The electric dipole moment of the neutron in chiral perturbation theory

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Abstract

We calculate the electric dipole moments of the neutron and the Λ within the framework of heavy baryon chiral perturbation theory. They are induced by strong CP-violating terms of the effective Lagrangian in the presence of the vacuum angle $\theta_0$. The construction of such a Lagrangian is outlined and we are able to give an estimate for $\theta_0$.

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1 Introduction

The axial $U(1)$ anomaly in QCD implies an additional term in the Lagrangian, which violates $P$, $T$ and $CP$. This new term is proportional to the so-called vacuum angle $\theta_0$, an unknown parameter, and its size may be determined from $CP$-violating effects, as e.g. $\eta \rightarrow \pi\pi$ or the electric dipole moments of the neutron and the $\Lambda$. The most recent measurements of the electric dipole moment of the neutron $d_n^e$ have constrained it to [1]

$$|d_n^e| < 6.3 \times 10^{-26} \text{e cm}.$$  \hspace{1cm} (1)

On the other hand, theoretical estimates for $d_n^e$ induced by the $\theta_0$-term can be given leading to an upper bound for $\theta_0$ [2, 3, 4]. Of particular interest here is the estimate of Pich and de Rafael [3] who used an effective chiral Lagrangian approach and came to the conclusion that it is possible to obtain an estimate for the size of the vacuum angle $\theta_0$ with an experimental upper limit of $|\theta_0| \leq 5 \times 10^{-10}$. However, the authors worked in a relativistic framework which does not have a systematic chiral counting scheme, so that higher loop diagrams contribute to lower chiral orders. This problem is avoided in heavy baryon chiral perturbation theory as proposed in [5], which allows for a consistent power counting. One should therefore study this effect within the heavy baryon formulation. Furthermore, the baryon Lagrangian in [3] which describes the interactions of the neutron with the pseudoscalar nonet $(\pi, K, \eta, \eta')$ does not contain explicitly $CP$-violating terms. They are rather induced by the vacuum alignment of the purely mesonic Lagrangian in the presence of the $\theta_0$-term. As shown in [6, 7] the most general baryonic Lagrangian taking the axial $U(1)$ anomaly into account does have explicitly $CP$-violating terms even at lower chiral orders. It has to be checked, if such terms lead to sizeable contributions for the present consideration. Finally, the authors of [3] proposed to estimate the contribution from unknown counterterms by varying the scale in the chiral logarithms. This procedure reveals the scale dependence of the involved coupling constants but not their absolute value, and it is desirable to have a somewhat more reliable estimate of the involved couplings. The aim of the present work is to reinvestigate the electric dipole moments of the neutron and the $\Lambda$ by taking the above mentioned points into consideration. One has to check, if it is still possible to give a reliable estimate of the vacuum angle $\theta_0$, and if it is different from the one given in [3].

The paper is organized as follows. In the next section we present the purely mesonic effective Lagrangian in the presence of the $\theta_0$-term and the vacuum alignment is discussed. Baryon fields are included in the effective theory in Sec. 3 by using the method outlined in [6]. We proceed by calculating in Sec. 4 the electric dipole moment of the neutron and the $\Lambda$ up to one-loop order within the heavy baryon framework. Numerical results and conclusions are given in Sec. 5.

2 The mesonic Lagrangian

In this section, we will consider the purely mesonic Lagrangian in the presence of the $\theta_0$-term. The derivation of this Lagrangian has been given elsewhere, see e.g. [8, 9, 10], so we will restrict ourselves to the repetition of some of the basic formulae which are needed in the present work. In [9, 10] the topological charge operator coupled to an external field is added to the QCD Lagrangian

$$\mathcal{L} = \mathcal{L}_{QCD} - \frac{g^2}{16\pi^2} \theta(x) \text{tr}_c(G_{\mu\nu} \tilde{G}^{\mu\nu})$$ \hspace{1cm} (2)
with \( \bar{G}_{\mu
u} = \epsilon_{\mu\nu\alpha\beta} G^{\alpha\beta} \) and \( \text{tr}_c \) is the trace over the color indices. Under \( U(1)_R \times U(1)_L \) the axial \( U(1) \) anomaly adds a term \( -(g^2/16\pi^2)2N_f \alpha \text{tr}_c(G_{\mu\nu}\bar{G}^{\mu\nu}) \) to the QCD Lagrangian, with \( N_f \) being the number of different quark flavors and \( \alpha \) the angle of the global axial \( U(1) \) rotation. The vacuum angle \( \theta(x) \) is in this context treated as an external field that transforms under an axial \( U(1) \) rotation as

\[
\theta(x) \rightarrow \theta'(x) = \theta(x) - 2N_f \alpha.
\]

Then the term generated by the anomaly in the fermion determinant is compensated by the shift in the \( \theta \) source and the Lagrangian from Eq. (2) remains invariant under axial \( U(1) \) transformations. The symmetry group \( SU(3)_R \times SU(3)_L \) of the Lagrangian \( \mathcal{L}_{QCD} \) is extended\(^3\) to \( U(3)_R \times U(3)_L \) for \( \mathcal{L} \). The Green functions of QCD are obtained by expanding the generating functional around \( \theta(x) = \theta_0 \) where the phase of the quark mass matrix emerging from the Yukawa couplings of the light quarks in the electroweak sector has been absorbed in \( \theta_0 \). The extended symmetry remains at the level of an effective theory and the additional source \( \theta \) also shows up in the effective Lagrangian. Let us consider the purely mesonic effective theory first. The lowest lying pseudoscalar meson nonet is summarized in a matrix valued field \( \bar{U}(x) \). The effective Lagrangian is formed with the fields \( \bar{U}(x) \), derivatives thereof and also includes both the quark mass matrix \( \mathcal{M} \) and the vacuum angle \( \theta \): \( \mathcal{L}_{\text{eff}}(\bar{U}, \partial \bar{U}, \ldots, \mathcal{M}, \theta) \). Under \( U(3)_R \times U(3)_L \) the fields transform as follows:

\[
\bar{U} = R \bar{U} L^\dagger \quad , \quad \mathcal{M} = R \mathcal{M} L^\dagger \quad , \quad \theta'(x) = \theta(x) - 2N_f \alpha \tag{4}
\]

with \( R \in U(3)_R \), \( L \in U(3)_L \), but the Lagrangian remains invariant. The phase of the determinant of \( \bar{U}(x) \) transforms under axial \( U(1) \) as \( \ln \det \bar{U}'(x) = \ln \det \bar{U}(x) + 2iN_f \alpha \) so that the combination \( \theta - i \ln \det \bar{U} \) remains invariant. It is more convenient to replace the variable \( \theta \) by this invariant combination, \( \mathcal{L}_{\text{eff}}(\bar{U}, \partial \bar{U}, \ldots, \mathcal{M}, \theta - i \ln \det \bar{U}) \). One can now construct the effective Lagrangian in these fields that respects the symmetries of the underlying theory. In particular, the Lagrangian is invariant under \( U(3)_R \times U(3)_L \) rotations of \( \bar{U} \) and \( \mathcal{M} \) at a fixed value of the last argument. The Lagrangian up to and including terms with two derivatives and one factor of \( \mathcal{M} \) reads

\[
\mathcal{L}_\phi = -V_0 + V_1 \langle \nabla_\mu \bar{U} \nabla^\mu \bar{U} \rangle + V_2 \langle \bar{\chi} \bar{U} + \bar{\chi} \bar{U} \rangle + iV_3 \langle \bar{\chi} \bar{U} - \bar{\chi} \bar{U} \rangle
\]

\[
+ V_4 \langle \bar{U} \nabla_\mu \bar{U} \rangle \langle \bar{U} \nabla^\mu \bar{U} \rangle. \tag{5}
\]

The expression \( \langle \ldots \rangle \) denotes the trace in flavor space and \( \bar{\chi} = \bar{\chi}^\dagger = 2B_0 \mathcal{M} \) with \( \mathcal{M} = \text{diag}(m_u, m_d, m_s) \) and \( B_0 = -\langle 0 | \bar{q} q | 0 \rangle / F_\pi^2 \) the order parameter of the spontaneous symmetry violation. The covariant derivative of \( \bar{U} \) is defined by

\[
\nabla_\mu \bar{U} = \partial_\mu \bar{U} - i(v_\mu + a_\mu) \bar{U} + i \bar{U} (v_\mu - a_\mu). \tag{6}
\]

The external fields \( v_\mu(x), a_\mu(x) \) represent Hermitian \( 3 \times 3 \) matrices in flavor space. Note that a term of the type \( iV_5 \langle \bar{U}^\dagger \nabla_\mu \bar{U} \rangle \nabla^\mu \theta \) can be transformed away \(^9\) and a term proportional to \( V_6 \nabla_\mu \theta \nabla^\mu \theta \) does not enter the calculations performed in the present work and will be neglected. The coefficients \( V_i \) are functions of the variable \( \theta - i \ln \det \bar{U} \), \( V_i(\theta - i \ln \det \bar{U}) \), and can be expanded in terms of this variable. The terms \( V_1, \ldots, 4 \) are of second chiral order, whereas \( V_0 \) is of zeroth chiral order. Parity conservation implies that the \( V_i \) are all even functions of

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\(^3\)To be more precise, the Lagrangian changes by a total derivative which gives rise to the Wess-Zumino term. We will neglect this contribution since the corresponding terms involve five or more meson fields which do not play any role for the discussions here.
\( \theta - i \ln \det \tilde{U} \) except \( V_3 \), which is odd, and \( V_1(0) = V_2(0) = F_\pi^2/4 \) gives the correct normalization for the quadratic terms of the Goldstone boson octet, where \( F_\pi \simeq 92.4 \text{ MeV} \) is the pion decay constant.

In order to use the effective Lagrangian, one must first determine the vacuum expectation value of \( \tilde{U} \) by minimizing the potential energy

\[
V(\tilde{U}) = V_0 - V_2(\tilde{\chi}^\dagger \tilde{U} + \tilde{\chi} \tilde{U}^\dagger) - i V_3(\tilde{\chi}^\dagger \tilde{U} - \tilde{\chi} \tilde{U}^\dagger). \tag{7}
\]

Since \( \tilde{\chi} \) is diagonal, one can assume the minimum \( U_0 \) to be diagonal as well and of the form

\[
U_0 = \text{diag}(e^{-i\varphi_u}, e^{-i\varphi_d}, e^{-i\varphi_s}). \tag{8}
\]

In terms of the angles \( \varphi_q \) the potential becomes

\[
V(U_0) = V_0(\tilde{\theta}_0) - 4 B_0 V_2(\tilde{\theta}_0) \sum_q m_q \cos \varphi_q - 4 B_0 V_3(\tilde{\theta}_0) \sum_q m_q \sin \varphi_q, \tag{9}
\]

where we have introduced the notation \( \tilde{\theta}_0 = \theta_0 - \sum_q \varphi_q \). The Taylor expansions of the functions \( V_i \) read

\[
V_i(\tilde{\theta}_0) = \sum_{n=0}^{\infty} V_i^{(2n)}(0) \tilde{\theta}_0^{2n} \quad \text{for } i = 0, 2
\]

\[
V_3(\tilde{\theta}_0) = \sum_{n=0}^{\infty} V_3^{(2n+1)}(0) \tilde{\theta}_0^{2n+1}
\]

for \( i = 0, 2 \) (10)

with coefficients not fixed by chiral symmetry. Minimizing the potential with respect to the angles \( \varphi_q \) leads to

\[
2B_0 m_q \sin \varphi_q = A + 2B_0 B m_q \cos \varphi_q \tag{11}
\]

with

\[
A = 2B_0 \left( \sum_{n=0}^{\infty} V_2^{(2n)}(0) \tilde{\theta}_0^{2n} \right)^{-1} \sum_{n=1}^{\infty} \left( \frac{1}{2B_0} n V_0^{(2n)}(0) \tilde{\theta}_0^{2n-1} - 2n V_2^{(2n)}(0) \tilde{\theta}_0^{2n-1} \sum_{j=u,d,s} m_j \cos \varphi_j - (2n-1) V_3^{(2n-1)}(0) \tilde{\theta}_0^{2n-2} \sum_{j=u,d,s} m_j \sin \varphi_j \right) \tag{12}
\]

and

\[
B = \left( \sum_{n=0}^{\infty} V_2^{(2n)}(0) \tilde{\theta}_0^{2n} \right)^{-1} \sum_{n=0}^{\infty} V_3^{(2n+1)}(0) \tilde{\theta}_0^{2n+1}. \tag{13}
\]

To lowest order both in the quark masses \( m_q \) and \( 1/N_c \) Eq. (11) reads

\[
\frac{1}{2} B_0 F_\pi^2 m_q \sin \varphi_q = V_0^{(2)}(0) \tilde{\theta}_0 \tag{14}
\]

which is the equation for the \( \varphi_q \) considered in [3, 11]. One then writes

\[
\tilde{U} = \sqrt{U_0} U \sqrt{U_0} \tag{15}
\]
and $U$ can be parametrized as

$$U(\phi, \eta_0) = \exp\{2i\phi/F_\pi + i\sqrt{2/3}\eta_0/F_0\},$$

(16)

where the singlet $\eta_0$ couples to the singlet axial current with strength $F_0$. The unimodular part of the field $U(x)$ contains the degrees of freedom of the Goldstone boson octet $\phi$

$$\phi = \frac{1}{\sqrt{2}} \left( \begin{array}{ccc} \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}} \eta_8 \end{array} \right),$$

(17)

while the phase $\det U(x) = e^{i\sqrt{6}\eta_0/F_0}$ describes the $\eta_0$. The diagonal subgroup $U(3)_V$ of $U(3)_R \times U(3)_L$ does not have a dimension-nine irreducible representation and consequently does not exhibit a nonet symmetry. We have therefore used the different notation $F_0$ for the decay constant of the singlet field.

One can now express the effective Lagrangian in terms of the Goldstone boson matrix $U$ and the angles $\varphi_q$

$$\mathcal{L}_\phi = -V_0 + V_1\langle \nabla_\mu U^\dagger \nabla^\mu U \rangle + [V_2 + BV_3]\langle \chi(U + U^\dagger) \rangle - iAV_2\langle U - U^\dagger \rangle
+ i[V_3 - BV_2]\langle \chi(U - U^\dagger) \rangle + AV_3\langle U + U^\dagger \rangle + V_4\langle U\nabla_\mu U^\dagger \rangle \langle U^\dagger \nabla^\mu U \rangle,$$

(18)

where we have absorbed the angles $\varphi_q$ in hermitian matrices $\chi$ and $H$ by defining

$$\sqrt{U_0^\dagger \chi \sqrt{U_0} = \chi + iH}, \quad \sqrt{U_0 \chi^\dagger \sqrt{U_0} = \chi - iH},$$

(19)

so that $\chi = 2B_0\text{diag}(m_q \cos \varphi_q)$ and $H = 2B_0\text{diag}(m_q \sin \varphi_q) = A + B\chi$. The $V_i$ are functions of $\sqrt{6}\eta_0/F_0 + \bar{\theta}_0$, $V_i(\sqrt{6}\eta_0/F_0 + \bar{\theta}_0)$ and we have assumed the external fields $v_\mu$ and $a_\mu$ to be diagonal which is the case if one considers electromagnetic interactions. Note that the Goldstone boson masses at lowest chiral order are not only functions of the current quark masses $m_q$, but also depend on the angles $\cos \varphi_q$. The kinetic energy of the $\eta_0$ singlet field obtains contributions from $V_1\langle \nabla_\mu U^\dagger \nabla^\mu U \rangle$ and $V_4\langle U\nabla_\mu U^\dagger \rangle \langle U^\dagger \nabla^\mu U \rangle$ which read

$$\left(\frac{F_\pi^2}{2F_0^2} + \frac{6}{F_0^2}V_4(0)\right) \partial_\mu \eta_0 \partial^\mu \eta_0,$$

(20)

We renormalize the $\eta_0$ field in such a way that the coefficient in brackets is $1/2$ in analogy to the kinetic term of the octet. By redefining $F_0$ and keeping for simplicity the same notation both for $\eta_0$ and $F_0$ one arrives at the same Lagrangian as in Eq. (18) but with $V_4(0) = (F_0^2 - F_\pi^2)/12$ in order to ensure the usual normalization for the kinetic term of a pseudoscalar particle.

### 3 CP-violating terms in the baryon Lagrangian

As mentioned before, another source for CP-violation is the baryon Lagrangian. The CP-violating terms can be divided into two groups. Firstly, the vacuum alignment of the mesonic Lagrangian induces $CP$ non-conserving meson-baryon interactions as considered in [3]. But secondly, there are also explicitly $CP$-violating terms in the most general Lagrangian in the presence of the $\theta$-vacuum angle which have been neglected in [3]. In order to construct the
Lagrangian in the baryon-sector, one has to adopt a non-linear representations for baryons. The main ingredient for a non-linear realization is the compensator field $K(\hat{U}, R, L) \in U(3)_V$, which appears in the chiral $U(3)_L \times U(3)_R$ transformation of the left and right coset representatives, $\xi_L(\hat{U})$ and $\xi_R(\hat{U})$:

$$
\begin{align*}
\tilde{\xi}_L(\hat{U}) & \to L \tilde{\xi}_L(\hat{U}) K^\dagger(\hat{U}, R, L) \\
\tilde{\xi}_R(\hat{U}) & \to R \tilde{\xi}_R(\hat{U}) K^\dagger(\hat{U}, R, L).
\end{align*}
$$

(21)

The field $\hat{U}$ from the last section is defined as

$$
\hat{U} = \tilde{\xi}_R \tilde{\xi}_L^\dagger.
$$

(22)

The baryon octet $B$ is given by the matrix

$$
B = \begin{pmatrix}
\frac{1}{\sqrt{2}} \Sigma^0 + \frac{1}{\sqrt{6}} \Lambda & \Sigma^+ & p \\
\Sigma^- & -\frac{1}{\sqrt{2}} \Sigma^0 + \frac{1}{\sqrt{6}} \Lambda & n \\
\Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}} \Lambda
\end{pmatrix}
$$

(23)

which transforms as a matter field

$$
B \to B' = KBK^\dagger.
$$

(24)

The covariant derivative of the baryon fields reads

$$
[\tilde{D}_\mu, B] = \partial_\mu B + [\tilde{\Gamma}_\mu, B]
$$

(25)

with $\tilde{\Gamma}_\mu$ being the chiral connection

$$
\tilde{\Gamma}_\mu = \frac{1}{2} [\tilde{\xi}_R^\dagger (\partial_\mu - i r_\mu) \tilde{\xi}_R + \tilde{\xi}_L^\dagger (\partial_\mu - i l_\mu) \tilde{\xi}_L]
$$

(26)

and $r_\mu = v_\mu + a_\mu$, $l_\mu = v_\mu - a_\mu$. For electromagnetic interactions the external fields are $a_\mu = 0$ and $v_\mu = -eQ A_\mu$ with the quark charge matrix $Q = \frac{1}{3} \text{diag}(2, -1, -1)$. In order to incorporate the interactions with the mesons into the effective theory it is convenient to form an object of axial-vector type with one derivative

$$
\tilde{\xi}_\mu = i [\tilde{\xi}_R^\dagger (\partial_\mu - i r_\mu) \tilde{\xi}_R - \tilde{\xi}_L^\dagger (\partial_\mu - i l_\mu) \tilde{\xi}_L].
$$

(27)

Further ingredients of the non-linear representation are

$$
\check{\chi}_\pm = \tilde{\xi}_L \chi^\dagger \tilde{\xi}_R \pm \tilde{\xi}_R \chi \tilde{\xi}_L
$$

(28)

and the quantity

$$
\bar{F}_{\mu \nu}^\pm = \tilde{\xi}_R^\dagger F_{\mu \nu}^R \tilde{\xi}_R \pm \tilde{\xi}_L^\dagger F_{\mu \nu}^L \tilde{\xi}_L,
$$

(29)

where $F_{\mu \nu}^{R/L}$ are the field strength tensors of $r_\mu / l_\mu$. The most general relativistic effective Lagrangian up to second order in the derivative expansion and contributing to the electric dipole moments of the neutron and $\Lambda$ reads

$$
\mathcal{L}_{\phi B} = iW_1 \langle [\tilde{D}_\mu, B] \gamma_\mu B \rangle - iW_1^* \langle B \gamma_\mu [\tilde{D}_\mu, B] \rangle + W_2 \langle BB \rangle \\
+ W_3 \langle \hat{B}_\mu \gamma_5 \tilde{\xi}_\mu B \rangle + W_4 \langle \hat{B}_\mu \gamma_5 \tilde{\xi}_\mu B \rangle + W_5 \langle \hat{B}_\mu \gamma_5 B \rangle \langle \tilde{\xi}_\mu \rangle \\
+iW_6 \langle \hat{B}_\mu \gamma_5 B \rangle + W_7 \langle \hat{B} \tilde{\chi}_+ B \rangle + W_8 \langle \hat{B} \tilde{\chi}_+ B \rangle + W_9 \langle \hat{B} B \rangle \langle \tilde{\chi}_+ \rangle \\
+iW_{10} \langle \hat{B} \tilde{\chi}_- B \rangle + iW_{11} \langle \hat{B} \tilde{\chi}_- B \rangle + iW_{12} \langle \hat{B} B \rangle \langle \tilde{\chi}_- \rangle \\
+iW_{13} \langle \hat{B} \sigma_{\mu \nu} \gamma_5 \tilde{\xi}_\mu \tilde{\xi}_\nu B \rangle + iW_{14} \langle \hat{B} \sigma_{\mu \nu} \gamma_5 \tilde{\xi}_\mu \tilde{\xi}_\nu B \rangle \\
+iW_{15} \langle \hat{B} \sigma_{\mu \nu} \gamma_5 B \rangle \langle \tilde{\xi}_\mu \rangle.
$$

(30)
The $W_i$ are functions of the combination $\sqrt{6}\eta_0/F_0 + \bar{\theta}_0$. From parity it follows that $W_{1, 5}$ and $W_{7,8,9}$ are even in this variable, whereas $W_6$ and $W_{10, 15}$ are odd. The latter have not been taken into account in [3]. One can further reduce the number of independent terms by making the following transformation. By decomposing the baryon fields into their left- and right-handed components

$$B_{R/L} = \frac{1}{2}(1 \pm \gamma_5)B$$

and transforming the left- and right-handed states separately via

$$B_{R/L} \rightarrow \sqrt{W_2 \pm iW_6}B_{R/L}$$

$$\bar{B}_{R/L} \rightarrow \sqrt{W_2 \pm iW_6}\bar{B}_{R/L}$$

one can eliminate the $\langle B\gamma_5B \rangle$ term and simplify the coefficient of $\langle B\bar{B} \rangle$. The details of this calculation are given in [6]. Note that this transformation leads to mixing of the terms of the type $B\bar{\chi}_+\pm B$ with $B\gamma_5\bar{\chi}_+\pm B$ which are of third chiral order and have been neglected here. Furthermore, the terms $W_{13, \ldots, 15}$ mix with the terms from the Lagrangian

$$\mathcal{L} = W_{16}\langle B(\sigma_{\mu\nu}\{\bar{F}_{\mu+}, B\} \rangle + W_{17}\langle B(\sigma_{\mu\nu}\{\bar{F}_{\mu+}, B\} \rangle + W_{18}\langle B(\sigma_{\mu\nu}B\{\bar{F}_{\mu+} \rangle, \quad (33)$$

but in both cases the form of the Lagrangian does not change and we can proceed by neglecting the $W_6$ term and setting $W_1 = W_1^*$ and $W_2 = -\bar{M}$ with $\bar{M}$ being the baryon mass in the chiral limit [6]. The expansion of the coefficients in terms of the $W_i$ read

$$W_1 = -\frac{1}{2} + \ldots, \quad W_2 = -\bar{M},$$

$$W_3 = -\frac{1}{2} D + \ldots, \quad W_4 = -\frac{1}{2} F + \ldots, \quad W_5 = \frac{1}{2} \lambda + \ldots,$$

$$W_7 = b_D + \ldots, \quad W_8 = b_F + \ldots, \quad W_9 = b_0 + \ldots,$$

$$W_i = w_i (\sqrt{6}\eta_0 + \bar{\theta}_0) + \ldots, \quad \text{for } i = 10, \ldots, 15 \quad (34)$$

where the ellipses denote higher orders in $\sqrt{6}\eta_0/F_0 + \bar{\theta}_0$ and we have only shown terms that can contribute to one-loop order. The axial-vector couplings $D$ and $F$ can be determined from semileptonic hyperon decays. A fit to the experimental data delivers $D = 0.80 \pm 0.01$ and $F = 0.46 \pm 0.01$ [12]. The third coupling, $\lambda$, is specific to the axial flavor-singlet baryonic current. The coefficients $b_{D, F, 0}$ have been determined from the calculation of the baryon masses and the $\pi N$ $\sigma$-term up to fourth chiral order [13] and their mean values are, in units of GeV$^{-1}$,

$$b_D = 0.079, \quad b_F = -0.316, \quad b_0 = -0.606. \quad (35)$$

The numerical values of the parameters $w_{10, \ldots, 15}$ are not known. We leave them undetermined for the time being and will give later an upper bound.

Again the correct vacuum has to be chosen. This is done by setting

$$\tilde{\xi}_R = \sqrt{U_0} \xi_R$$

$$\tilde{\xi}_L = \sqrt{U_0} \xi_L$$

$$7$$
and we will choose the coset representatives such that
\[ \xi_R = \xi_L^+ = u = \sqrt{U}. \] (37)

For diagonal external fields \( v_\mu \) and \( a_\mu \) one can write
\[ \tilde{\Gamma}_\mu = \Gamma_\mu = \frac{1}{2}[u^\dagger (\partial_\mu - ir_\mu) u + u (\partial_\mu - il_\mu) u^\dagger] \] (38)
and
\[ \tilde{\xi}_\mu = u_\mu = i[u^\dagger (\partial_\mu - ir_\mu) u - u (\partial_\mu - il_\mu) u^\dagger]. \] (39)

Furthermore, one obtains
\[ \tilde{\chi}_+ = \chi_+ - iA(U - U^\dagger) - iB\chi_- \]
\[ \tilde{\chi}_- = \chi_- - iA(U + U^\dagger) - iB\chi_+, \] (40)
where the quark mass matrix enters in the combinations
\[ \chi_\pm = u\chi^\dagger u \pm u^\dagger \chi u^\dagger. \] (41)

Finally, \( \tilde{F}_{\mu\nu}^+ \) simplifies to
\[ \tilde{F}_{\mu\nu}^+ = F_{\mu\nu}^+ = u^\dagger F_{\mu\nu}^R u + u F_{\mu\nu}^L u^\dagger. \] (42)

The relativistic baryon Lagrangian reads to the order we are working
\[ \mathcal{L}_{\phi B} = \ i(\bar{\phi}\gamma_\mu [D_\mu, B]) - \bar{M}^0 \langle \bar{B}B \rangle \\
- \frac{1}{2}D\langle \bar{B}\gamma_\mu \gamma_5 \{u^\mu, B\} \rangle - \frac{1}{2}F\langle \bar{B}\gamma_\mu \gamma_5 [u^\mu, B] \rangle + \frac{1}{2}\lambda \langle \bar{B}\gamma_\mu \gamma_5 B \rangle \langle u^\mu \rangle \\
- ib_D A\langle \bar{B}\{U - U^\dagger, B\} \rangle - ib_F A\langle \bar{B}[U - U^\dagger, B] \rangle - ib_0 A\langle \bar{B}B \rangle \langle U - U^\dagger \rangle \\
+ 4Aw_1^0 \frac{\sqrt{6}}{F_0} \eta_0 \langle \bar{B}B \rangle + 6Aw_2^0 \frac{\sqrt{6}}{F_0} \eta_0 \langle \bar{B}B \rangle \\
+ i(w_{13}^0 \bar{B}_0 + w_{13} \frac{\sqrt{6}}{F_0} \eta_0) \langle \bar{B}\sigma_{\mu\nu} \gamma_5 \{F_{\mu\nu}^+, B\} \rangle \\
+ i(w_{14}^0 \bar{B}_0 + w_{14} \frac{\sqrt{6}}{F_0} \eta_0) \langle \bar{B}\sigma_{\mu\nu} \gamma_5 [F_{\mu\nu}^+, B] \rangle \\
+ i(w_{15}^0 \bar{B}_0 + w_{15} \frac{\sqrt{6}}{F_0} \eta_0) \langle \bar{B}\sigma_{\mu\nu} \gamma_5 B \rangle \langle F_{\mu\nu}^+, B \rangle, \] (43)
where we have neglected meson-baryon interactions with more than one meson field since they do not contribute at one-loop order and terms of \( \mathcal{O}(\theta_0^2) \) or higher orders are omitted throughout this work. Note that there exist terms of fourth chiral order of the type \( \bar{B}\sigma_{\mu\nu} \gamma_5 \bar{\chi}_- F_{\mu\nu}^+ B \). Using Eq. (40) for \( \bar{\chi}_- \) they induce \( CP \)-violating terms of the form \( \bar{\theta}_0 \bar{B}\sigma_{\mu\nu} \gamma_5 \bar{F}_{\mu\nu}^+ B \) which are already accounted for by the terms \( w_{13,14,15} \). This amounts to a renormalization of the couplings \( \bar{\theta}_0 w_{13,14,15} \). We have therefore introduced the notation \( w_{13,14,15} \) for these interaction terms, in order to distinguish them from the unrenormalized \( w_{13,14,15} \) of the interactions proportional to \( \eta_0 \).

The drawback of the relativistic framework including baryons is that due to the existence of a new mass scale, namely the baryon mass in the chiral limit \( \bar{M} \), there exists no strict
chiral counting scheme, i.e. a one-to-one correspondence between the meson loops and the chiral expansion. In order to overcome this problem one integrates out the heavy degrees of freedom of the baryons, similar to a Foldy-Wouthuysen transformation, so that a chiral counting scheme emerges. To this end, one constructs eigenstates of the velocity projection operator $P_v = (1 + v)/2$

$$B_v(x) = e^{i M v \cdot x} P_v B(x).$$

(44)

The Dirac algebra simplifies considerably. It allows to express any Dirac bilinear $B_v \Gamma B_v(\Gamma = 1, \gamma \mu \gamma_5, \ldots)$ in terms of the velocity $v$, and the spin operator $2 S = i \gamma_5 \sigma \mu \nu v$. One can rewrite the Dirac bilinears which appear in the present calculation as

$$B_v \gamma_\mu \gamma_5 B_v = 2 B_v S_\mu B_v, \quad B_v \sigma_\mu \nu \gamma_5 B_v = 2i (v_\mu B_v S_\nu B_v - v_\nu B_v S_\mu B_v).$$

(45)

In the following, we will drop the index $v$. The Lagrangian of the heavy baryon formulation reads

$$L_{\Delta B} = i \langle \bar{B}[v \cdot D, B] \rangle - D \langle \bar{B} S_\mu [u^\mu, B] \rangle - F \langle \bar{B} S_\mu [u^\mu, B] \rangle + \lambda \langle \bar{B} S_\mu B \rangle [u^\mu] - i b_D A [B [U - U^\dagger, B]] - i b_F A [B [U - U^\dagger, B]] - i b_0 A [BB] [U - U^\dagger] + 4 A w_10 \frac{\sqrt{6}}{F_0} \eta_0 \langle \bar{B} B \rangle + 6 A w_12 \frac{\sqrt{6}}{F_0} \eta_0 \langle \bar{B} B \rangle$$

$$- 4 (w_{13}^0 + w_{13} \frac{\sqrt{6}}{F_0} \eta_0) \langle \bar{B} v_\mu S_\nu [F_+^{\mu \nu}, B] \rangle$$

$$- 4 (w_{14}^0 + w_{14} \frac{\sqrt{6}}{F_0} \eta_0) \langle \bar{B} v_\mu S_\nu [F_+^{\mu \nu}, B] \rangle$$

$$- 4 (w_{15}^0 + w_{15} \frac{\sqrt{6}}{F_0} \eta_0) \langle \bar{B} v_\mu S_\nu B \langle F_+^{\mu \nu} \rangle \rangle. \quad (46)$$

No relativistic corrections are needed to the order we are working.

4 The electric dipole moments $d_n^\gamma$ and $d_\Lambda^\gamma$

In this section, we will calculate the electric dipole moments of the neutron and the $\Lambda$ at lowest order in chiral perturbation theory, i.e. $O(p^2)$. Due to the vacuum alignment the baryon Lagrangian contains interaction terms $b_{D,F,0}$ of zeroth chiral order and one-loop diagrams with these vertices will also contribute at $O(p^2)$. In the relativistic framework of [3] the electric dipole moment of the neutron $d_n^\gamma$ has been defined via

$$L_{n EDM} = \frac{1}{2} d_n^\gamma \bar{n} i \sigma_\mu \nu \gamma_5 n F^{\mu \nu}, \quad (47)$$

where $F^{\mu \nu}$ is the field strength tensor of the photon field $A^\mu$. We prefer to rewrite this as a form factor

$$D_n^\gamma(q^2) \bar{u}(p') \sigma_\mu \nu \gamma_5 u(p) q^\mu,$$

(48) with $q = p' - p$ being the momentum transfer. The electric dipole moment is given by

$$d_n^\gamma = D_n^\gamma(q^2 = 0).$$

(49)
For the calculation of the form factor in the heavy baryon approach we set \( v_\mu = (1, 0) \) and use the Breit frame \( v \cdot p = v \cdot p' \) since it allows a unique translation of Lorentz-covariant matrix elements into non-relativistic ones. In this frame the form factor reads

\[
-2iD_n^\gamma(q^2)H v\nu S \cdot qH + \ldots,
\]

where \( H \) is the large component of \( u \) and the ellipsis stands for a similar expression in the small components of \( u \), which is of higher chiral order and will be omitted. The electric dipole moment \( d_n^\nu \) receives contributions from the \( w_{13} \)-term and the loops (we work in the isospin limit \( m_u = m_d = \hat{m} \))

\[
d_n^\nu = d_n^{\nu (tree)} + d_n^{\nu (loop)}
\]

with

\[
d_n^{\nu (tree)} = -8e\theta_0 \left[ \frac{1}{3}w_{13}^{\nu} + \frac{16}{F_\pi^2 F_0^2 m_{\eta_0}^2}V_3^{(1)}V_0^{(2)} w_{13} \right],
\]

where the pertinent diagrams are shown in Figure 1. Diagram 1b) is missing in [3] since the mesonic Lagrangian used within this work does not have a term linear in the singlet field \( \eta_0 \). Such an interaction originates from the term \( V_3(\chi_-) \) which is not considered in [3]. The chiral logarithms of the diagrams shown in Fig. 2 read

\[
d_n^{\nu (loop)} = \frac{1}{\pi^2 F_\pi^4} eV_0^{(2)} \tilde{\theta}_0 \left[ -(b_D + b_F)(D + F) \ln \frac{m_\pi^2}{\mu^2} + (b_D - b_F)(D - F) \ln \frac{m_K^2}{\mu^2} \right]
\]

with \( \mu \) being the scale introduced in dimensional regularization. In the Breit frame only diagrams 2a) and b) contribute to the electric dipole moment. Diagrams 2c), d) and e), f) are proportional to \( S_\nu \) and therefore do not contribute to \( d_n^\nu \). The loop integral for diagrams 2a) and b) contains also analytic and divergent pieces which can be absorbed by redefining \( w_{13}^{\nu} \). The divergent pieces of \( w_{13}^{\nu} \) cancel the divergencies from the loops and render the final expression finite. We summarize the remaining analytical contributions in \( w_{13}^{\nu} \), so that \( w_{13}^{\nu} \) in Eq. (52) is understood to be finite. The results in Eqs. (52) and (53) are not in contradiction with the fact that the electric dipole moment induced by the \( \theta_0 \)-term tends to zero if any of the quark masses vanish. If, e.g., \( m_u = 0 \) then a solution for Eq. (11) is given by \( \varphi_u = \theta_0 \) and \( \varphi_d,s = 0 \) leading to \( \theta_0 = \theta_0 - \varphi_u = 0 \). Therefore, \( d_n^\nu \) vanishes in this case.

The results for the \( \Lambda \) are

\[
d_\Lambda^{\gamma (tree)} = -4e\tilde{\theta}_0 \left[ \frac{1}{3}w_{13}^{\nu} + \frac{16}{F_\pi^2 F_0^2 m_{\eta_0}^2}V_3^{(1)}V_0^{(2)} w_{13} \right]
\]

and

\[
d_\Lambda^{\gamma (loop)} = -\frac{1}{\pi^2 F_\pi^4} eV_0^{(2)} \tilde{\theta}_0 [b_DF + b_FD] \ln \frac{m_K^2}{\mu^2}.
\]

Note that the relation \( d_n^\nu = 2d_\Lambda^\gamma \) is only valid at tree level and not for the chiral logarithms as claimed in [3]. The discrepancy is due to the lack of pion loops in the present work. Once one accounts for the mistake made in [3] by replacing \( \ln m_\pi \) by \( \ln m_K \) in the chiral loop contribution for \( d_\Lambda^\gamma \) one obtains our result (55).

## 5 Numerical results and conclusions

In order to compute the numerical results for the electric dipole moments shown in the last section, we use the central values for the parameters \( b_D = 0.079 \text{ GeV}^{-1} \), \( b_F = -0.316 \text{ GeV}^{-1} \),
$D = 0.80$ and $F = 0.46$. To lowest order in the angles $\varphi_0$ and using $\hat{m} \ll m_s$ one can express $\hat{\theta}_0$ in terms of $\theta_0$ via

$$\theta_0 \simeq [1 + \frac{8V_0^{(2)}}{F^{2}_\pi m^2_{\pi}}] \hat{\theta}_0. \quad (56)$$

From the calculation of the $\eta$ and $\eta'$ masses and decay constants one can extract the value for $V_0^{(2)} \ [14]$

$$V_0^{(2)} \simeq -\frac{27}{4} F^{4}_\pi \simeq -5.0 \times 10^{-4} \text{ GeV}^4, \quad (57)$$

so that

$$\hat{\theta}_0 \simeq \frac{F^{2}_\pi m^{2}_\pi}{8V_0^{(2)}} \theta_0 \simeq -0.04 \theta_0. \quad (58)$$

Inserting this into the loop contribution and using $\mu = 1 \text{ GeV}, m_{\eta_0} \simeq m_{\eta'} = 958 \text{ MeV}$ we obtain

$$d^{(\text{loop})}_n = -7.5 \times 10^{-16} \theta_0 \text{ cm}$$

$$d^{(\text{loop})}_\Lambda = -1.7 \times 10^{-16} \theta_0 \text{ cm}. \quad (59)$$

The numerical result for $d^{(\text{loop})}_n$ is in agreement with the one given in [3] once one accounts for the different values of the parameters $b_2, b_F, D, F$ and the scale $\mu$ used within that work. The result for the chiral logarithm of $d^{(\text{loop})}_\Lambda$ is considerably smaller than for $d^{(\text{loop})}_n$ since there is no contribution from the pion loops which dominate in the case of $d^{(\text{loop})}_n$.

A precise numerical value for the tree contribution to the electric dipole moments cannot be given since the parameters $w_{13}^{r}$ and $w_{13}^{r}$ are not known. However, we will give an upper bound for their contribution based on large $N_c$ arguments which seem to work well in the purely mesonic sector $\ [14, 15]$. In the present investigation we are only interested in an order of magnitude estimate for the vacuum angle $\theta$ and for this purpose it is sufficient to give a numerical range for the tree level contribution. We will first estimate the ratio of diagrams 1a) and 1b) using $1/N_c$ arguments. Applying large $N_c$ counting rules, see e.g. $[9, 10]$, both $w_{13}^r$ and $w_{13}^r$ are of order $\mathcal{O}(N^0_c)$ so that we can assume $|w_{13}^r/w_{13}^r| = \mathcal{O}(1)$. One obtains the ratio

$$\frac{|d^{(1a)}_\Lambda|}{|d^{(1a)}_n|} \simeq \frac{|d^{(1b)}_\Lambda|}{|d^{(1b)}_n|} \simeq \frac{48}{F^{4}_\pi m^{2}_{\pi}} |V^{(2)}_0 V^{(1)}_3| \simeq 0.12, \quad (60)$$

where we have used $F^{4}_\pi/F_0 = 1 + \mathcal{O}(N^{-1}_c)$ and taken the value for $V^{(1)}_3$ from $[14]$.

$$V^{(1)}_3 \simeq 0.04 F^{2}_\pi \simeq 3.5 \times 10^{-4} \text{ GeV}^2. \quad (61)$$

Diagram 1b) turns out to be insignificant in our estimate.

Based on large $N_c$ arguments one can also give an upper bound for $w_{13}^r$. The contact terms of the Lagrangian in Eq. (33), which contribute to the magnetic moments of the baryons, describe – similar to the $w_{13}^r$-term – the coupling of the field strength tensor $F^{\mu \nu}$ to the baryons. But the leading coefficient of the Taylor expansion of $W_{16}, w_{16}$, is of order $\mathcal{O}(N^1_c)$, whereas $|w_{13}^r| \simeq |w_{13}| = \mathcal{O}(N^0_c)$, so that we can assume $|w_{13}^r| < |w_{16}|$. In a calculation of the baryon magnetic moments to fourth chiral order $w_{16}$ has been determined to be $[16]$

$$w_{16} \simeq 0.4 \text{ GeV}^{-1}. \quad (62)$$
Inserting this upper limit for $w_{13}$ one obtains

\[
|d_n^{(\text{tree})}| < 9.6 \times 10^{-16} \theta_0 \, e \text{cm} \\
|d_\Lambda^{(\text{tree})}| < 4.8 \times 10^{-16} \theta_0 \, e \text{cm}. \quad (63)
\]

Since we have taken quite conservative limits, the tree level contributions could be dominating if the extreme values are chosen. A more realistic estimate might be obtained by setting $|w_{13}^{(r)} / w_{16}| \simeq \frac{1}{3}$ for $N_c = 3$, so that

\[
|d_n^{(\text{tree})}| \simeq 3.2 \times 10^{-16} \theta_0 \, e \text{cm} \\
|d_\Lambda^{(\text{tree})}| \simeq 1.6 \times 10^{-16} \theta_0 \, e \text{cm}. \quad (64)
\]

While the chiral logarithms dominate for $d_n^r$, the tree level contribution can be substantial for the $\Lambda$. This leads to

\[
d_n = (-7.5 \pm 3.2) \times 10^{-16} \theta_0 \, e \text{cm} \\
d_\Lambda = (-1.7 \pm 1.6) \times 10^{-16} \theta_0 \, e \text{cm}, \quad (65)
\]

where the theoretical uncertainty is given by the estimate of the tree level contribution in Eq. (64). From a comparison of the central value for $d_n^r$ with the experimental upper limit in Eq. (1) we obtain

\[
|\theta_0| < 8.4 \times 10^{-11}. \quad (66)
\]

In the case of the $\Lambda$ the experimental constraint is given by

\[
d_\Lambda < 1.5 \times 10^{-16} \, e \text{cm} \quad (67)
\]

which leads to

\[
|\theta_0| < 0.9. \quad (68)
\]

We have shown that it is possible to obtain a reliable limit for the vacuum angle $\theta_0$ by calculating the electric dipole moment of the neutron within the framework of heavy baryon chiral perturbation theory. To this end, we have constructed the most general effective Lagrangian up to one-loop order in the presence of the vacuum angle $\theta_0$ with the method proposed in [6]. The theoretical uncertainty from unknown parameters at tree level has been estimated by using large $N_c$ arguments. While the chiral loops dominate for the electric dipole moment of the neutron, the counterterms can be substantial in the case of the $\Lambda$.

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**References**


**Figure captions**

Fig.1 Shown are the tree diagrams for the electric dipole moment. Solid and dashed lines denote baryons and pseudoscalar mesons, respectively. The wavy line represents a photon and the dot is a $CP$-violating vertex.

Fig.2 Loop diagrams contributing to the electric dipole moment. Solid and dashed lines denote baryons and pseudoscalar mesons, respectively. The wavy line represents a photon and the dot is a $CP$-violating vertex.