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# Weak transition matrix elements from finite-volume correlation functions

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## Abstract

The two-body decay rate of a weakly decaying particle (such as the kaon) is shown to be proportional to the square of a well-defined transition matrix element in finite volume. Contrary to the physical amplitude, the latter can be extracted from finite-volume correlation functions in euclidean space without analytic continuation. The  $K \rightarrow \pi\pi$  transitions and other non-leptonic decays thus become accessible to established numerical techniques in lattice QCD.

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## 1. Introduction

The computation of the non-leptonic kaon decay rates from first principles, using lattice QCD and numerical simulations, meets a number of technical difficulties (see ref. [?], for example). Apart from the operator renormalization, which must be controlled at the non-perturbative level, the central problem is that the computational framework is limited to correlation functions in euclidean space and that there is apparently no simple relation between the behaviour of these functions at large time separations and the desired transition matrix elements [?,?].

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This statement (which is often referred to as the *Maiani–Testa no-go theorem*) applies to very large or infinite lattices, where the spectrum of final states is continuous. One might think that having a finite volume (as is unavoidable when numerical simulations are employed) makes it even more difficult to extract the transition amplitudes. In the present paper we wish to show that this is actually not so. The key observation is that the two-pion energy spectrum is far from being continuous when the lattice is only a few fermis wide. Under these conditions, a kaon at rest cannot decay into two pions unless one of these energy levels happens to be close to its mass. This is the case for certain lattice sizes, and a simple formula then relates the square of the corresponding transition amplitude in finite volume to the physical decay rate in infinite volume.

The problem is thus reduced to calculating the required finite-volume transition amplitudes. Since the initial and final states are isolated energy eigenstates, these matrix elements can in principle be computed using established techniques, such as those commonly employed to determine form factors. An additional difficulty is that the relevant two-pion states are not the lowest ones in the specified sector. Two-particle states in finite volume have, however, previously been studied [?–?] and practical methods have been devised to calculate the higher levels.

To keep the presentation as transparent as possible, we shall consider a simplified generic theory with two kinds of spinless particles, referred to as the kaon and the pion. Details are given in the next section, and we then first discuss the form of the two-pion energy spectrum in finite volume. This is essentially a summary of the relevant results of refs. [?–?]. In sect. 4 we define the transition amplitudes in finite volume and state the formula that relates them to the corresponding decay rates in infinite volume. The following sections contain the proof of this relation and a discussion of its application to the physical kaon decays.

## 2. Preliminaries

As announced above, we consider a generic situation where there are two particles, the “kaon” and the “pion”, with spin zero and masses such that

$$2m_\pi < m_K < 4m_\pi. \tag{2.1}$$

We assume that the symmetries of the theory are such that the kaon is stable in the absence of the weak interactions and that the pions scatter purely elastically

below the four-pion threshold. The weak interactions, described by a local effective lagrangian  $\mathcal{L}_w(x)$ , then allow the kaon to decay into two pions. The corresponding transition amplitude is

$$T(K \rightarrow \pi\pi) = \langle \pi p_1, \pi p_2 \text{ out} | \mathcal{L}_w(0) | K p \rangle, \quad (2.2)$$

with  $p_1$ ,  $p_2$  and  $p$  the four-momenta of the pions and the kaon. We shall only be interested in the physical case where the total momentum  $p = p_1 + p_2$  is conserved. Lorentz invariance and the kinematical constraints then imply that the transition amplitude is independent of the momentum configuration.

The meson states in eq. (2.2) are normalized according to the standard relativistic conventions (appendix A) and their phases are constrained by the LSZ formalism. In the case of the pions, for example, one assumes that there exists an interpolating hermitian field  $\varphi(x)$  such that

$$\langle 0 | \varphi(x) | \pi p \rangle = \sqrt{Z_\pi} e^{-ipx} \quad (2.3)$$

for some positive constant  $Z_\pi$ . If the phase of the kaon states is chosen in the same way, the CPT symmetry implies

$$T(K \rightarrow \pi\pi) = A e^{i\delta_0} \quad (2.4)$$

with  $A$  real and  $\delta_0$  the  $S$ -wave scattering phase shift of the outgoing pion state. The decay rate is then given by the usual expression

$$\Gamma = \frac{k_\pi}{16\pi m_K^2} |A|^2, \quad k_\pi \equiv \frac{1}{2} \sqrt{m_K^2 - 4m_\pi^2}, \quad (2.5)$$

proportional to the pion momentum  $k_\pi$  in the centre-of-mass frame.

### 3. Two-pion states in finite volume

In a spatial box of size  $L \times L \times L$  with periodic boundary conditions, the eigenvalues of the total momentum operator are integer multiples of  $2\pi/L$ . The energy spectrum is also discrete in this situation, with level spacings that can be appreciable. In the following we consider the subspace of states with zero total momentum and trivial transformation behaviour under cubic rotations and reflections.

The energy spectrum of the two-pion states in this sector below the inelastic threshold  $W = 4m_\pi$  has been studied in detail in refs. [?–?]. In particular, for the lowest energy value the expansion

$$W = 2m_\pi - \frac{4\pi a_0}{m_\pi L^3} \left\{ 1 + c_1 \frac{a_0}{L} + c_2 \frac{a_0^2}{L^2} \right\} + \mathcal{O}(L^{-6}), \quad (3.1)$$

$$c_1 = -2.837297, \quad c_2 = 6.375183, \quad (3.2)$$

has been obtained, where

$$a_0 = \lim_{k \rightarrow 0} \frac{\delta_0(k)}{k} \quad (3.3)$$

is the  $S$ -wave scattering length (here and below the scattering phase is considered to be a function of the pion momentum  $k$  in the centre-of-mass frame). The higher energy values in the elastic region are determined through

$$W = 2\sqrt{m_\pi^2 + k^2}, \quad (3.4)$$

$$n\pi - \delta_0(k) = \phi(q), \quad q \equiv \frac{kL}{2\pi}, \quad (3.5)$$

where  $n = 1, 2, \dots$  labels the energy levels in increasing order and the angle  $\phi(q)$  is a known kinematical function (appendix B). Apart from the lowest level, the energy spectrum at any given value of  $L$  is thus obtained by inserting the solutions  $k$  of eq. (3.5) in eq. (3.4)†.

All these results are valid up to terms that vanish exponentially at large  $L$ . Box sizes a few times larger than the diameter of the pion should be safe from these corrections. Equation (3.5) moreover assumes that the scattering phases  $\delta_l$  for angular momenta  $l \geq 4$  are small in the elastic region, which is usually the case since  $\delta_l$  is proportional to  $k^{2l+1}$  at low energies.

For illustration, let us consider QCD with three flavours of quarks, unbroken isospin symmetry and quark masses such that the masses of the charged pions and kaons coincide with their physical values. In the subspace with isospin 0, the two-pion energy spectrum is then given by eqs. (3.1)–(3.5), with  $\delta_0$  the appropriate pion

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† Similar formulae have been derived for the spectrum in the subspaces of states with non-zero total momentum [?]. The extension of our results to these sectors could give further insight into the connection between finite and infinite volume matrix elements and may prove useful in practice.

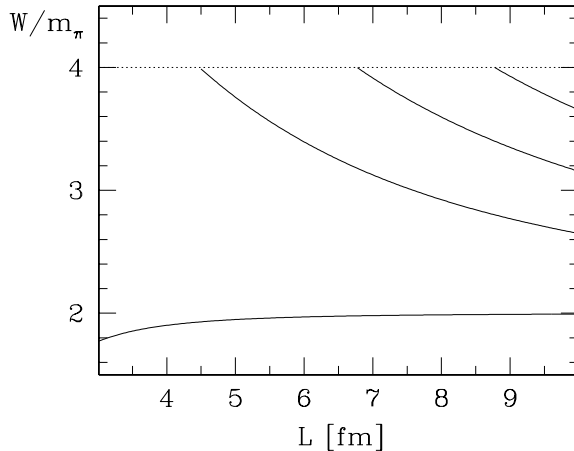


Fig. 1. Two-pion energy spectrum in QCD below the inelastic threshold, in the sector with isospin 0, calculated from eqs. (3.1)–(3.5) with the scattering phase shift given by next-to-leading order chiral perturbation theory. The levels shown in this plot are all non-degenerate.

scattering phase. If we insert the phase shift that is obtained at one-loop order of chiral perturbation theory [?–?], this yields the curves shown in fig. 1. For any other reasonable choice of the scattering phase the plot would look essentially the same, because the interaction effects are proportional to  $1/L^3$  and thus tend to be small. Note that the spacing between successive levels is quite large. One is clearly very far away from having a continuous spectrum when  $L \leq 10$  fm.

#### 4. Kaon decays in finite and infinite volume

Let us imagine that a state  $|K\rangle$  describing a kaon in finite volume with zero momentum has been prepared at time  $x_0 = 0$ . In the absence of the weak interactions, this is an energy eigenstate (and thus a stationary state) with energy  $m_K$ . However, through the interaction hamiltonian

$$H_w = \int_{x_0=0} d^3x \mathcal{L}_w(x), \quad (4.1)$$

the time evolution of the state becomes non-trivial and it starts to mix with the other

eigenstates of the unperturbed hamiltonian. It is straightforward to work this out using ordinary time-dependent perturbation theory. For the transition probability at time  $x_0 = t$  to any finite-volume two-pion state  $|\pi\pi\rangle$  with energy  $W$ , the result

$$P(K \rightarrow \pi\pi) = 4 |\langle \pi\pi | H_w | K \rangle|^2 \frac{\sin^2(\frac{1}{2}\omega t)}{\omega^2}, \quad \omega \equiv W - m_K, \quad (4.2)$$

is then obtained (in this equation the states are assumed to be normalized to unity and higher-order weak-interaction effects have been neglected).

From eq. (4.2) one infers that the transition probabilities tend to be very small unless the energy of one of the two-pion final states happens to be close to the kaon mass. Recalling fig. 1, it is clear that this will be the case only for certain box sizes  $L$ . In the following we focus on these special values of  $L$  and introduce the associated transition matrix element

$$M = \langle \pi\pi | H_w | K \rangle, \quad (4.3)$$

where *both states are normalized to unity* as before, while their phase will not matter and can be chosen arbitrarily. Since  $W = m_K$  in this case, eq. (4.2) becomes

$$P(K \rightarrow \pi\pi) = |M|^2 t^2 \quad (4.4)$$

and the kaon will thus have an appreciable probability to decay into the two-pion state if one waits long enough (the formula breaks down at very large times, because the higher-order terms are then no longer negligible).

The central result obtained in the present paper is that the finite-volume matrix element  $M$  is related to the decay rate of the kaon in infinite volume through

$$|A|^2 = 8\pi \left\{ q \frac{\partial \phi}{\partial q} + k \frac{\partial \delta_0}{\partial k} \right\}_{k=k_\pi} \left( \frac{m_K}{k_\pi} \right)^3 |M|^2 \quad (4.5)$$

[cf. eqs. (2.5),(3.5)]. The relation holds under the same premises as eq. (3.5) and the comments made in sect. 3 thus apply here too. Another restriction is that the two-pion final state has to be non-degenerate in the specified sector of the unperturbed theory. This condition is satisfied for  $n < 8$  [?], but degeneracies can occur at higher level numbers and the formula then ceases to be valid.

In principle eq. (4.5) allows one to compute the kaon decay rate in infinite volume by studying the theory in finite volume. Note that in the course of such a calculation it should also be possible to determine the two-pion energy spectrum and thus the scattering phase  $\delta_0$  in the elastic region.

The proportionality factor in eq. (4.5) essentially accounts for the different normalizations of the particle states in finite and infinite volume. One can easily check this in the free theory, where the pion self-interactions are neglected. In this case and for  $n \leq 6$ , the  $n$ -th two-pion energy level passes through  $m_K$  at

$$L = \frac{2\pi}{k_\pi} \sqrt{n}. \quad (4.6)$$

Equation (4.5) then assumes the form

$$|A|^2 = \frac{4}{\nu_n} (m_K L)^3 |M|^2, \quad (4.7)$$

$$\nu_n \equiv \text{number of integer vectors } \mathbf{z} \text{ with } \mathbf{z}^2 = n, \quad (4.8)$$

which is precisely what is derived from the relative normalizations of the plane waves in finite and infinite volume that describe the (non-interacting) kaon and pion states (sect. 6).

## 5. Proof of eq. (4.5)

The interpretation of the proportionality factor in eq. (4.5) given above also applies in the interacting case. This follows from the fact that the transition matrix elements probe the  $S$ -wave component of the two-pion wave function near the origin and that this component is the same in finite and infinite volume apart from its phase and normalization. The latter can be worked out explicitly in the framework of refs. [?–?], but the calculation is rather involved and will not be presented here.

Instead we shall go through a different argument, where one studies the influence of the weak interaction on the energy spectrum in finite volume. This can be done directly, using ordinary perturbation theory, or one may start from eq. (3.5) and take the weak-interaction effects on the scattering phase into account. The combination of the results of these calculations then yields eq. (4.5).

As already mentioned in sect. 2, the kaon is assumed to carry a quantum number (alias strangeness) that forbids its decay into pions in the unperturbed theory. Since only the strangeness-changing part of the weak interaction lagrangian contributes to the kaon transition amplitudes, all other terms may be dropped without loss. The matrix elements of the weak hamiltonian  $H_w$  between states with the same

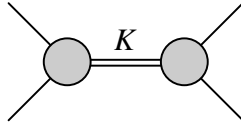


Fig. 2. Kaon resonance contribution to the elastic pion scattering amplitude in the  $s$ -channel. The diagram appears at second order of the expansion in powers of the weak interaction, with the bubbles representing the first-order  $K\pi\pi$  vertex function.

strangeness are then all equal to zero. As a consequence most energy values in finite volume are affected by the weak interaction only to second order.

First order energy shifts do occur, however, if there are degenerate states at lowest order that mix under the action of  $H_w$ . This is the case at the values of  $L$  where one of the two-pion energy values coincides with the kaon mass, i.e. at the special points considered in the preceding section. Degenerate perturbation theory then yields

$$W = m_K \pm |M| + \dots \quad (5.1)$$

for the first order change of these energy values (here and below the ellipses denote higher-order terms that do not contribute to the final results).

The energy shifts (5.1) can also be calculated by including the weak corrections to the scattering phase on the left-hand side of eq. (3.5). From the above one infers that the solutions of eq. (3.5) we are interested in are given by

$$k = k_\pi \pm \Delta k + \dots, \quad \Delta k \equiv \frac{m_K}{4k_\pi} |M|. \quad (5.2)$$

Compared to the kaon resonance width (which is of second order in the weak interaction), these values of  $k$  are far away from the kaon pole. The weak corrections to the pion scattering amplitude in the relevant range of energies are hence small and can be safely computed by working out the perturbation expansion in powers of the interaction lagrangian.

One might think that these corrections are all of second or higher order, because the interaction is strangeness-changing. The reason this is not so is that the kaon propagator in a diagram like the one shown in fig. 2 evaluates to

$$\frac{iZ_K}{p^2 - m_K^2} = \pm \frac{iZ_K}{2m_K |M|} + \dots \quad (5.3)$$

at the energies (5.1) and thus reduces the effective order of the term by 1. This diagram is in fact the only one that yields a first-order contribution to the scattering



amplitude. It can be calculated by noting that the momenta flowing into the three-point vertices are all on shell up to higher-order corrections. The vertices are hence proportional to the kaon decay amplitude  $A$ . Together with eq. (5.3) this leads to the result

$$\bar{\delta}_0(k) = \delta_0(k) \mp \frac{k_\pi |A|^2}{32\pi m_K^2 |M|} + \dots \pmod{\pi} \quad (5.4)$$

for the scattering phase in the full theory at the point (5.2) (as in the previous section,  $\delta_0$  stands for the phase shift in the unperturbed theory).

We now replace  $\delta_0$  in eq. (3.5) by  $\bar{\delta}_0$  and expand all terms in powers of the weak interaction. The lowest-order terms cancel while, at first order, the equation implies

$$-\Delta k \left\{ \frac{\partial \delta_0(k)}{\partial k} \right\}_{k=k_\pi} + \frac{k_\pi |A|^2}{16\pi m_K^2 |M|} = \Delta k \left\{ \frac{\partial \phi(q)}{\partial k} \right\}_{k=k_\pi}. \quad (5.5)$$

This is easily seen to be equivalent to eq. (4.5) after substituting the expression (5.2) for  $\Delta k$  and we have thus proved this relation.

## 6. Verification of eq. (4.5) in perturbation theory

In a low-energy effective theory, such as the chiral non-linear  $\sigma$ -model, it is possible to obtain an independent check on eq. (4.5) by working out the transition amplitudes in finite and infinite volume in perturbation theory. Since this calculation does not rely on any of the results presented above, it can provide additional confidence in the correctness of the equation. Perturbation theory may also prove helpful when considering more complicated situations, where one has several decay channels or particles with non-zero spin. In this section we describe how such a calculation goes, without giving too many details.

### 6.1 Specification of the model

The two-pion energy spectrum and the proportionality factor in eq. (4.5) depend on the final-state interactions only through the phase shift  $\delta_0$ . All other properties of the pion interactions do not matter and to check the equation we may thus consider an arbitrary effective meson theory with the correct particle spectrum.

For the pion interaction lagrangian the simplest choice is

$$\mathcal{L}_{\text{int}}(x) = \frac{1}{4!} \lambda \varphi(x)^4, \quad (6.1)$$

where  $\varphi(x)$  denotes the pion field and  $\lambda$  the bare coupling. To make the perturbation expansion completely well-defined, we introduce a Pauli–Villars cutoff  $\Lambda$ . At tree level the euclidean pion propagator is then given by

$$\int d^4x e^{-ipx} \langle \varphi(x) \varphi(0) \rangle = \frac{1}{m^2 + p^2} - \frac{1}{\Lambda^2 + p^2}, \quad (6.2)$$

with  $m$  the bare mass of the pion (its physical mass is denoted by  $m_\pi$  as before). The cutoff should be large enough so that ghost particles cannot be produced at energies below the four-pion threshold, but in view of the universality of eq. (4.5) there is no need to take  $\Lambda$  to infinity at the end of the calculation.

As far as the kaon is concerned, the least complicated possibility is to describe it by a hermitian free field  $\theta(x)$  with mass  $m_K$  and to take

$$\mathcal{L}_w(x) = \frac{1}{2} g \theta(x) \varphi(x)^2 \quad (6.3)$$

as the weak-interaction lagrangian. One then first has to expand the transition amplitude (2.2) in powers of  $\lambda$ , but we shall not discuss this here since the calculation is completely standard. The way to obtain the perturbation expansion of the finite-volume matrix element (4.3) may be less obvious, however, and we thus proceed to explain this in some detail.

## 6.2 Two-pion states

In finite volume the low-lying two-pion energy eigenstates with zero total momentum and trivial transformation behaviour under the cubic group may be labelled by an integer  $n = 0, 1, 2, \dots$  such that the associated energies  $W_n$  increase monotonically with  $n$ . We denote these states by  $|\pi\pi n\rangle$  and assume that they have unit norm.

To lowest order in  $\lambda$ , the energy values are determined through the free energy-momentum relation and the relative momentum of the pions. Since we only consider cubically invariant states, any two momenta that are related to each other by a cubic transformation describe the same state. For  $n \leq 6$  the momenta in the set

$$\Omega_n = \{ \mathbf{k} = 2\pi\mathbf{z}/L \mid \mathbf{z} \in \mathbb{Z}^3, \mathbf{z}^2 = n \} \quad (6.4)$$

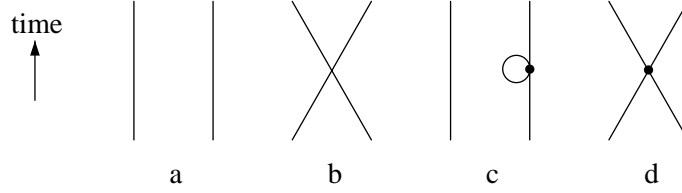


Fig. 3. Feynman diagrams contributing to the correlation function (6.7). The lines represent the free pion propagator in the time-momentum representation (6.8) and the filled circles the self-interaction vertex. All external lines end at times  $x_0$  or  $y_0$ .

are all equivalent in this sense. The corresponding state is thus non-degenerate and one concludes from this that

$$W_n = 2\sqrt{m^2 + n(2\pi/L)^2} + \mathcal{O}(\lambda), \quad 0 \leq n \leq 6. \quad (6.5)$$

In the following, our attention will be restricted to these levels.

The corresponding energy eigenstates  $|\pi\pi n\rangle$  can be created from the vacuum by applying the operators

$$\mathcal{O}_n(x_0) = \sum_{\mathbf{k} \in \Omega_n} \int_0^L d^3x d^3y e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \varphi(x_0, \mathbf{x}) \varphi(x_0, \mathbf{y}). \quad (6.6)$$

Note that  $\mathcal{O}_n(x_0)$  couples to all two-pion states in the given sector, since there are no quantum numbers that would forbid this. In euclidean space and at large time separations  $x_0 - y_0$ , its connected two-point function is thus given by

$$\langle \mathcal{O}_n(x_0) \mathcal{O}_n(y_0) \rangle_{\text{con}} = \sum_{l=0}^6 |\langle 0 | \mathcal{O}_n(0) | \pi\pi l \rangle|^2 e^{-W_l(x_0 - y_0)} + \dots, \quad (6.7)$$

where the ellipses stand for more rapidly decaying terms.

The perturbation expansion of the two-pion energy  $W_n$  and the associated matrix element  $|\langle 0 | \mathcal{O}_n(0) | \pi\pi n \rangle|$  may now be obtained by expanding the left-hand side of eq. (6.7) in Feynman diagrams in the standard way. If one uses the time-momentum representation

$$\int_0^L d^3x e^{-i\mathbf{p}\mathbf{x}} \langle \varphi(x) \varphi(0) \rangle = \frac{e^{-\omega_{\mathbf{p}}|x_0|}}{2\omega_{\mathbf{p}}} - (m \leftrightarrow \Lambda), \quad \omega_{\mathbf{p}} \equiv \sqrt{m^2 + \mathbf{p}^2}, \quad (6.8)$$

for the tree-level pion propagator, the diagrams evaluate to a sum of exponentials. The desired expansions can then be read off from the coefficients of the exponential factor that corresponds to the  $n$ -th level.

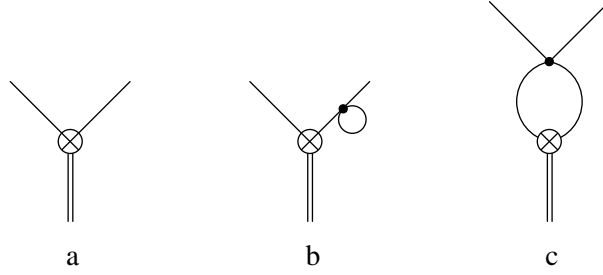


Fig. 4. Diagrams contributing to the correlation function (6.10). The double line represents the kaon propagator and the circled cross the weak interaction vertex at the origin. All other graphical elements are as in fig. 3.

To leading order the diagrams a and b in fig. 3 yield the expected expression (6.5) for the two-pion energy and

$$|\langle 0 | \mathcal{O}_n(0) | \pi\pi n \rangle| = \sqrt{2\nu_n} L^3 / W_n + \mathcal{O}(\lambda) \quad (6.9)$$

for the matrix element [cf. eq. (4.8)]. At the next order in the coupling, there are two types of diagrams. Diagram c and three further diagrams of this kind amount to an additive renormalization of the pion mass by a term that is independent of  $L$  up to exponentially small corrections [?]. Such contributions are neglected here and the renormalization is thus equivalent to replacing  $m$  by  $m_\pi$  in the tree-level expressions. One is then left with the diagram d, which can be worked out analytically in a few lines.

### 6.3 Transition matrix element

The finite-volume transition matrix element (4.3) can be computed by studying the euclidean correlation function

$$\int_0^L d^3y \langle \mathcal{O}_n(x_0) \mathcal{L}_w(0) \theta(y) \rangle_{\text{con}} = \sum_{l=0}^6 e^{-W_l x_0 + m_\kappa y_0} \langle 0 | \mathcal{O}_n(0) | \pi\pi l \rangle \langle \pi\pi l | H_w | K \rangle \langle K | \theta(0) | 0 \rangle + \dots \quad (6.10)$$

at large  $x_0$  and large negative  $y_0$ . As in the case of the two-pion states, the terms we are interested in are found by looking for the appropriate exponential factor.

To lowest order diagram a in fig. 4 yields

$$|\langle \pi \pi n | H_w | K \rangle| = \frac{g\sqrt{\nu_n}}{2W_n\sqrt{m_K}L^3} \{1 + O(\lambda)\}. \quad (6.11)$$

The pion mass in this expression is renormalized by the tadpole insertions at the next order (diagram b and its mirror image). Diagram c, the only other diagram at this order, may be evaluated by inserting the time-momentum representation for the external and also the internal lines. Apart from various simple terms, one then ends up with the momentum sum

$$S_n = L^{-3} \sum_{\mathbf{p} \notin \Omega_n} \left\{ \frac{1}{\omega_{\mathbf{p}}(\mathbf{p}^2 - \mathbf{k}^2)} - R_\Lambda(\mathbf{p}^2, \mathbf{k}^2) \right\}, \quad \mathbf{k} \in \Omega_n, \quad (6.12)$$

where  $R_\Lambda$  is an expression that arises from the Pauli-Villars regularization.

A general summation formula proved in ref. [?] allows one to compute such sums up to terms that vanish more rapidly than any power of  $1/L$ . The precise form of  $R_\Lambda$  is not important for this. One only needs to know that it is a smooth function of  $\mathbf{p}$  and  $\mathbf{k}$  and that it makes the sum absolutely convergent. The result

$$S_n = \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2\omega_{\mathbf{p}}} \left[ \frac{1}{\mathbf{p}^2 - \mathbf{k}^2 + i\epsilon} + \frac{1}{\mathbf{p}^2 - \mathbf{k}^2 - i\epsilon} \right] - R_\Lambda(\mathbf{p}^2, \mathbf{k}^2) \right\} \\ + \frac{z_n}{4\pi^2\omega_{\mathbf{k}}L} + \frac{\nu_n}{2(\omega_{\mathbf{k}}L)^3} + \frac{\nu_n}{L^3} R_\Lambda(\mathbf{k}^2, \mathbf{k}^2) \quad (6.13)$$

is then obtained, with the constant  $z_n$  given by

$$z_n = \lim_{q^2 \rightarrow n} \left\{ \sqrt{4\pi} \mathcal{Z}_{00}(1; q^2) + \frac{\nu_n}{q^2 - n} \right\} \quad (6.14)$$

(the zeta function  $\mathcal{Z}_{00}(s; q^2)$  is defined in appendix B).

#### 6.4 Final steps

To check eq. (4.5) one has to tune the box size so that  $W_n = m_K$  for a specified level number  $n$ . This condition determines  $L$  order by order in the coupling. The perturbation expansion of the right-hand side of eq. (4.5) is then obtained by inserting this series in the proportionality factor and the perturbative expressions for the matrix element  $|\langle \pi \pi n | H_w | K \rangle|$ .

To lowest order, the box size is given by eq. (4.6) and the function  $\phi'(q)$  in the proportionality factor is thus to be expanded around  $q = \sqrt{n}$ . This generates a term proportional to  $z_n$ , which cancels the corresponding term in eq. (6.13). The integral in this equation matches with the contribution to the transition amplitude  $A$  of the infinite-volume diagram with the topology of diagram c. All other terms that occur at first order in the coupling cancel and one finds that eq. (4.5) holds as expected.

## 7. Application to the physical kaon decays

Compared to the generic theory considered so far, the situation in the case of the physical kaon decays is complicated by the fact that there are several decay channels. To a first approximation we may however assume that isospin is an exact symmetry in the absence of the weak interactions. The decay channels can then be separated from each other by passing to a basis of states with definite quantum numbers.

As an example we discuss the CP-conserving decays of the neutral kaon into two-pion states with isospin 0 and 2. The corresponding decay amplitudes,  $A_0$  and  $A_2$ , are related to the physical transition matrix elements through

$$T(K_S^0 \rightarrow \pi^+\pi^-) = \frac{2}{\sqrt{6}} A_0 e^{i\delta_0^0} + \frac{1}{\sqrt{3}} A_2 e^{i\delta_0^2}, \quad (7.1)$$

$$T(K_S^0 \rightarrow \pi^0\pi^0) = -\frac{2}{\sqrt{6}} A_0 e^{i\delta_0^0} + \frac{2}{\sqrt{3}} A_2 e^{i\delta_0^2}. \quad (7.2)$$

In these equations  $\delta_0^I$  denotes the  $S$ -wave pion scattering phase in the channel with isospin  $I$  and the normalization and phase conventions are as in sect. 2.

In the sector of two-pion states with isospin  $I$ , zero electric charge, zero total momentum and trivial transformation behaviour under cubic rotations and reflections, the energy spectrum in finite volume is determined by the equations that we have previously discussed, with  $\delta_0$  replaced by  $\delta_0^I$ . At the points where one of these energy levels passes through  $m_K$ , we define the associated transition matrix element

$$M_I = \langle (\pi\pi)_I | H_w | K^0 \rangle, \quad (7.3)$$

where it is understood that the states are normalized to unity and that  $H_w$  is the CP-conserving part of the effective weak hamiltonian. With these conventions, the

Table 1. Calculation of the proportionality factor in eq. (7.4) at the first level crossing

$I$	$L$ [fm]	$q$	$q\partial\phi/\partial q$	$k\partial\delta_0^I/\partial k$
0	5.34	0.89	4.70	1.12
2	6.09	1.02	6.93	-0.09

physical amplitudes are given by

$$|A_I|^2 = 8\pi \left\{ q \frac{\partial\phi}{\partial q} + k \frac{\partial\delta_0^I}{\partial k} \right\}_{k=k_\pi} \left( \frac{m_K}{k_\pi} \right)^3 |M_I|^2. \quad (7.4)$$

Note that  $A_0$  and  $A_2$  are real and only their relative sign is observable. Up to this sign, the complete information can thus be retrieved from the matrix elements and the energy spectrum in finite volume.

For illustration, let us suppose that the scattering phases  $\delta_0^I$  are accurately described by the one-loop formulae of chiral perturbation theory [?–?]. The two-pion energy spectrum in the subspaces with isospin  $I$  and the box sizes  $L$ , where the next-to-lowest levels in these sectors (the ones with level number  $n = 1$ ) coincide with the kaon mass, can then be calculated. After that the proportionality factor in eq. (7.4) is easily evaluated (table 1) and one ends up with

$$|A_0| = 44.9 \times |M_0|, \quad (7.5)$$

$$|A_2| = 48.7 \times |M_2|, \quad (7.6)$$

$$|A_0/A_2| = 0.92 \times |M_0/M_2|. \quad (7.7)$$

As can be seen from these figures, the large difference between the scattering phases in the two isospin channels (about  $45^\circ$  at  $k = k_\pi$ ) does not lead to a big variation in the proportionality factors. In fact, if we set the scattering phases to zero altogether, eqs. (4.6)–(4.8) give  $|A_I| = 47.7 \times |M_I|$  for  $n = 1$ , which is not far from the results quoted above.

The proportionality factor in eq. (7.4) thus appears to be only weakly dependent on the final-state interactions. In particular, if the theory is to reproduce the  $\Delta I = 1/2$  enhancement, the large factor has to come from the ratio of the finite-volume matrix elements  $M_I$ .

## 8. Concluding remarks

Finite-volume techniques have been used in lattice field theory for many years and have long proved to be a most effective tool. It may well be that weak transition matrix elements are also best approached in this way. For two-body decays a concrete proposition along this line has been made here, which is conceptually satisfactory and which we believe has a fair chance to work out in practice.

In the case of the physical kaon decays, the proportionality factor relating the transition matrix elements in finite and infinite volume turned out to be nearly the same in the two isospin channels. This may be surprising at first sight, since the interactions of the pions in the isospin 0 state are much stronger than in the isospin 2 state. One should, however, take into account the fact that the comparison is made at box sizes  $L$  greater than 5 fm. It is hence quite plausible that the finite-volume matrix elements already include most of the final-state interaction effects (such as the ones recently discussed in refs. [?–?]). Apart from a purely kinematical factor, an only small correction is then required to pass to the matrix elements in infinite volume.

Since the unitarity of the underlying field theory has been essential for our argumentation, it is not obvious that eq. (4.5) holds in quenched QCD. As usual, however, one expects to be safe from the deficits of the quenched approximation when the quark masses are not too small and our results should then be applicable. An investigation of the problem in quenched chiral perturbation theory, following refs. [?,?], may be worth while at this point to find out where precisely the unphysical effects set in.

As a final comment we note that the ideas developed in this paper may also be applied to baryon decays, such as  $\Lambda \rightarrow N\pi$ ,  $\Sigma \rightarrow N\pi$  and  $\Xi \rightarrow \Lambda\pi$ , as well as to any other decay where the particles in the final state scatter only elastically. Depending on the kinematical details, the relation between the finite and infinite volume transition matrix elements may, however, assume a slightly different form.



## Appendix A

The components of four-vectors in real and euclidean space are labelled by an index running from 0 to 3. Bold-face types denote the spatial parts of the corresponding four-vectors and scalar products are always taken with euclidean metric, except for Lorentz vectors in real space where  $xy = x_0y_0 - \mathbf{x}\mathbf{y}$ .

States  $|p\rangle$  in infinite volume describing a spinless particle, with mass  $m$  and four-momentum

$$p = (p_0, \mathbf{p}), \quad p_0 = \sqrt{m^2 + \mathbf{p}^2} > 0, \quad (\text{A.1})$$

are normalized in such a way that

$$\langle p | p' \rangle = 2p_0 (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}'). \quad (\text{A.2})$$

Particle states in finite volume are always normalized to unity.

In the centre-of-mass frame, the elastic scattering amplitude of two spinless particles of mass  $m$  may be expanded in partial waves according to

$$T = 16\pi W \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) t_l(k), \quad W = 2\sqrt{m^2 + k^2}, \quad (\text{A.3})$$

where  $W$  denotes the total energy of the particles,  $\theta$  the scattering angle and  $P_l(z)$  the Legendre polynomials [?]. Below the inelastic threshold, unitarity implies

$$t_l = \frac{1}{2ik} (e^{2i\delta_l} - 1), \quad (\text{A.4})$$

with  $\delta_l$  the (real) scattering phase for angular momentum  $l$ .

## Appendix B

For all  $q \geq 0$  the angle  $\phi(q)$  is determined through

$$\tan \phi(q) = -\frac{\pi^{3/2}q}{\mathcal{Z}_{00}(1; q^2)}, \quad \phi(0) = 0, \quad (\text{B.1})$$

and the requirement that it depends continuously on  $q$ . The zeta function in this equation is defined by

$$\mathcal{Z}_{00}(s; q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\mathbf{n} \in \mathbb{Z}^3} (\mathbf{n}^2 - q^2)^{-s} \quad (\text{B.2})$$

if  $\text{Re } s > \frac{3}{2}$  and elsewhere through analytic continuation.

Numerical methods to compute the zeta function are described in ref. [?] and a table of values of  $\phi(q)$  is included in ref. [?]. The source code of a set of ANSI C programs for these functions can be obtained from the authors.

## References

- [1] C. Dawson, G. Martinelli, G. C. Rossi, C. T. Sachrajda, S. Sharpe, M. Talevi, M. Testa, Nucl. Phys. B514 (1998) 313
- [2] L. Maiani, M. Testa, Phys. Lett. B245 (1990) 585
- [3] M. Ciuchini, E. Franco, G. Martinelli, L. Silvestrini, Phys. Lett. B380 (1996) 353
- [4] I. Montvay, P. Weisz, Nucl. Phys. B290 [FS20] (1987) 327
- [5] Ch. Frick, K. Jansen, J. Jersák, I. Montvay, G. Münster, P. Seufferling, Nucl. Phys. B331 (1990) 515
- [6] M. Lüscher, U. Wolff, Nucl. Phys. B339 (1990) 222
- [7] M. Guagnelli, E. Marinari, G. Parisi, Phys. Lett. B240 (1990) 188
- [8] C. R. Gatttringer, C. B. Lang, Nucl. Phys. B391 (1993) 463
- [9] R. Gupta, A. Patel, S. Sharpe, Phys. Rev. D48 (1993) 388
- [10] H. R. Fiebig, A. Dominguez, R. M. Woloshyn, Nucl. Phys. B418 (1994) 649
- [11] M. Göckeler, H. A. Kastrup, J. Westphalen, F. Zimmermann, Nucl. Phys. B425 (1994) 413
- [12] M. Fukugita, Y. Kuramashi, M. Okawa, H. Mino, A. Ukawa, Phys. Rev. D52 (1995) 3003
- [13] S. Aoki et al. (JLQCD Collab.), Phys. Rev. D58 (1998) 054503
- [14] C. Gutsfeld, H. A. Kastrup, K. Stergios, Nucl. Phys. B560 (1999) 431

- [15] M. Lüscher, Commun. Math. Phys. 104 (1986) 177
- [16] M. Lüscher, Commun. Math. Phys. 105 (1986) 153
- [17] M. Lüscher, Nucl. Phys. B354 (1991) 531
- [18] M. Lüscher, Nucl. Phys. B364 (1991) 237
- [19] K. Rummukainen, S. Gottlieb, Nucl. Phys. B450 (1995) 397
- [20] J. Gasser, H. Leutwyler, Phys. Lett. B125 (1983) 325; Ann. Phys. (NY) 158 (1984) 142; Nucl. Phys. B250 (1985) 465
- [21] J. Gasser, U.-G. Meissner, Phys. Lett. B258 (1991) 219
- [22] M. Knecht, B. Moussallam, J. Stern, N. H. Fuchs, Nucl. Phys. B457 (1995) 513
- [23] E. Pallante, A. Pich, Strong enhancement of  $\epsilon'/\epsilon$  through final state interactions, hep-ph/9911233
- [24] E. A. Paschos, Rescattering effects for  $\epsilon'/\epsilon$ , hep-ph/9912230
- [25] A. J. Buras, M. Ciuchini, E. Franco, G. Isidori, G. Martinelli, L. Silvestrini, Final state interactions and  $\epsilon'/\epsilon$ : a critical look, hep-ph/0002116
- [26] C. W. Bernard, M. F. L. Golterman, Phys. Rev. D53 (1996) 476
- [27] M. F. L. Golterman, K. C. Leung, Phys. Rev. D56 (1997) 2950; *ibid.* D57 (1998) 5703; *ibid.* D58 (1998) 097503
- [28] I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals, Series and Products (Academic Press, New York, 1965)