) + \eta \cdot A \right] \text{ in both the gauge-fixing and the ghost term. We put } O[\phi] = I, \text{ the identity operator, in (5.34) to obtain,}
\]
\[
\langle \langle I \rangle \rangle_p = \langle \langle I \rangle \rangle_{\text{landau}}
\]  
(5.36)
We then obtain for the connected part \(\langle O[\phi] \rangle_p = \langle \langle O[\phi] \rangle \rangle_p / \langle \langle I \rangle \rangle_p \), etc the final result\(^7\): \(\langle O[\phi] \rangle_p = \langle O[\phi] \rangle_{\text{landau}} + i \int_0^1 \frac{d\kappa}{\kappa^2} \int \mathcal{D} \phi \exp \{ i \int S_{G}^\gamma \mathcal{P}^*(y) \left( \partial \cdot A^\gamma \right) / \kappa \} \cdot \sum_i \left[ \delta_{1i}^2 \left[ \phi^2 + \kappa \left( \delta_{2i} \left[ \phi^2 \right] \right) \right] \right] \left( \frac{\delta O[A,c,\kappa]}{\delta \phi_i} \right)\)

\(^7\)We need not be alarmed by \(\kappa^2\) in the denominator: near \(\kappa = 0\), the ghost propagator and the longitudinal gauge propagator yield enough factors of \(\kappa\).
\[ \bullet \sum_i (\delta_{1i}[\phi] + \kappa(\delta_{2i}[\phi]) \int d^4y \, \bar{\psi}(y) \left( \partial \cdot A_{\gamma} \right) \frac{\delta L_{[A,c,\kappa]}^{L}}{\delta \phi_i} \|_{\text{conn}} \]

\[ = \langle O[\phi] \rangle_{\text{landau}} \]

\[ + i \int_0^1 \frac{ds}{s^2} \sum_i (\delta_{1i}[\phi] + \kappa(\delta_{2i}[\phi]) \int d^4y \bar{\psi}(y) \left( \partial \cdot A_{\gamma} \right) \frac{\delta L_{[A,c,\kappa]}^{L}}{\delta \phi_i} \|_{\text{mixed, conn}} \]

(5.37)

We note here that in the last term, we have the connected Green’s functions in the mixed gauge with the appropriate \( \epsilon \)-term.

### 6 An Alternate and procedure for Planar Gauges

In this section, we shall present an alternate, though somewhat heuristic, procedure that uses directly the familiar results of the section 3. This procedure applies directly to the vacuum expectation values of gauge-invariant observables, if not for their arbitrary Green’s functions. The conclusions drawn here are however easily directly verified by an algebra similar to that in Section 3; and without any need for the lack of rigor in the following derivation. We present the former way here.

To use the results of section 3, we first establish an interpolating route. We define

\[ \tilde{F}^M[A] = \partial \cdot A (1 - \kappa) + \kappa \eta \cdot A \]

(6.1)

Then with

\[ S_{gf}^M = -\frac{1}{2\lambda M} \int d^4x \, \tilde{F}^M[A] \partial^2 \tilde{F}^M[A] \]

(6.2)

and

\[ S_{gh}^M = -\int d^4x \, \bar{\psi} \partial^2 [M(1 - \kappa) + \kappa M'] c \]

(6.3)

and the BRS transformation

\[ \delta \bar{\psi} = \left\{ \tilde{F}^M/\lambda \eta^2 \right\} \delta \Lambda \]

(6.4)

we have the interpolating route from the pseudo-Lorentz gauges \( (\kappa = 0) \) to the planar gauges \( (\kappa = 1) \). Now, for the purpose of treating the extra singularity introduced in the
propagator on account of the $\partial^2$ in (6.2), we interpret it as $\partial^2-\text{i}\epsilon$ both there and in (6.3) as pointed out earlier in (5.1). In the momentum space, we write the equation (6.2) as,

$$S_{gf}^M = \frac{1}{2\lambda} \int d^4k \ F^M[k][k^2 + \text{i}\epsilon] \ F^M[k]$$

$$= -\frac{1}{2\lambda} \int d^4k \ \{F^M[k]\sqrt{\frac{k^2 + \text{i}\epsilon}{\eta^2}}\}^2$$  \hspace{1cm} (6.5)

where $F^M[k]$ is the Fourier transform of $F^M[A].$ We define

$$\text{F.T.}\{F^M[k]\sqrt{\frac{k^2 + \text{i}\epsilon}{\eta^2}}\} = \sqrt{\frac{\partial^2 - \text{i}\epsilon}{\eta^2}} F^M[A] = F_M[A]$$  \hspace{1cm} (6.6)

We then have,

$$S_{gf}^M = -\frac{1}{2\lambda} \int F_M[A]^2 d^4x$$ \hspace{1cm} (6.7)

and the ghost term as

$$S_{gh} = -\int d^4x \ \tau \sqrt{[\partial^2 - \text{i}\epsilon] \eta^2} \delta F_{M}^{\mu} D_{\mu} c$$

$$= -\int d^4x \ \sqrt{[\partial^2 - \text{i}\epsilon] \eta^2} \tau \delta F_{M}^{\mu} D_{\mu} c$$ \hspace{1cm} (6.8)

The last step is seen by going to the Fourier space and comparing the expressions. We now make a linear change of variables, leading to a constant Jacobian that can be ignored,

$$\tau' = \sqrt{[\partial^2 - \text{i}\epsilon] \eta^2} \tau$$ \hspace{1cm} (6.9)

Then

$$S_{gh} = -\int d^4x \ \tau' \delta F_{M}^{\mu} D_{\mu} c$$ \hspace{1cm} (6.10)

Now the system of gauge fixing term (6.7) and the ghost term (6.10) has been cast in the standard FP form. We can now apply the results of the Section 3 to interpolate between the planar and the pseudo-Lorentz gauges with

$$\Theta[\phi] = -i \int d^4y \ \tau' \gamma(y) [F_M(k = 1) - F_M(k = 0)] \gamma$$

$$= -i \int d^4y \ \tau' \gamma(y) \sqrt{\frac{\partial^2 - \text{i}\epsilon}{\eta^2}} \eta.A - \partial.A \gamma$$

$$= -i \int d^4y \ \tau' \gamma(y) [\partial^2 - \text{i}\epsilon] \eta.A - \partial.A \gamma$$ \hspace{1cm} (6.11)

Thus, under the field transformation

$$\phi' = \phi + \delta^L_{BRS}[\phi] \Theta [\phi]$$ \hspace{1cm} (6.12)

with $\Theta$ given in terms of $\Theta[\phi]$ of (6.11) through relation in section 2, relates the planar gauge to the pseudo-Lorentz gauge. Further as far as the expectation values of gauge-invariant observables are concerned, they have been shown to have the same value in the pseudo-Lorentz and the Lorentz gauges in Section 5.
Of course, as mentioned at the beginning of this section, the above heuristic argument that allows one to make a direct use of the results of Section 3, can be avoided and we can, with additional labor, verify the result for the field transformation given by (6.12) directly from the Jacobian condition (2.14). We have in fact carried out this verification.

References

[1] See e.g. S.D.Joglekar in “Finite field-dependent BRS transformations and axial gauges”, invited talk at the conference titled “Theoretical Physics today: Trends and Perspectives held at IACS calcutta, April 1998, appeared in Ind. J.Phys.73B,137(1999)[also see references in 8,9,and 18 below]


