We explore the idea of a network of defects to live inside a domain wall in models of three real scalar fields, engendering the $Z_2 \times Z_3$ symmetry. The host domain wall appears from the field that governs the $Z_3$ symmetry, and entraps the hexagonal network formed by the three-junctions of the model of two scalar fields that describes the remaining $Z_2$ symmetry. We show that if the host domain wall bends to the spherical form, there may appear non-topological structures hosting networks that accept diverse patterns. In particular, if $Z_3$ is also broken, the model may generate a buckyball containing sixty junctions, a fullerene-like structure.

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Domain walls appear in diverse branches of physics, as for instance in systems of condensed matter that present ferromagnetic [1], ferroelectric [2] and other properties [3], and also in cosmology [4,5]. They arise in systems with at least two isolated degenerate minima, and in field theory they usually live in three spatial dimensions as bidimensional objects, seen as immersions into $(3,1)$ dimensions of static solutions of $(1,1)$ dimensional models that engender the $Z_2$ symmetry. The standard domain wall presents no internal structure, but there are models where they may entrap field configurations that engender non-trivial behavior. This idea follows as in Refs. [6-8], and in the more recent Refs. [9,10]. Other investigations include for instance supersymmetry [11], supergravity [12], and applications to polymers [13].

Although domain walls may be dangerous [4,5] to cosmological applications, they have found their way into cosmology as for instance seeds for the formation of non-topological structures. This possibility appears in Refs. [14-17], where the discrete symmetry is changed to an approximate symmetry, or in Ref. [18], with the discrete symmetry biased so that domains of distinct but degenerate vacua spring unequally. The non-topological structures may be stable, but now stability requires the presence of conserved charges, of bosonic and/or fermionic origin.

Domain walls may also be of interest when they host non-trivial structures. We illustrate this point using the model introduced in the first work of Ref. [10], describing the pair of fields $(\phi, \chi)$ via the superpotential $\phi^3/3 - \phi + r \phi \chi^2$. Here $r$ is a parameter, real and dimensionless, that couples the two fields. The system is described by a quartic potential, and we use natural units, working with dimensionless space and time variables, and fields. In this model, the sector connecting the minima $(\pm 1, 0)$ is a BPS sector, with energy density or tension $t = 4/3$. For $r > 0$ this BPS sector admits two different types of static solutions: the one-field solutions $\phi_1(z) = -\tanh(z)$ and $\chi_1 = 0$, and the two-field solutions $\phi_2(z) = -\tanh(2rz)$ and $\chi_2(z) = a(r)/\cosh(2rz)$, with $a^2(r) = 1/r - 2$, valid for $0 < r < 1/2$. We are working in $(3,1)$ space-time dimensions, so the one-field solution represents a standard domain wall, while the two-field solution appears as a domain wall having internal structure. As $z$ varies in $(-\infty, \infty)$, in configuration space the vectors $(\phi_i, \chi_i)$, $i = 1, 2$, describe a straight line segment $(i = 1)$ and an elliptic arc $(i = 2)$. These solutions also appear in condensed matter, and there they are named Ising and Bloch walls, respectively [1-3]. The Bloch walls are chiral interfaces, and are used to describe more complex phenomena, as for instance in the applications where chirality is also broken [19].

In Ref. [20] the idea of nesting a network of defects inside a domain wall has been presented. This possibility may appear in models with three real scalar fields, engendering the $Z_2 \times Z_3$ symmetry. In the present work we offer a model that contains the basic mechanisms behind this idea. The model will ultimately lead to the scenario of a domain wall hosting a network of defects, which may have direct interest to physics, as we show when we explore the pattern of the nested network in the case we allow the underlying $Z_2 \times Z_3$ symmetry to be broken.

We first develop the idea of a domain wall hosting a network of defects. We consider the model described by the three real scalar fields $\sigma$, $\phi$, and $\chi$, defined by the (dimensionless) potential,

$$V(\sigma, \phi, \chi) = \frac{2}{3} \left( \sigma^2 - \frac{9}{4} + \left( r \sigma^2 - \frac{9}{4} \right) \left( \phi^2 + \chi^2 \right) \right) + \frac{1}{2} \left( \phi^2 + \chi^2 \right)^2 - \phi \left( \phi^2 - 3 \chi^2 \right)$$

Here $r$ couples $\sigma$ to the pair of fields $(\phi, \chi)$. This potential is polynomial, and contains up to the fourth order power in the fields. Thus, it behaves standardly in $(3,1)$ space-time dimensions. Also, it presents discrete $Z_2 \times Z_3$ symmetry. We set $(\phi, \chi) \to (0,0)$, to get the projection $V(\sigma, 0, 0) \to V(\sigma) = (2/3) \left( \sigma^2 - 9/4 \right)^2$. The projected potential presents $Z_3$ symmetry, and can be written with the superpotential $W(\sigma) = (2\sqrt{3}/9)\sigma^3 - (3\sqrt{3}/2)\sigma$, in the
form $V = (1/2)(dW/da)^2$. The reduced model supports the explicit configurations $\sigma_h(z) = \pm (3/2) \tanh(\sqrt{3}z)$. The tension of the host wall is $t_h = 3 \sqrt{3} = (3/2)m_h$, where $m_h$ represents the mass of the elementary $\sigma$ meson. Also, the width of the wall is such that $t_h \sim 1/\sqrt{3}$.

The potentials projected inside ($\sigma \to 0$) and outside ($\sigma \to \pm 3/2$) the host domain wall are $V_{in}(\phi, \chi)$ and $V_{out}(\phi, \chi)$. Inside the wall we have

$$V_{in}(\phi, \chi) = (\phi^2 + \chi^2) - \phi(\phi^2 - 3\chi^2) - \frac{9}{4}(\phi^2 + \chi^2)^2 + \frac{27}{8}$$

This potential engenders the $Z_3$ symmetry, and there are three global minima, at the points $v_{in}^0 = (3/2)(1, 0)$ and $v_{in}^{\pm 3} = (3/4)(-1, \pm \sqrt{3})$, which define an equilateral triangle. Outside the wall we get

$$V_{out}(\phi, \chi) = (\phi^2 + \chi^2)^2 - \phi(\phi^2 - 3\chi^2) + \frac{9}{4}(r - 1)(\phi^2 + \chi^2)$$

$V_{out}$ also engenders the $Z_3$ symmetry, but now the minima depend on $r$. We can adjust $r$ such that $r > 9/8$, which is the condition for the fields $\phi$ and $\chi$ to develop no non-zero vacuum expectation value outside the host domain wall, ensuring that the model supports no domain defect outside the host domain wall. The restriction of considering quartic potentials forbids the possibility of describing the $Z_3$ portion of the model with the complex superpotential used in [21]; see also Ref. [22].

We investigate the masses of the elementary $\phi$ and $\chi$ mesons. Inside the wall they degenerate to the single value $m_{in} = 3\sqrt{3}/2$. Outside the wall, for $r > 9/8$ they also degenerate to a single value, $m_{out}(r) = 3 \sqrt{r - 1}/2$, which depends on $r$. We see that $m_{out}(r = 4) = m_{in}$. Also, $m_{out}(r > 4)$ for $r$ bigger than $4$, and $m_{out}(r < m_{in}$ for $r$ in the interval $(9/8, 4)$.

We study linear stability of the classical solutions $\sigma = \sigma_h(z)$ and $(\phi, \chi) = (0, 0)$. The fields $\phi$ and $\chi$ vanish classically, and their fluctuations ($\eta_n, \xi_n$) decouple. The procedure leads to two equations for the fluctuations, that degenerate to the single Schrödinger-like equation

$$-\frac{d^2 \psi_n(z)}{dz^2} + \frac{9}{2} V(z) \psi_n(z) = W_n^2 \psi_n(z)$$

Here $V(z) = -1 + r \tanh^2 \sqrt{3}z$. This equation is of the modified Pöschl-Teller type, and can be examined analytically. The lowest eigenvalue is $W_0^2 = (3/2) \sqrt{6}r + 1 - 6$. There is instability for $r < 5/2$, showing that the host domain wall with $(\phi, \chi) = (0, 0)$ is unstable and therefore relax to lower energy configurations, with $(\phi, \chi) \neq (0, 0)$ for $r < 5/2$. Inside the host domain wall the sigma field vanishes, and the model is governed by the potential $V_{in}(\phi, \chi)$, which consequently may allow the presence of non-trivial $(\phi, \chi)$ configurations. The host domain wall entraps the system described by $V_{in}(\phi, \chi)$ for the parameter $r$ in the interval $(9/8, 5/2)$. In this interval we have $m_{out} < m_{in}$, showing that it is not energetically favorable for the elementary $\phi$ and $\chi$ mesons to live inside the wall for $r \in (9/8, 5/2)$. The model automatically suppresses backreactions of the $\phi$ and $\chi$ mesons into the defects that may appear inside the host domain wall.

In Ref. [20] the potential inside the wall was shown to admit a network of domain walls, in the form of a hexagonal array of domain walls. In the thin wall approximation the network may be represented by the solutions

$$\phi_n = \frac{3}{8} \sqrt{3} \tanh \left( \frac{1}{2} \sqrt{\frac{27}{8}} (y + \sqrt{3}x) \right)$$

and

$$\chi_n = \frac{3}{8} \sqrt{3} \tanh \left( \frac{1}{2} \sqrt{\frac{27}{8}} (y + \sqrt{3}x) \right)$$

and by $(\phi_n, \chi_n)$, obtained by rotating the pair $(\phi_1, \chi_1)$ by $2(n - 1)\pi/3$, for $n = 2, 3$. We identify the space $(\phi, \chi)$ with $(x, y)$, so rotations in $(\phi, \chi)$ also rotates the plane $(x, y)$ accordingly. The energy or tension of the individual defects in the network is given by, in the thin wall approximation, $t_n = (27/8) \sqrt{3}/2 = (9/8) m_{in}$. In the nested network, the width of each defect obeys $l_n \sim \sqrt{8/27}$. This shows that $l_h/l_n = 3/\sqrt{27}$, and so the host domain wall is slightly thicker than the defects in the nested network. In the thin wall approximation, the potential $V_{in}(\phi, \chi)$ allows the formation of three-junctions as reactions that occur exothermically, and the nested array of thin wall configurations is stable. In FIG. 1 we depict the hexagonal network of defects inside the domain wall, in the thin wall approximation. The dashed lines show equilateral triangles, that belong to the dual lattice. Both the hexagonal network and the dual triangular network are composed of equilateral polygons, a fact that follows in accordance with the $Z_3$ symmetry.

![FIG. 1. The equilateral hexagonal network of defects, that may live inside the host domain wall. The dashed lines show the dual lattice, formed by equilateral triangles.](image)

We now explore the breaking of the $Z_3 \times Z_3$ symmetry of the model. The simplest case refers to the breaking of the $Z_3$ symmetry, without breaking the remaining $Z_2$ symmetry. We consider the case of breaking the internal $Z_3$ symmetry in the following way. We take for instance the vacuum state $v_{in}^0 = (3/2)(1, 0)$, and change its position to a location farther from or closer to the other minima of the system, increasing or decreasing the angle between two of the three defects; see FIG. 2. We can do...
this with the inclusion in the potential of another term, proportional to the second-order power on \( \phi \). We notice that the energy of the defect depends on the distance between the two minima the defect connects, and goes with the cube of it. Thus, if the vacuum state deviates significantly from its \( Z_3 \)-symmetric position, we cannot neglect the correction to the energy of the defects. This changes the regular hexagonal pattern of FIG. 1 to two other hexagonal patterns, composed of thicker or thinner hexagons. We recall that hexagonal patterns may appear in chemical systems [23], and in fluid convection [24] where they may also involve non-equilateral hexagons.

![FIG. 2. The vacuum states (black dots) and the junction that forms the nested network. (a) and (b) illustrate the only two ways of breaking the \( Z_3 \rightarrow Z_2 \) symmetry.](image)

We can also explore the presence of local defects in the hexagonal network. We do this introducing pent-hepta pair of cells, in a local deformation of the network that disorganize its otherwise regular pattern. The mechanism is similar to that of Ref. [25]. However, if the \( Z_2 \) symmetry that governs the host domain wall is effective, local deformations may only appear in a flat surface, requiring the pentagons and heptagons are not regular polygons. This possibility may be seen in Benard-Marangoni convection; see Ref. [26] for a report on the experimental observation of such patterns. But if together with the slight breaking of the \( Z_3 \) symmetry of the internal network, one slightly breaks the \( Z_2 \) symmetry of the host domain wall locally, this will ultimately favor the appearance of local deformations composed of pair of equilateral pentagons and heptagons. Since in the network of equilateral hexagons, the presence of equilateral pentagons and heptagons introduce local curvature, positive and negative, respectively, we can understand these local defects as a mechanism for roughening the planar surface that contains the network. To break the symmetry of the nested network, one requires a slight change of position of one of the three minima of the nested system, so we can neglect the difference in energy and consider the tension as in the regular hexagonal network. We see that the roughening springs to generate higher energy states from the planar regular hexagonal structure.

We now concentrate on breaking the \( Z_2 \) symmetry of the host domain wall. We can do this with the inclusion in the potential of a term odd in \( \sigma \), that slightly removes the degeneracy of the two minima \( \sigma = \pm 3/2 \). Thus, the host domain wall bends trying to involve the local minimum, the false vacuum. To stabilize the non-topological structure we include charged fields into the system. The way one couples the charged fields is not unique, but if we choose to add fermions, we can couple them to the \( \sigma \) field in a way such that the projection with \((\phi, \chi) \to (0,0)\) may leave the model supersymmetric. This is obtained with the superpotential \( W(\sigma) \), with the Yukawa coupling \( dW/d\sigma^2 = (4/3)\sqrt{3} \sigma \). In this case massless fermions bind [27] to the host domain wall, and contribute to stabilize [17] the non-topological defect that emerges with the breaking of the \( Z_2 \) symmetry.

The breaking of the \( Z_2 \) symmetry can be done breaking or not the remaining \( Z_3 \) symmetry of the model. We examine these two possibilities supposing that the host domain wall bends under the assumption of spherical symmetry, becoming a non-topological defect with the standard spherical shape. This is the minimal surface of genus zero, and according to the Euler theorem we can only tile the spherical surface with three-junctions as a regular polygonal network in the three different ways: with 4 triangles, or 6 squares, or yet 12 pentagons. These three cases preserve the \( Z_3 \) symmetry of the original network, locally, at the three-junction points. However, if locally one slightly breaks the \( Z_2 \) symmetry of the network to the \( Z_2 \) one, the three-junctions can now tile the spherical surface with 12 pentagons and 20 hexagons. We think of breaking the \( Z_2 \) symmetry minimally, to the \( Z_2 \) symmetry, through the same mechanism presented in FIG. 2. Thus, if the symmetry is broken slightly we can consider the defect tensions as in the regular hexagonal network.

The tiling with 12 pentagons and 20 hexagons generates a spherical structure that resembles the fullerene, the buckyball composed of sixty carbon atoms. We visualize the symmetries involved in the spherical structures thinking of the corresponding dual lattices, which are triangular lattices, but in the three first cases the triangles are equilateral, while in the fourth case they are isosceles. We recall that regular heptagons introduce negative curvature, so they cannot appear when the genus zero surface is minimal. However, they may for instance spring to generate higher energy states from the fullerene-like structure, locally roughening the otherwise smooth spherical surface.

We write the energy of the non-topological structure as \( E_{nt} = E_{nt} \pm E_n \), where \( E_{nt} \) stands for the energy of the standard non-topological defect, and \( E_n \) is the portion due to the nested network. We use \( E_{nt} = E_q + E_h \), which shows the contributions of the charged fields and of the host domain wall, respectively. We have \( E_h = S t_h \), and \( E_n = N d t_n \), where \( S \) is the area of the spherical surface, and \( N \) and \( d \) are the number and length of segments in the nested network. We introduce the ratio \( E_q/E_h = 1 + [N/(1+r)](t_n/t_h)(d/S) \), with \( r = E_q/E_h \). The non-topological structure nests a network of defects, which modifies the scenario one gets with the standard domain wall. The modification depends on the way one couples charged bosons and fermions to the \( \sigma, \phi \), and \( \chi \) fields. However, if the \( Z_3 \) symmetry is locally
broken to the $Z_2$ one, the most probable defect corresponds to the fullerene or buckyball structure. But if the $Z_3$ symmetry is locally effective, there may be three equilateral structures, the most probable arising as follows. We consider the simpler case of plane polygonal structures, identifying the tetrahedron ($i = 3$), cube ($i = 4$), and dodecahedron ($i = 5$). We introduce $r_{ij}$ as the energy ratio for the $i$ and $j$ structures. We get $r_{ij} = (1+r+t_n/h_i t_k)/(1+r+t_n/h_j t_k)$, for $i, j = 3, 4, 5$. Here $h_3, h_4,$ and $h_5$ stand for the radius of the incircle of the triangle, square, and pentagon, respectively. Energy favors the triangular lattice as the nested network. This configuration is self-dual, because the network and its dual are the very same triangular lattice. The two other configurations, the octahedron, dual to the cube, and the icosahedron, dual to the dodecahedron, do not appear in the $Z_2 \times Z_3$ model because they require four- and five-junctions, respectively.

The present work can be extended in several directions. For instance, we could use the $Z_2 \times Z_k$ symmetry ($k = 4, 5, 6$), getting to $k$-junctions. This allows to tile the plane with squares ($k = 4$), or triangles ($k = 6$), and the spherical surface with triangles, as the octahedron ($k = 4$) or the icosahedron ($k = 5$). This direction seems appropriate to model the recent experimental observations of squares in specific Rayleigh-Bénard and Bénard-Marangoni convections [28]. Also, in the $Z_2 \times Z_3$ model, if the host domain wall bends cylindrically, one may get to nanotube-like configurations [29]. Another line follows [30], that investigates pattern formation within the cosmological scenario. Our investigation provides direct generalization to more realistic scenarios, involving pattern formation in the early universe.

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