Single-Particle Spin-Orbit Strengths of the Nucleon and Hyperons by $SU_6$ Quark-Model

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The quark-model hyperon-nucleon interaction suggests an important antisymmetric spin-orbit component. It is generated from a color analogue of the Fermi-Breit interaction dominating in the one-gluon exchange process between quarks. We discuss the strength $S_B$ of the single-particle spin-orbit potential, following the Scheerbaum's prescription. Using the $SU_6$ quark-model baryon-baryon interaction which was recently developed by the Kyoto-Niigata group, we calculate $NN$, $ΛN$ and $ΣN$ G-matrices in symmetric nuclear matter and apply them to estimate the strength $S_B$. The ratio of $S_B$ to the nucleon strength $S_N \sim -40$ MeV·fm$^5$ is $S_Λ/S_N \sim 1/5$ and $S_Σ/S_N \sim 1/2$ in the Born approximation. The $G$-matrix calculation of the model FSS modifies $S_Λ$ to $S_Λ/S_N \sim 1/12$. For $S_N$ and $S_Σ$, the effect of the short-range correlation is comparatively weak against meson-exchange potentials with a short-range repulsive core. The significant reduction of the $Λ$ single-particle potential arises from the combined effect of the antisymmetric $LS$ force, the flavor-symmetry breaking originating from the strange to up-down quark-mass difference, as well as the effect of the short-range correlation. The density dependence of $S_B$ is also examined.

Key words: $YN$ interaction, quark model, $G$-matrix, hyperon single-particle potential, spin-orbit interaction

1 Introduction

Though the quantum chromodynamics (QCD) is believed to be the fundamental theory of the strong interaction, it is still too difficult to apply the QCD
directly to two-baryon systems. At this stage a number of effective models have been proposed to understand the nucleon-nucleon (NN) and hyperon-nucleon (YN) interactions from basic elements of quarks and gluons [1]. Among them the non-relativistic quark model has a unique feature that enables us to take full account of the dynamical motion of the two composite baryons within a framework of the resonating-group method (RGM) [2]. The model describes confinement with a phenomenological potential and uses the quark-quark (qq) residual interaction consisting of a color analogue of the Fermi-Breit (FB) interaction. In the last several years, it was found that such a naive model does not produce medium- and long-range interactions, but can give a realistic description of the NN and YN interactions if meson-exchange effects are properly taken into account in the model.

A simultaneous description of the NN and YN interactions has recently been achieved by two groups. One is the SU₆ quark model, RGM-F [3,4], FSS [5–7] and RGM-H [6,7], by the Kyoto-Niigata group,¹ and the other is the SU₃-chiral symmetry quark model [8–10] by the Beijing-Tübingen group. In these models, the spin-flavor SU₆ or chiral-symmetric effective meson-exchange potentials (EMEP) generated from scalar and pseudo-scalar meson exchanges between quarks are incorporated. It was found that the flavor-nonet scalar mesons play an important role in describing the NN and YN interactions in a single framework with a unique set of model parameters. We stress that a simultaneous and realistic description of the NN and YN interactions is very important, since the experimental data for the YN interaction are at present very limited, and thus one has to rely on the theoretical consistency of the framework in order to make best use of the rich experimental information on the NN interaction.

One of the features of the quark-model description for the NN and YN interactions is that the antisymmetric LS force (LS(−) force) originating from the FB spin-orbit interaction is considerably strong in the strangeness S = −1 and the isospin I = 1/2 channel [11–13]. Since the signs of the ordinary LS force and the LS(−) force are opposite in the ΛN interaction, this strong LS(−) force is vital to produce very small spin-orbit (ℓs) splitting for the Λ single-particle (s.p.) states. This is consistent with the early experimental observation that the s.p. spin-orbit term in 40 ≥ A ≥ 12 nuclei is almost zero from the analysis of the recoilless (K⁻, π⁻) reaction. [14] More recently, preliminary results of the γ-ray spectroscopy for ⁹ΛBe and ¹³ΛC hypernuclei

¹ Difference of these three models lies only in how to deal with the spin-flavor (-color) factors of the quark-exchange kernel in EMEP. In FSS and most of RGM-H these factors are exactly calculated, while in RGM-F they are approximated to be proportional to those of the exchange normalization kernel. RGM-H uses the latter approximation partly: i.e., solely for the isoscalar-type scalar-meson (ε and S*) exchanges.
seem to indicate very small $\ell s$ splitting in these nuclei. [15] In view of the recent progress of experimental techniques, a quantitative analysis of s.p. $\ell s$ potentials appears important. The purpose of this paper is to extend the Scheerbaum’s formulation [16] for the nucleon s.p. $\ell s$ potentials to hyperons interacting with nucleons via the non-local interaction, and to examine in detail the s.p. $\ell s$ strengths of $N$, $\Lambda$ and $\Sigma$, first in the Born approximation of the quark-exchange kernel, and then in the $G$-matrix calculation for our realistic quark-model $NN$ and $YN$ interactions.

Since the spin-orbit interaction between baryons is essentially short-ranged, a number of authors have paid attention to the FB $LS$ force, trying to understand its microscopic origin from the quark degree of freedom. Here we briefly review some typical investigations, in which the spin-orbit forces of the $NN$ and $YN$ interactions are treated in the $(3q)-(3q)$ RGM. In the WKB-RGM localization techniques of the quark-exchange kernel, Suzuki and Hecht [11] calculated $LS$ potentials, originating from the symmetric ($sLS$) and antisymmetric ($aLS$) pieces of the FB interaction. They assumed the same strange and up-down quark masses and neglected the flavor symmetry breaking (FSB). This restriction was removed in [12]. After the correction of the sign error of the original paper, they found that the $sLS$ and $aLS$ spin-orbit terms have same sign and therefore reinforce each other, giving rise to an attractive spin-orbit potential in the $^3O$ state and a repulsive potential in the $^3E$ state for the $NN$ interaction.

Morimatsu et al. [13] used only the $sLS$ piece, but took into account the effect of FSB in a simple approximation. In these studies, a main interest is naturally the $LS^{(-)}$ force which involves the simultaneous spin-flip and the flavor exchange of the hyperon and the nucleon, a typical feature of the non-identical baryon systems. The potential concept used in [13] is not based on the RGM kernel, but on the energy surface of the so-called generator-coordinate method (GCM) kernel. Using the folding procedure for the GCM kernel, they calculated, for the first time, the quark-model predictions for the s.p. $\ell s$ potentials of the nucleon and hyperons in symmetric nuclear matter. Although their absolute values of the s.p. $\ell s$ strengths are somewhat too large, they obtained the relative ratio, $U_N : U_\Lambda : U_\Sigma = 1 : 0.21 : 0.55$, which is very close to our prediction $1 : 1/5 : 1/2$ in the Born approximation given in this paper. On the other hand, He, Wang and Wong [18] compared the quark-model potentials with the Paris (for $NN$) and the Nijmegen potentials in the form of the Born amplitudes. They explicitly introduced a core radius $c$, in order to take into account the effect of the short-range correlation in the Nijmegen hard-core model D [19] and model F [20]. This procedure was also adopted by the Jülich group to show the relative strength of the $LS$ and $LS^{(-)}$ forces in their one-boson exchange potential (OBEP) model [21]. Through all of these studies, it is now generally recognized

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2 Here we follow the notation given in Eq. (5.2) of [23].

that the quark-exchange kernel from the FB interaction leads to the spin and flavor dependence which is qualitatively very similar to that of OBEP, thus yielding a possible alternative to the meson-exchange description of the $LS$ interaction by vector and scalar mesons.  

Strictly speaking, the $LS$ force cannot be discussed independently of other pieces of interaction. Apparently the $LS$ force is influenced by the description of the short-range correlation which is different between the meson-exchange model and the quark model. The $LS$ force also depends on how to derive the s.p. $\ell s$ potentials in the finite nuclei from the original $NN$ and $YN$ interactions in the free space. In fact, the first issue is the major motivation for any realistic quark models for the $NN$ and $YN$ interactions. For example, Yang et al. [10] discussed the difference of the one-gluon exchange (OGE) process and the scalar-meson nonet exchange (OSE) introduced between quarks in the framework of the chiral $SU_3$ quark model. Since their $LS$ force is too weak in the $NN$ sector because of several reasons, they reinforced the $sLS$ term of OGE by a factor of 3.1 and that of OSE by a factor of 4.8. Through this prescription, they argued that a sizable OGE component, which would definitely result in a quite strong $LS^{(-)}$ force in the $I = 1/2$ channel, is not favorable, since it leads to an unphysical resonance in the $\Lambda N$ channel. This is more or less a correct statement as long as the $LS$ components of the FB interaction is concerned. However, our result in the model RGM-H [6,7] implies that there exists a solution which reproduces the necessary $LS$ force in the $NN$ interaction without introducing any enhancement factor, and still reproduces the observed $\Lambda p$, $\Sigma^+ p$ and $\Sigma^- p$ differential cross sections reasonably well. The main difference between the two models lies in the choice of the harmonic oscillator constant $b$ of the $(3q)$ clusters and the magnitude of the quark-gluon coupling constant $\alpha_S$. The Beijing, Tübingen and Salamanca groups use $b \sim 0.5$ fm$^{-1}$ and $\alpha_S \sim 0.5$, while RGM-F, FSS and RGM-H use $b \sim 0.6$ fm$^{-1}$ and $\alpha_S \sim 2$. Since the $LS$ force is short-ranged, it is very sensitive to the magnitude of the size parameter $b$. It is sometimes claimed that our $\alpha_S$ is too big, compared with the QCD coupling constant, and is contradictory to the experimental fact that there seems no spin-orbit splitting existing in the negative-parity excited states of baryons (especially, the nucleon and $\Delta$). We should, however, keep in mind that our $\alpha_S$ is merely a model parameter in the

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4 There exists, however, appreciable quantitative difference between predictions by various versions of OBEP and the quark model. For example, the Nijmegen potentials generally predict a rather small $LS^{(-)}$ force, compared with that of the quark model.

5 The model RGM-H (nor the other versions, RGM-F and FSS) does not include the $LS$ component from the scalar-meson exchange, while it is included in the chiral $SU_3$ quark model. However, the incorporation of even more sophisticated EMEP involving vector mesons does not change this situation. A new version of our quark model in this direction will be published elsewhere.
nonperturbed region, which has very little to do with the real QCD coupling constant. The explicit value is determined from the best fit to the experimental data in the present framework. The second point is the so-called “missing \(LS\) force” problem in the \(P\)-wave baryons. Fujiwara [22] has shown that the seemingly small spin-orbit splitting of the \(P\)-wave baryon spectrum can be explained by the dispersive effect due to the resonance nature of these \(P\)-wave baryons embedded in the baryon-meson continua. In other words, the missing \(LS\) force problem of the \(P\)-wave baryons does not necessarily indicate that the FB interaction is inappropriate as a residual interaction of the non-relativistic quark model.

In this paper, we apply the Scheerbaum’s discussion [16] for the strength of the nucleon s.p. \(\ell s\) potential to the formulation of the quark-model invariant amplitudes developed in [23]. The strength factor \(S_B\) for the hyperon s.p. \(\ell s\) potential in the Thomas form is explicitly derived. Two different kinds of approaches are attempted in the Born approximation. One is to use the Wigner transform at \(p = 0\) in the WKB-RGM formalism as an effective local potential in low-energy processes, and the other is the \(P\)-wave approximation for the dominant contribution to the \(LS\) invariant amplitudes. Both methods involve some kind of averaging procedure for the spatial integrals and leave one momentum as an input parameter. This momentum dependence, however, is generally very weak. One can thus adopt the zero-momentum limit, in which these two methods give the same result, yielding very simple expressions for \(S_B\). We consider spin-saturated (s.s.) finite nuclei or symmetric nuclear matter. The most reliable description of \(S_B\) is therefore formulated through the nuclear-matter approximation of the \(G\)-matrix invariant amplitudes.

We present in Section 2 basic formulae to calculate \(S_B\). After introducing two kinds of approximations to the spatial integrals for the \(LS\) Born amplitudes in Subsection 2.2, a method of \(G\)-matrix calculation is discussed in Subsection 2.3. In Section 3 we give analytic expressions of \(S_B\) in the simplest Born approximation, and use them to examine the characteristic structure of the s.p. \(\ell s\) potentials. The \(G\)-matrices calculated in nuclear matter are used to obtain a more realistic estimate for \(S_B\). The strength \(S_\Lambda\) turns out to be very small because of the cancellation between \(LS\) and \(LS^{(-)}\) components. The short-range correlation is found to further reduce \(S_\Lambda\) to be less than \(1/10)S_N\). Section 4 is devoted to a summary.
2 Formulation

2.1 Strengths of hyperon single-particle spin-orbit potentials

We start from the RGM equation for the $(3\bar{q})-(3\bar{q})$ system [3,6]:

$$
\left[ \varepsilon_{\alpha} + \frac{\hbar^2}{2\mu_{\alpha}} \left( \frac{\partial}{\partial R} \right)^2 \right] \chi_{\alpha}(\mathbf{R}) = \sum_{\alpha'} \int d\mathbf{R}' \ G_{\alpha\alpha'}(\mathbf{R}, \mathbf{R}'; E) \ \chi_{\alpha'}(\mathbf{R}') , \quad (1)
$$

where the $G_{\alpha\alpha'}(\mathbf{R}, \mathbf{R}'; E)$ is composed of various pieces of the interaction kernels as well as the direct potentials of EMEP. The subscript $\alpha$ stands for a set of quantum numbers of the channel wave function; $\alpha = [1/2(11)a_1, 1/2(11)a_2]$ $SSY_{1}\Sigma; \mathcal{P}$, where $1/2(11)a$ is the spin and $SU_3$ quantum number in the Elliott notation ($\lambda\mu$), $a = YI$ the flavor label of the octet baryons ($N = 1(1/2), \Lambda = 00, \Sigma = 01$ and $\Xi = -1(1/2)$), and $\mathcal{P}$ is the flavor-exchange phase. In the $NN$ system with $a_1a_2 = NN$, $\mathcal{P}$ is actually redundant since $\mathcal{P} = (-1)^{I-I}$. The relative energy $\varepsilon_{\alpha}$ in the channel $\alpha$ is related to the total energy $E$ of the system through $\varepsilon_{\alpha} = E - E_{a_1} - E_{a_2}$. According to [23], we introduce the basic Born kernel of Eq. (1) through

$$
M_{\alpha\alpha'}(\mathbf{q}_f, \mathbf{q}_i; E) = \langle e^{i\mathbf{q}_f \cdot \mathbf{R}} | G_{\alpha\alpha'}(\mathbf{R}, \mathbf{R}'; E) | e^{i\mathbf{q}_i \cdot \mathbf{R}'} \rangle = \langle e^{i\mathbf{q}_f \cdot \mathbf{R}} \eta_{a}^{SF} | G(\mathbf{R}, \mathbf{R}'; E) | e^{i\mathbf{q}_i \cdot \mathbf{R}'} \eta_{a'}^{SF} \rangle , \quad (2)
$$

where $\eta_{a}^{SF}$ is the spin-flavor wave function at the baryon level, defined in Eq. (2.9) of [23].

In the following we restrict ourselves to the spin-saturated (s.s.) nuclei and apply the Scheerbaum’s prescription for the s.p. spin-orbit strengths, first to the quark-exchange kernel $G_{\alpha\alpha'}(\mathbf{R}, \mathbf{R}'; E)$, secondly to the $G$ matrices obtained by solving the corresponding Bethe-Goldstone equation [25]. We call the first prescription the Born approximation, and the second one a realistic calculation. Suppose $G$ is the quark-exchange kernel $G(\mathbf{R}, \mathbf{R}'; E)$ or the $G$-matrix. We calculate s.p. energy

$$
E_{v}^{s.s.} = \sum_{c} \langle vc | G | vc - cv \rangle , \quad (3)
$$

for the spin-orbit interaction. The two-particle interaction $G$ is assumed to be expressed as

$$
\langle \mathbf{k}_1\mathbf{k}_2 | G | \mathbf{k}_1'\mathbf{k}_2' \rangle = \delta(\mathbf{K}_{12} - \mathbf{K}_{12}') \langle \mathbf{k}_1 | G | \mathbf{k}_1' \rangle
$$
\[
\delta(K_{12} - K'_{12}) \frac{1}{(2\pi)^3} M(k_{12}, k'_{12}) ,
\]

where \( K_{12} = k_1 + k_2 \) and \( k_{12} = (\xi k_1 - k_2)/(1 + \xi) \) with \( \xi = (M_2/M_1) \) are the center-of-mass and relative momenta, respectively. In the case of the \( G \)-matrix, \( M(k_{12}, k'_{12}) \) may depend on \( (K_{12})^2 \) and the starting energy as well.

It is convenient to use the invariant kernel \( M(\Omega) \), by which the Born kernel Eq. (2) is expressed as

\[
M(k_{12}, k'_{12}) = \sum_{\Omega} M^{\Omega}(k_{12}, k'_{12}) \mathcal{O}^{\Omega}(k_{12}, k'_{12}) .
\]

Here we only consider \( \Omega = LS, LS^(-) \) and \( LS^(-)\sigma \) components [23], which are represented by the Pauli-spinor invariants

\[
\mathcal{O}^{LS} = i n \cdot S , \quad \mathcal{O}^{LS^-} = i n \cdot S^(-) , \quad \mathcal{O}^{LS^-\sigma} = i n \cdot S^(-) P_\sigma ,
\]

with \( n = [k'_{12} \times k_{12}] \), \( S = \frac{1}{2}(\sigma_1 + \sigma_2) \), \( S^(-) = \frac{1}{2}(\sigma_1 - \sigma_2) \), and \( P_\sigma = \frac{1 + \sigma_1 \cdot \sigma_2}{2} \).

The invariant kernel \( M^{\Omega}(k_{12}, k'_{12}) \) in Eq. (5) consists of various types of spin-flavor factors \( X_T^{\Omega} \) and the spatial functions \( f_T^\Omega(\theta) \) calculated for the quark-exchange kernel of the FB interaction. These are explicitly given in [23] and Appendix A. When the contribution from the exchange Feynman diagram is incorporated with the exchange operator \( P_\sigma P_F P_r \), the total Born kernel is expanded as

\[
M(k_{12}, k'_{12}) - M(k_{12}, -k'_{12}) P_\sigma P_F
\]

\[
= \sum_{\Omega} M^{\Omega \text{ total}}(k_{12}, k'_{12}) \mathcal{O}^{\Omega}(k_{12}, k'_{12}) ,
\]

with the matrix element in the isospin basis

\[
\langle [BN]_{II} | M(k_{12}, k'_{12}) - M(k_{12}, -k'_{12}) P_\sigma P_F | [BN]_{II} \rangle = \sum_{\Omega} M^{\Omega \text{ total}}_{BB}(k_{12}, k'_{12}) \mathcal{O}^{\Omega}(k_{12}, k'_{12}) .
\]

Here the \( LS \) components \( M^{\Omega \text{ total}}_{BB}(k_{12}, k'_{12}) \) for the \( NN \) and \( YN \) systems are explicitly given by

\[
M^{LS \text{ total}}_{NN}(k_{12}, k'_{12}) = \sum_T (X_T^{LS})_{NN} \left[ f_T^{LS}(\theta) - (-1)^T f_T^{LS}(\pi - \theta) \right] ,
\]

7
\[ M_{YY}^{LS \text{ total}}(k_{12}, k'_{12}) = \sum_{\mathcal{T}} \left[ (X_T^{LS})^{ud}_{YY} f_\mathcal{T}^{LS}(\theta) + (X_T^{LS})^{ud}_{YY} f_\mathcal{T}^{LS}(\pi - \theta) \right] , \]
\[ M_{YY}^{LS(\sigma) \text{ total}}(k_{12}, k'_{12}) = \sum_{\mathcal{T}} \left[ (X_T^{LS(\sigma)})^{ud}_{YY} f_\mathcal{T}^{LS}(\theta) + (X_T^{LS(\sigma)})^{ud}_{YY} f_\mathcal{T}^{LS}(\pi - \theta) \right] , \]
\[ M_{YY}^{LS(\sigma) \sigma \text{ total}}(k_{12}, k'_{12}) = \sum_{\mathcal{T}} \left[ (X_T^{LS(\sigma)\sigma})^{ud}_{YY} f_\mathcal{T}^{LS}(\theta) + (X_T^{LS(\sigma)\sigma})^{ud}_{YY} f_\mathcal{T}^{LS}(\pi - \theta) \right] . \] 

(9)

We should note that the spin-flavor factors depend on isospin and the LS function \( f_\mathcal{T}^{LS}(\theta) \) given in Eq. (A.3) is a function of \( k_{12}^2 \) and \((k'_{12})^2\), in addition to the relative angle: \( \cos \theta = \mathbf{k}_{12} \cdot \mathbf{k}'_{12} \). The sum over \( \mathcal{T} \) in Eq. (9) is with respect to the quark-exchange interaction types \( \mathcal{T} = S, S', D_+, D_- \) [26], where the former two come from the aLS term of the FB interaction and the latter two from the sLS term (see Eq. (5.2) of [23]). For the YN system, the exchange term (i.e., \( f_\mathcal{T}^{LS(\pi - \theta)} \)) in Eq. (9) originates from the strangeness exchange process and the spin-flavor factors of the \( LS^{(-)} \) and \( LS^{(-)}\sigma \) types are interchanged between the \( LS^{(-)} \) and \( LS^{(-)}\sigma \) terms. The s.p. wave functions are expressed as

\[
\begin{align*}
\psi(k, s) &= \frac{1}{(2\pi)^3} \int d\mathbf{r} \ e^{-ikr} \psi(r, s), \\
\psi(r, s) &= \sum_{m_r m_s} \langle \ell m_r \frac{1}{2} m_s | j m \rangle \ \phi_{n\ell m_r}(r) \chi_{\frac{1}{2} m_s}(s), \\
\phi_{n\ell m_r}(r) &= R_{n\ell}(r) Y_{m_r}(\hat{r}),
\end{align*}
\]

(10)

for the valence particle and the core nucleons; \( v, c = n\ell j \). By noting

\[
\langle k_1 k_2 | BN|I_z, vc \rangle = \psi_v(k_1, s_1) \psi_c(k_2, s_2) \langle I_v I_{vz} \frac{1}{2} \tau | I I_z \rangle ,
\]

(11)

with \( \tau = 1/2 \) for \( c = p \) and \( \tau = -1/2 \) for \( c = n \) and taking a sum over \( c \) for the core protons \( (c_r = c_{1/2}) \) and neutrons \( (c_r = c_{-1/2}) \) separately, Eq. (3) becomes \( E_{v,s}^{s,s,} = \sum_{\tau} E_{v,s,\tau}^{s,s,} \) with

\[
E_{v,s,\tau}^{s,s,} = \sum_{c_r} \sum_{I} C_{v,\tau}^I(B) \frac{1}{(2\pi)^3} \sum_{k_1, k_2, k'_1, k'_2} \langle v_1 | k_1, s_1 \rangle \langle c_r | k_2, s_2 \rangle \delta(k_1 - k'_{12}) \times M_{BB}^{\Omega, \text{ total}}(k_{12}, k'_{12}) \mathcal{O}_{\Omega}(k_{12}, k'_{12}) \langle v_1, s_1 | c_r, k'_{2}, s_2 \rangle .
\]

(12)

The isospin factor defined by

\[
C_{v,\tau}^I(B) = \sum_{I_z} \langle I_v I_{vz} \frac{1}{2} \tau | I I_z \rangle^2
\]

(13)
is given in Eq. (A.1). The implicit spin sum over \( s_2 \) is easily carried out:

\[
\sum c\tau \psi^\dagger c\tau (k_2^2, s_2) = \frac{1}{2} \rho_\tau (k_2, k_2') \sigma_1 ,
\]

\[
\sum c\tau \psi^\dagger c\tau (k_2^2, s_2) S^{(-)} P_\sigma \psi_v (k_1', s_1) \psi_c (k_2^2, s_2) = 0 ,
\]

where the core-density of the protons or neutrons are defined by

\[
\rho_\tau (k_2, k_2') = 2 \sum_{(n\ell m) \in c\tau} \phi^*_{n\ell m} (k_2) \phi_{n\ell m} (k_2') .
\]

We find that the \( LS^{(-)}\sigma \) term does not contribute due to the spin-averaging.

After all, we have obtained

\[
E_{s.s.} = \sum I C_I^B \frac{1}{2(2\pi)^3} \sum_{k_1, k_2, k_1', k_2'} \psi^d_v (k_1, s_1) \delta (K_{12} - K'_{12}) \rho_\tau (k_2, k_2') \times \left[ M^{LS \text{ total}}_{BB} (k_{12}, k'_{12}) + M^{LS^{(-)} \text{ total}}_{BB} (k_{12}, k'_{12}) \right] \\
\times \left[ k'_{12} \times k_{12} \right] \cdot \sigma_1 \psi_v (k_1', s_1) .
\]

So far we have made no approximation.

We can eliminate the \( k_2' \) sum in Eq. (16) through \( k_1 + k_2 = k_1' + k_2' \); i.e., \( k_2' = k_1 + k_2 - k_1' \). If we use the momenta \( q \) and \( p \) defined by

\[
q = k_{12} - k_{12}' = k_1 - k_1' ,
\]

\[
p = k_{12}' + \frac{1}{\xi} k_{12} = k_1' - \frac{1}{\xi} k_2 ,
\]

the outer product \([k'_{12} \times k_{12}]\) in Eq. (16) can be expressed as

\[
[k'_{12} \times k_{12}] = -\frac{\xi}{1 + \xi} \left\{ [k_1 \times k_1'] + \frac{1}{\xi} [k_2 \times q] \right\} .
\]

The essential point of the Scheerbaum’s discussion [16] is that his space integrals \( D(q)/q \) and \( E(p)/p \) are very smooth functions with respect to the small values of momentum transfers \( q = |q| \) and \( p = |p| \). From this observation, he replaced the integral by a constant value \( \langle D(q)/q + E(p)/p \rangle \) evaluated at an appropriate averaged value \( p = q = \bar{q} \) and carried out the summation over
$k_1$, $k_2$ and $k'_1$ in Eq. (16). This behavior of the integral is related to the short-range character of the LS interaction and the assumption of the locality of the effective spin-orbit potential adopted there. Though the non-locality of the exchange kernel in the present case makes it difficult to follow his argument directly, we can make use of the short-range character of the LS force and assume that the amplitudes $M_{BB}^{\text{total}}(k_{12}, k'_{12})$ in Eq. (16) have a very weak $k_1$, $k_2$ and $k'_1$ dependence. As we will see later, this assumption turns out to be fairly good even in our case. Following the same procedure as developed by Scheerbaum, we can eventually arrive at the s.p. spin-orbit potential of the Thomas type:

$$E_{v,\tau}^{s.s.} = \int d\mathbf{r}_1 \psi_\tau^\dagger(\mathbf{r}_1, s_1) U_\tau(\mathbf{r}_1) \psi_\tau(\mathbf{r}_1, s_1),$$

$$U_\tau(\mathbf{r}) = K_\tau \frac{1}{r} \frac{dp_\tau(r)}{dr} \ell \cdot \sigma_1,$$  \hspace{1cm} (19)

where the proton ($\tau = 1/2$) and neutron ($\tau = -1/2$) densities are defined by

$$\rho_\tau(r) = 2 \sum_{(n\ell) \in \epsilon_\tau} \frac{4\pi}{2\ell + 1} \left[ R_{n\ell}(r) \right]^2,$$ \hspace{1cm} (20)

and the strength factor is given by

$$K_\tau = -\frac{1}{2} \frac{\xi}{1 + \xi} \sum_I C_I^\dagger(B) \times \left[ M_{BB}^{LS} \text{total}(k_{12}, k'_{12}) + M_{BB}^{LS(-)} \text{total}(k_{12}, k'_{12}) \right].$$ \hspace{1cm} (21)

For the s.s. nuclei with equal proton and neutron numbers (i.e., $Z = N$), the formulae in Eqs. (19) and (21) are further simplified into

$$U(\mathbf{r}) = -\frac{\pi}{2} S_B \frac{1}{r} \frac{d\rho(r)}{dr} \ell \cdot \sigma,$$

$$S_B = \frac{1}{2\pi} \frac{\xi}{1 + \xi} \sum_I \frac{2I + 1}{2I_B + 1}$$

$$\times \left[ M_{BB}^{LS} \text{total}(k_{12}, k'_{12}) + M_{BB}^{LS(-)} \text{total}(k_{12}, k'_{12}) \right],$$ \hspace{1cm} (22)

where $\rho(r) = \rho_n(r) + \rho_p(r)$ with $\rho_n(r) = \rho_p(r)$ is the total density and the sum formula (A.2) for $C_I^\dagger(B)$ is used. We call $S_B$ in Eq. (22) the Scheerbaum factor.
Table 1

The spin-flavor factors $X_B^T$ as a function of $\lambda = (m_s/m_{ud})$. Note that $X_B^S = X_B^T$.

$X_B^T$ values for $\lambda = 1$ are given in the second line. The last row implies off-diagonal factors for the $\Lambda N$-$\Sigma N$ coupling.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$X_B^{T\pm}$</th>
<th>$X_B^{S\pm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>$\frac{14}{9}$</td>
<td>$-\frac{10}{27}$</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>$\frac{2}{9\lambda} (2 + \frac{1}{\lambda})$</td>
<td>$-\frac{1}{9\lambda} (2 + \frac{1}{\lambda})$</td>
</tr>
<tr>
<td>$\lambda = 1$</td>
<td>$\frac{2}{3}$</td>
<td>$-\frac{1}{2}$</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>$\frac{2}{38\lambda} (106 - \frac{6}{\lambda} - \frac{1}{\lambda^2})$</td>
<td>$-\frac{1}{8\lambda} (18 - \frac{10}{\lambda} - \frac{3}{\lambda^2})$</td>
</tr>
<tr>
<td>$\lambda = 1$</td>
<td>$\frac{22}{27}$</td>
<td>$-\frac{5}{8\lambda}$</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>$-\frac{2}{9}$</td>
<td>$-\frac{1}{8\lambda} (1 + \frac{14}{\lambda} + \frac{6}{\lambda^2})$</td>
</tr>
<tr>
<td>$\lambda = 1$</td>
<td>$-\frac{2}{9}$</td>
<td>$-\frac{7}{27}$</td>
</tr>
<tr>
<td>$(\Lambda-\Sigma)$</td>
<td>$-\frac{2}{27} (7 + \frac{2}{\lambda})$</td>
<td>$-\frac{1}{8\lambda} (5 - \frac{2}{\lambda})$</td>
</tr>
<tr>
<td>$(\Sigma-\Lambda)$</td>
<td>$-\frac{2}{27}$</td>
<td>$-\frac{1}{27}$</td>
</tr>
</tbody>
</table>

2.2 Born approximation

Let us calculate the Scheerbaum factor for the s.s. symmetric nuclei in the Born approximation. For $B = N$ with $I_B = 1/2$, we have two possible isospin values $I = 0$ and $1$ in Eq. (22). Then the invariant parts of the Born kernel in Eq. (9) yield

$$S_N = \frac{1}{8\pi} \left\{ \sum_T (X_T^{LS})_{I=0}^N \left[ f_T^{LS}(\theta) - f_T^{LS}(\pi - \theta) \right] 
+ 3 \sum_T (X_T^{LS})_{I=1}^N \left[ f_T^{LS}(\theta) + f_T^{LS}(\pi - \theta) \right] \right\}. \tag{23}$$

Here $I = 0$ corresponds to the $^3E$ state and $I = 1$ to the $^3O$ state. If we assume

$$f_T^{LS}(\theta) = f_T^{LS}(\pi - \theta), \tag{24}$$

we find that only the $^3O$ state contributes to $S_N$ and obtain

$$S_N = \frac{3}{4\pi} \sum_T (X_T^{LS})_{I=1}^{1} \overline{f_T^{LS}(\theta)}. \tag{25}$$
This expression corresponds to Scheerbaum’s Eq. (3.57) [16]. Thus we have a correspondence
\[
\sum_T (X_{T,LS}^{\theta})_{NN}^I = \frac{4\pi}{q} \int_0^\infty s^3 j_1(\tilde{q}s) g^{3O}(s) \, ds .
\] (26)

Under the same assumption as Eq. (24) the Scheerbaum factors for the hyperons are given by
\[
S_B = \frac{1}{2\pi} \frac{\xi}{1 + \xi} \sum_T X_B^T \frac{f_{LS}^T(\theta)}{f^T_S(\theta)} ,
\] (27)

where \(X_B^T\) for \(B = Y\) are defined by
\[
X_B^T = \sum_I \frac{2I + 1}{2I_B + 1} \left[ (X_{T,LS}^{\theta})_{BB}^u (X_{T,LS}^{\sigma})_{BB}^d + (X_{T,LS}^{\theta})_{BB}^d (X_{T,LS}^{\sigma})_{BB}^u \right] .
\] (28)

In this notation, \(X_N^T\) is given by (\(\xi=1\))
\[
X_N^T = 3 (X_{T,LS}^{\theta})_{NN}^{I=1} = 3 (X_{T,LS}^{3O})_{NN} .
\] (29)

The spin-flavor factors \(X_B^T\) can be easily obtained from the explicit expressions of \(X_{T,LS}^{\theta}\), \(X_{T,LS}^{\sigma}\), and \(X_{T,LS}^{\sigma}\), which are given in Appendix C of [23]. They are tabulated in Table 1. When we derive these results, we should note that \(\cal{P}' = 1\) in
\[
(X_{T,LS}^{\theta})_{\cal{P},d,\cal{P}'} = (X_{T,LS}^{\theta})_{\cal{P},d}^u + (X_{T,LS}^{\sigma})_{\cal{P},d}^e \cal{P}'
\] (30)
corresponds to the \(3^O\) contribution. For \(B = \Sigma\), the isospin sum in Eq. (28) gives
\[
\text{isoscalar term} = \sum_I \frac{2I + 1}{2I_B + 1} \cdot 1 = \frac{1}{2I_B + 1} \sum_{I_B}^1 1
\]
\[
= \frac{1}{2I_B + 1} \sum_{I_B}^1 \sum_I^1 1 = 2 ,
\]
\[
\text{isovector term} = \sum_I \frac{2I + 1}{2I_B + 1} (\tau_1 \cdot \tau_2)_I = \frac{1}{2I_B + 1} \sum_{I_B}^1 \langle II_z | \tau_B \cdot \tau_N | II_z \rangle
\]
\[
\lambda = m_{ud}/m_s \text{ in Appendix C of [23] is a misprint of } \lambda = m_s/m_{ud}.
\]
<table>
<thead>
<tr>
<th>model</th>
<th>$b$ (fm)</th>
<th>$m_{ud}$ (MeV/$c^2$)</th>
<th>$\alpha_S$</th>
<th>$\lambda = m_s/m_{ud}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RGM-F</td>
<td>0.6</td>
<td>313</td>
<td>1.5187</td>
<td>1.25</td>
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<tr>
<td>FSS</td>
<td>0.616</td>
<td>360</td>
<td>2.1742</td>
<td>1.526</td>
</tr>
<tr>
<td>RGM-H</td>
<td>0.667</td>
<td>389</td>
<td>2.1680</td>
<td>1.490</td>
</tr>
</tbody>
</table>

The factor 2 for the isoscalar term is the sum over the proton and neutron, and the isovector term does not contribute since we have assumed $Z = N$ (the total isospin is zero for $\alpha$, $^{16}$O and $^{40}$Ca).

Next we discuss the averaging procedure in $f_{LS}^T(\theta)$. A possible approximation to obtain $\bar{f}_{LS}^T(\theta)$ is to use the Wigner transform $G_{W}^{LS}(r, p)$ (which is given in Eq. (2.16) of [23]) at $p = 0$, and to follow the Scheerbaum’s prescription for the local potential $G_{WS}^{LS}(r, 0)$. We can show that this procedure is equivalent to set $q = 0$ in Eq. (A.3). In this case the $\theta$-dependence in $f_{LS}^T(\theta)$ disappears ($\theta = \pi$) and $f_{WS}^{LS}(\theta)$ becomes a function of the momentum transfer $k = |k|$ ($k = q_f - q_i$). This corresponds to the Scheerbaum’s parameter $\tilde{q}$. We call this the Scheerbaum approximation.

The relationship between the basic Born kernel in Eq. (5) and the Wigner transform for the $LS$ component is given by

$$M(q_f, q_i) = \int dr \ e^{-i k \cdot r} \ G_W(r, q)$$

$$= \sum_T X_T^{LS} \int dr \ e^{-i k \cdot r} \ G_{WT}^{LS}(r, q) \ [r \times q] \cdot S,$$

where $k = q_f - q_i$ and $q = (1/2)(q_f + q_i)$. The spatial function $G_{WT}^{LS}(r, q = 0)$ becomes a function of $r = |r|$ only, and thus we can carry out the angle integral $\int d\hat{r}$ after the partial wave expansion of the plane wave. Then the component with the angular momentum $\ell = 1$ only survives, and we obtain

$$M(q_f, q_i)|_{q=0} = \sum_T X_T^{LS} \frac{4\pi}{k} \int_0^\infty dr \ j_1(kr) \ G_{WT}^{LS}(r, 0) \ O^{LS}(q_f, q_i),$$

(33)
or
\begin{equation}
f_T^{LS}(\theta)|_{q=0} = \frac{4\pi}{k} \int_0^\infty r^3 \, dr \, j_1(kr) \, G_{W,T}^{LS}(r,0) \, .
\end{equation}

Note that \( |q=0 \) implies setting \( q = 0 \) except for the \( LS \) operator part. If we call \( G_{W}^{LS}(r,0) = \sum_{T} X_T^{LS} G_{W,T}^{LS}(r,0) \) the \( LS \) potential, we find
\begin{equation}
\sum_{T} X_T^{LS} f_T^{LS}(\theta)|_{q=0} = \frac{4\pi}{k} \int_0^\infty r^3 \, dr \, j_1(kr) \, G_{W}^{LS}(r,0) \, ,
\end{equation}

which is nothing but Eq. (26) if we assign
\begin{equation}
G_{W}^{LS}(r,0) \sim g^{3O}(r) \, .
\end{equation}
Then we find
\begin{equation}
\overline{f_T^{LS}(\theta)} \sim f_T^{LS}(\theta)|_{q=0} \, ,
\end{equation}

with \( k = \overline{q} \) (see Eq. (A.5)). We will discuss the choice of the value \( k = \overline{q} \) and a further simplification in the next section.

Another approximation for the \( LS \) function \( f_T^{LS}(\theta) \) in Eq. (27) is obtained by taking only \( P \)-wave components in the partial wave expansion of the Born kernel [24]. Suppose the partial wave expansion of Eq. (8) is
\begin{equation}
\sum_{\Omega} M_{\alpha\alpha'}^{\Omega, \text{total}}(q_f, q_i) \, O^{\Omega}(q_f, q_i) \\
= \sqrt{(1 + \delta_{a_1,a_2})(1 + \delta_{a_1',a_2'})} \sum_{J\Omega S S' J' \alpha S'\alpha} 4\pi \, R_{\alpha S S '}^{\Omega, J} (q_f, q_i) \\
\times \mathcal{Y}_{(\ell S)JM}(q_f; \text{spin}) \, \mathcal{Y}^{*\Omega,(\ell' S')JM}(q_i; \text{spin}) \, ,
\end{equation}

where \( \mathcal{Y}_{(\ell S)JM}(q; \text{spin}) = [Y_{\ell}(\hat{q}) \chi_S(\text{spin})]_{JM} \) is the standard space-spin wave function. The front factor \( \sqrt{(1 + \delta_{a_1,a_2})(1 + \delta_{a_1',a_2'})} \) is 2 for \( NN \) and 1 for \( YY \). The partial-wave amplitudes \( R_{\alpha S S '}^{\Omega, J} (q_f, q_i) \) for \( \Omega = LS \) and \( LS(-) \) are explicitly given by
\begin{equation}
R_{\alpha S S '}^{\Omega, J} (q_f, q_i) = \delta_{\ell \ell'} \delta_{S S'} \delta_{S_1} \, \frac{1}{2(2\ell + 1) \, [\ell(\ell + 1) + 2 - J(J + 1)]} \, [\ell(\ell + 1) - 2] \\
\times \sum_{T} \left( X_T^{LS}\right)_{\alpha\alpha'} \left( f_T^{LS}_{\ell + 1} - f_T^{LS}_{\ell - 1} \right) \, ,
\end{equation}

where
\begin{equation}
\sum_{\Omega} M_{\alpha\alpha'}^{\Omega, \text{total}}(q_f, q_i) \, O^{\Omega}(q_f, q_i) \\
= \sqrt{(1 + \delta_{a_1,a_2})(1 + \delta_{a_1',a_2'})} \sum_{J\Omega S S' J' \alpha S'\alpha} 4\pi \, R_{\alpha S S '}^{\Omega, J} (q_f, q_i) \\
\times \mathcal{Y}_{(\ell S)JM}(q_f; \text{spin}) \, \mathcal{Y}^{*\Omega,(\ell' S')JM}(q_i; \text{spin}) \, .
\end{equation}

The part of the sum for \( \Omega = LS \) and \( LS(-) \) is explicitly given by
\begin{equation}
R_{\alpha S S '}^{\Omega, J} (q_f, q_i) = \delta_{\ell \ell'} \delta_{S S'} \delta_{S_1} \, \frac{1}{2(2\ell + 1) \, [\ell(\ell + 1) + 2 - J(J + 1)]} \, [\ell(\ell + 1) - 2] \\
\times \sum_{T} \left( X_T^{LS}\right)_{\alpha\alpha'} \left( f_T^{LS}_{\ell + 1} - f_T^{LS}_{\ell - 1} \right) \, ,
\end{equation}

where
\[ R_{\alpha S\alpha' S'}^{LS(-)}(q_f, q_i) = \delta_{\ell,\ell'} \delta_{J, J'} q_f q_i \frac{\sqrt{J(J+1)}}{2J+1} \sum_T \left[ \left( X_T^{LS(-)} \right)_{\alpha\alpha'} \right. \\
+ \left( X_T^{LS(-)\sigma} \right)_{\alpha\alpha'} (-1)^{1-S'} \left( f_T^{LS\ell-1} - f_T^{LS\ell+1} \right) \right] \]
\((S, S' = 1, 0 \text{ or } 0, 1 \text{ only}) \), \( (39) \)

where
\[ f_T^{LS\ell} = \frac{1}{2} \int_0^{\pi} f_T^{LS}(\theta) P_\ell(\cos \theta) \sin \theta \, d\theta \] \( (40) \)

is the angular-momentum projection of the \( LS \) function Eq. (A.3). (See Eq. (2.29) of [24].) If we take \( \ell = \ell' = 1 \) only in Eq. (38) and use the formulae \((n = [q_i \times q_f])\)

\[
O^{LS} = \mathbf{n} \cdot \mathbf{S} = -\frac{2\pi}{3} q_f q_i \sum_{JM} \left[ 4 - J(J+1) \right] \\
\times \mathcal{Y}_{(11)JM}(\hat{q}_f; \text{spin}) \mathcal{Y}_{(11)JM}^*(\hat{q}_i; \text{spin}) ,
\]
\[
O^{LS(-)} = \mathbf{n} \cdot \mathbf{S}^{(-)} = 4\pi \frac{\sqrt{2}}{3} q_f q_i \sum_{JM} \left\{ \mathcal{Y}_{(11)JM}(\hat{q}_f; \text{spin}) \mathcal{Y}_{(10)JM}^*(\hat{q}_i; \text{spin}) \\
+ \mathcal{Y}_{(10)JM}(\hat{q}_f; \text{spin}) \mathcal{Y}_{(11)JM}^*(\hat{q}_i; \text{spin}) \right\} ,
\]
\[
O^{LS(-)\sigma} = \mathbf{n} \cdot \mathbf{S}^{(-)} P_\sigma = 4\pi \frac{\sqrt{2}}{3} q_f q_i \sum_{JM} \left\{ - \mathcal{Y}_{(11)JM}(\hat{q}_f; \text{spin}) \mathcal{Y}_{(10)JM}^*(\hat{q}_i; \text{spin}) \\
+ \mathcal{Y}_{(10)JM}(\hat{q}_f; \text{spin}) \mathcal{Y}_{(11)JM}^*(\hat{q}_i; \text{spin}) \right\} , \quad (41)
\]

we eventually find that this prescription corresponds to the approximation \(^7\)

\[
M^{LS \text{ total}}_{aa'}(q_f, q_i) \sim \left\{ \begin{array}{ll}
2 \sum_T \left( X_T^{LS} \right)_{u^d}^{l=1} & \left( f_T^{LS\ell_0} - f_T^{LS\ell_2} \right) \\
\sum_T \left[ \left( X_T^{LS} \right)_{aa'}^{ud} + \left( X_T^{LS} \right)_{aa'}^{s} \right] & \left( f_T^{LS\ell_0} - f_T^{LS\ell_2} \right)
\end{array} \right. \\
\text{for} \begin{cases} \mathcal{N} \mathcal{N} \\ \mathcal{Y} \mathcal{N} \end{cases},
\]
\[
M^{LS(-) \text{ total}}_{aa'}(q_f, q_i) \sim \sum_T \left[ \left( X_T^{LS(-)} \right)_{aa'}^{ud} + \left( X_T^{LS(-)\sigma} \right)_{aa'}^{s} \right] \\
\times \left( f_T^{LS\ell_0} - f_T^{LS\ell_2} \right) \quad \text{for} \quad \mathcal{Y} \mathcal{N} . \quad (42)
\]

\(^7\) Note that \( O^{LS(-)\sigma} \) part disappears because of the second equation of Eq. (14).
If we compare these with Eq. (9), we find that this approximation corresponds to setting

$$ f_T^S(\theta) \sim f_{T0}^{LS} - f_{T2}^{LS}. $$

(43)

Note that we still have two parameters \( q_f = |q_f| \) and \( q_i = |q_i| \), the choice of which will be discussed in the next section.

2.3 Realistic calculation

Here we consider a realistic calculation of \( S_B \), based on the \( G \)-matrices for the \( NN \) and \( YN \) interaction [25]. In this case, \( M_{BB}^{\text{total}}(k_{12}, k'_{12}) \) in Eq. (16) now depends on some averaged value \( K \), \( \sqrt{(K_{12})^2} \) and the starting energy \( \omega = E_B(k'_{1}) + E_N(k'_{2}) \); i.e., \( G_{BB}^{\text{total}}(k_{12}, k'_{12}; K, \omega) \). We assume the density \( \rho_r(k_2, k'_2) \) in Eq. (15) is well approximated by the local form

$$ \rho_r(k_2, k'_2) = \delta(k_2 - k'_2) \rho_r(k_2), \quad (44) $$

where \( \rho_r(k) \) is given by the Fermi distribution

$$ \rho_r(k) = 2 \Theta(k_F^r - |k|), \quad (45) $$

with \( \Theta \) being the Heaviside's step function. Under this assumption, we can approximately set \( k_2 \sim k'_2 = k_1 + k_2 - k'_1 \). This implies \( k_1 = k'_1 \) and \( k_2 = k'_2 \). This approximation, however, makes the \( \ell s \) factor in Eq. (18) disappear. Thus we must take the density average by Eqs. (44) and (45) only for the invariant part just as in Eq. (21) and (22). After the change of the notation \( k_2 \rightarrow q_2 \) and \( k_{12} \rightarrow q \) etc., the expression we use is

Table 3
Contributions of symmetric (\( sLS \)) and antisymmetric (\( aLS \)) \( LS \) terms of the FB interaction [23] to the nucleon Scheerbaum factor \( S_N \) in the simplest approximation with \( \bar{q} = 0 \). The unit is MeV \( \cdot \) fm\(^5\).

<table>
<thead>
<tr>
<th>model</th>
<th>( sLS )</th>
<th>( aLS )</th>
<th>( S_N )</th>
<th>( aLS/sLS )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( D_- + D_+ )</td>
<td>( S + S' )</td>
<td>total</td>
<td>ratio</td>
</tr>
<tr>
<td>RGM-F</td>
<td>-28.2</td>
<td>-9.7</td>
<td>-37.8</td>
<td>0.344</td>
</tr>
<tr>
<td>FSS</td>
<td>-32.2</td>
<td>-11.0</td>
<td>-43.2</td>
<td>0.342</td>
</tr>
<tr>
<td>RGM-H</td>
<td>-32.2</td>
<td>-11.0</td>
<td>-43.2</td>
<td>0.342</td>
</tr>
</tbody>
</table>
Table 4
The Scheerbaum factor $S_B$ in the simplest $\bar{q} = 0$ approximation by the $p = 0$ Wigner transform $G_W(0)$. In $S_{\Lambda-\Sigma} = S_{\Sigma-\Lambda}$, the average mass of $\Lambda$ and $\Sigma$ is used for $\xi$. The unit is MeV · fm$^5$. $\lambda = (m_s/m_{ud})$ implies the FSB. When $\lambda = 1$, we also assume $\xi = 1$.

<table>
<thead>
<tr>
<th></th>
<th>RGM-F</th>
<th></th>
<th>FSS</th>
<th></th>
<th>RGM-H</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lambda = 1$</td>
<td>$\lambda = 1.25$</td>
<td>$\lambda = 1$</td>
<td>$\lambda = 1.526$</td>
<td>$\lambda = 1$</td>
</tr>
<tr>
<td>$N$</td>
<td>20.5</td>
<td>16.8</td>
<td>23.4</td>
<td>18.5</td>
<td>23.5</td>
</tr>
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<td>$\Lambda$</td>
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<td>-9.0</td>
<td>-14.1</td>
<td>-8.3</td>
<td>-14.1</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>-22.7</td>
<td>-19.3</td>
<td>-26.0</td>
<td>-21.5</td>
<td>-26.0</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>12.3</td>
<td>9.2</td>
<td>14.0</td>
<td>9.5</td>
<td>14.0</td>
</tr>
</tbody>
</table>

\[
\overline{G_{BB}^\Omega}(k_{12}, k_{12}', K, \omega) = \frac{1}{\rho_\tau(q_2)} \frac{\int \rho_\tau(q_2) G_{BB}^\Omega(q, q'; K, \omega) \, dq_2}{\int \rho_\tau(q_2) \, dq_2} = \frac{1}{4\pi (k_F)^3} \int_{|q_2| < k_F} G_{BB}^\Omega(q, q'; K, \omega) \, dq_2 ,
\]

where we have assumed symmetric nuclear matter and used $\rho_\tau(q_2) \to \Theta(k_F - |q_2|)$. Here we make use of the same treatment of angle-averaging just as used for the derivation of the s.p. potentials in [25]. We first change the integral variable $q_2$ in Eq. (46) to $q_2 = \xi q_1 - (1 + \xi)q$. We assume $q_1$ and determine $K$ through $K = \sqrt{(K_{12})^2(q_1, q)}$. Since $q_2 = |q_2|$ is given by $q_1$ and $K$, it is also determined from $q_1$ and $q$. The starting energy $\omega$ is determined as $\omega = E_B(q_1) + E_N(q_2)$. Then by using the weight function $W(q_1, q)$ introduced in Eq. (21) of [25], we find

\[
\overline{G_{BB}^\Omega}(k_{12}, k_{12}', K, \omega) = \frac{3}{4\pi} \frac{1}{(k_F)^3} (1 + \xi)^3 \int_0^{q_{max}} q^2 \, dq \ W(q_1, q) \times \int d\bar{q} \ G_{BB}^\Omega(q, q; K, \omega) .
\]

In order to reduce the angular integral in Eq. (47) further, we need explicit formulae for the partial-wave decomposition of the invariant amplitudes. For the $LS$ and $LS^(-)$ components, these are given by $M_{aa'}^{LS \text{ total}}(q_f, q_i) = (2/|n|) h_{aa'}^0$ and $M_{aa'}^{LS^(-) \text{ total}}(q_f, q_i) = (2/|n|) h_{aa'}^-$, where $h_{aa'}^0$ and $h_{aa'}^-$ are the flavor matrix elements of $\lambda$.

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The full expression of the partial-wave decomposition of the invariant amplitudes is given in Appendix D of [24].
\[ h^0 = -\frac{1}{4} \sum_J \frac{(2J+1)}{J(J+1)} \left[ G^I_{J,1,J} P^I_J(\cos \theta) \right. \\
- (J+1) \left. G^I_{J-1,1,J-1} P^I_{J-1}(\cos \theta) + J G^I_{J+1,1,J+1} P^I_{J+1}(\cos \theta) \right] , \\
\]
\[ h^- = \frac{1}{4} \sum_J \frac{2J+1}{\sqrt{J(J+1)}} \left[ G^I_{J,0,J} + G^I_{0,J,1,J} \right] P^I_J(\cos \theta) . \] (48)

Here \( G^I_{S\ell,S'\ell'} = G^I_{S\ell,S' \ell'}(q_f, q_i; K, \omega) \) and \( P^I_J(\cos \theta) \) is the associated Legendre function of the first kind with degree 1. We set \( q_f = q_i = q \) and \( \theta = 0 \), and use \( (1/\sin \theta)P^I_J(\cos \theta) = P^I_{J'}(\cos \theta) \) and \( P^I_{J'}(1) = J(J+1)/2. \) Then we easily find

\[ G^{LS}(q, q; K, \omega) = -\frac{1}{4q^2} \sum_{\ell=1}^{\infty} \left[ (2\ell-1)(\ell+1) G^{\ell-1}_{1\ell,1\ell} \\
+ (2\ell+1) G^\ell_{1\ell,1\ell} - (2\ell+3) \ell G^\ell+1_{1\ell,1\ell} \right] , \\
G^{LS(-)}(q, q; K, \omega) = \frac{1}{4q^2} \sum_{\ell=1}^{\infty} (2\ell+1)\sqrt{\ell(\ell+1)} \left[ G^\ell_{1\ell,0\ell} + G^\ell_{0\ell,1\ell} \right] . \] (49)

Combining Eqs. (22), (47) and (49), we obtain

\[ S_B(q_1) = -(1 + \delta_{B,N}) \frac{1}{2\pi} \frac{3}{4(k_F)^3} \xi(1+\xi)^2 \sum_{I,J} \frac{2I+1}{2IB+1}(2J+1) \\
\times \int_0^{q_{max}} dq \ W(q_1, q) \left\{ (J+2)G^J_{B1J+1,B1J+1}(q, q; K, \omega) \\
+ G^J_{B1J,B1J}(q, q; K, \omega) - (J-1)G^J_{B1J-1,B1J-1}(q, q; K, \omega) \\
- \sqrt{J(J+1)} \left[ G^J_{B1J,B0J}(q, q; K, \omega) + G^J_{B0J,B1J}(q, q; K, \omega) \right] \right\} . \] (50)

### 3 Result and discussion

For the value of \( k = \bar{q} \) in Eq. (37), we follow the suggestion by Scheerbaum [16] and take the value corresponding to the ”wavelength” of the density distribution. If we take this to be \( \sim 4t \) with \( t \sim 2.4 \) fm being the nuclear surface thickness, we arrive at the estimate\(^9\)

\[ k = \bar{q} \sim \frac{2\pi}{4t} \sim 0.7 \text{ fm}^{-1} . \] (51)

\(^9\) This is almost half of the Fermi momentum \( k_F = (9\pi/8)^{1/3}/r_0 = 1.36 \text{ fm}^{-1} \) for \( r_0 = 1.12 \) fm, which corresponds to \( \rho = (3/4\pi)/r_0^3 = 0.170 \text{ fm}^{-3}. \)
Table 5

$LS$ and $LS(-) + LS(-)\sigma$ contributions to the Scheerbaum factor $S_\Lambda$ in the simplest approximation of Eq. (52). The model is FSS. $\lambda = (m_s/m_{ud})$ implies the FSB. When $\lambda = 1$, we also assume $\xi = 1$.

<table>
<thead>
<tr>
<th>$\Lambda$</th>
<th>$X_{D_\pm}^\Lambda$</th>
<th>$X_{S_\pm}^\Lambda$</th>
<th>$S_\Lambda$</th>
<th>$\tilde{S}_\Lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$LS$</td>
<td>$\frac{8}{9}$</td>
<td>$-\frac{2}{9}$</td>
<td>$\frac{1}{9}$</td>
<td>$0.998$</td>
</tr>
<tr>
<td>$LS(-) + LS(-)\sigma$</td>
<td>$-\frac{2}{9}$</td>
<td>$-\frac{1}{9}$</td>
<td>$-\frac{1}{18}$</td>
<td>$-0.421$</td>
</tr>
<tr>
<td>sum</td>
<td>$\frac{2}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>$\frac{1}{18}$</td>
<td>$0.577$</td>
</tr>
<tr>
<td>$\lambda = 1.526$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$LS$</td>
<td>$0.6723$</td>
<td>$-0.1395$</td>
<td>$0.1014$</td>
<td>$0.813$</td>
</tr>
<tr>
<td>$LS(-) + LS(-)\sigma$</td>
<td>$-0.2856$</td>
<td>$-0.0539$</td>
<td>$-0.0525$</td>
<td>$-0.440$</td>
</tr>
<tr>
<td>sum</td>
<td>$0.3867$</td>
<td>$-0.1934$</td>
<td>$0.0489$</td>
<td>$0.373$</td>
</tr>
</tbody>
</table>

The simplest approximation is to set $k = q = 0$ in Eq. (A.5). In this case we can write down an analytic expression for $S_B$:

\[
S_B = -\alpha_S x^3 m_{ud} c^2 b^5 \frac{\xi}{1 + \xi} \tilde{S}_B ,
\]

\[
\tilde{S}_N = \frac{14}{9} - \frac{10}{27} \left( \frac{3}{4} \right)^{\frac{3}{2}} + \frac{32}{81} \left( \frac{12}{11} \right)^{\frac{3}{2}} ,
\]

\[
\tilde{S}_\Lambda = \frac{2}{9\lambda} \left( 2 + \frac{1}{\lambda} \right) - \frac{1}{9\lambda} \left( 2 + \frac{1}{\lambda} \right) \left( \frac{3}{4} \right)^{\frac{3}{2}} + \frac{1}{9\lambda} \left( 2 - \frac{1}{\lambda} \right) \left( \frac{12}{11} \right)^{\frac{3}{2}} ,
\]

\[
\tilde{S}_\Sigma = \frac{2}{3 \cdot 81} \left( 106 - \frac{6}{\lambda} - \frac{1}{\lambda^2} \right) - \frac{1}{81} \left( 18 - \frac{10}{\lambda} - \frac{3}{\lambda^2} \right) \left( \frac{3}{4} \right)^{\frac{3}{2}}
\]

\[
+ \frac{1}{3 \cdot 81} \left( 26 + \frac{24}{\lambda} - \frac{7}{\lambda^2} \right) \left( \frac{12}{11} \right)^{\frac{3}{2}} ,
\]

\[
\tilde{S}_\Xi = -\frac{2}{9} - \frac{1}{81} \left( 1 + \frac{14}{\lambda} + \frac{6}{\lambda^2} \right) \left( \frac{3}{4} \right)^{\frac{3}{2}} - \frac{1}{81} \left( 1 + \frac{18}{\lambda} - \frac{6}{\lambda^2} \right) \left( \frac{12}{11} \right)^{\frac{3}{2}} ,
\]

\[
\tilde{S}_{\Lambda-\Sigma} = -\frac{2}{27} \left( 7 + \frac{2}{\lambda} \right) - \frac{1}{81} \left( 5 - \frac{2}{\lambda} \right) \left( \frac{3}{4} \right)^{\frac{3}{2}} - \frac{1}{81} \left( 13 + \frac{6}{\lambda} \right) \left( \frac{12}{11} \right)^{\frac{3}{2}} ,
\]

where $x = (h/m_{ud} cb)$, $\lambda = (m_s/m_{ud})$ and $\xi = (M_N/M_B)$. The value of $S_B$ in this simplest approximation is given in Table 4. The parameters of our three models, RGM-F, FSS and RGM-H, are given in Table 2. We note that FSS and RGM-H produce very similar results for the s.p. $\ell s$ force, because the strength factor, $\alpha_S x^3 m_{ud} c^2 b^5 \xi/(1 + \xi)$, and the quark-mass ratio, $\lambda = (m_s/m_{ud})$, are very similar to each other. In particular, $\sqrt{2/\pi} \alpha_S x^3 m_{ud} c^2$ is constrained to be 440 MeV for RGM-F and FSS, in order to reproduce the $N-\Delta$ mass splitting through the color-magnetic term of the FB interaction.
Table 6
The Scheerbaum factors $S_B$ predicted by FSS in various types of approximations.
1) $G_W(0)$: Born approximation with $p = 0$ Wigner transform with $\bar{q} = 0$, 2) $G_W$: Born approximation with $p = 0$ Wigner transform with $\bar{q} = 0.7$ fm$^{-1}$, 3) $P$: $P$-wave Born approximation with $q_f = q_i = 0.35$ fm$^{-1}$, 4) $G$-matrix: $QTQ$ or continuous choice with $q_1 = 0$ and $k_F = 1.35$ fm$^{-1}$. The unit is MeV $\cdot$ fm$^5$.

<table>
<thead>
<tr>
<th>$B$</th>
<th>Born $G_W(0)$</th>
<th>Born $G_W$</th>
<th>Born $P$</th>
<th>$G$-matrix $QTQ$</th>
<th>cont. ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>$-43.2$</td>
<td>$-40.5$</td>
<td>$-41.7$</td>
<td>$-40.4$</td>
<td>$-41.6$</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>$-8.3$</td>
<td>$-7.8$</td>
<td>$-8.0$</td>
<td>$-3.8$</td>
<td>$-3.4$</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>$-21.5$</td>
<td>$-20.1$</td>
<td>$-20.7$</td>
<td>$-27.5$</td>
<td>$-22.4$</td>
</tr>
<tr>
<td>$\Xi$</td>
<td>$9.5$</td>
<td>$9.0$</td>
<td>$9.2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The $S_N (= S_N^{3Q})$ value, $\sim -40$ MeV $\cdot$ fm$^5$, is rather small, compared with the $S_N^{3Q}_{\text{free}}$ value, $-53 \sim -61$ MeV $\cdot$ fm$^5$, given in Table 1 of the Scheerbaum’s paper [16] for the Reid soft-core and other potentials. In his calculation, the effect of the short-range correlation reduces this value to $-34 \sim -47$ MeV $\cdot$ fm$^5$ in the same Table 1. These values were obtained with $\bar{q} = 0.7$ fm$^{-1}$ in the Scheerbaum approximation. However, we will see in the following that the effect of the short-range correlation obtained by solving the $G$-matrix equation is very small in our case. This is probably because our short-range repulsion is not represented by the hard core but by the quark-exchange kernel.

Table 3 shows that the Galilean non-invariant $aLS$ term of the FB interaction [23] has a fairly large contribution to $S_N^{3Q}$. The magnitude of $aLS$ contribution is almost $1/3$ of the $sLS$ contribution and they reinforce each other with the same sign [11].

We note that the $S_\Lambda$ value changes significantly by the FSB, which is easily understood from the spin-flavor factors in Table 1. All the $X_{\lambda f}^A$ factors contain the factor $1/\lambda$. If we assume $S_N^{3Q} = 1$, then $S_\Lambda$ is about $1/3$ for $\lambda = 1$, while it is $\sim 1/5$ for $\lambda = 1.69$ (the maximum FSB).\(^\text{10}\) On the other hand, $S_\Sigma$ does not change very much by the FSB: it changes from $3/5$ to $1/2$ as $\lambda$ changes from 1 to 1.69. The sign of $S_\Xi$ is positive and its value changes from $-1/3$ to $-1/4$. The $\Lambda$-$\Sigma$ coupling term is not small, and is about half of $S_N$ both in $\lambda = 1$ and $\lambda \neq 1$ cases. The sign of $S_{\Lambda-\Sigma} = S_{\Sigma-\Lambda}$ depends on the phase convention of the $\Lambda$ and $\Sigma$ flavor functions.

Table 5 shows the decomposition of $S_\Lambda$ into $LS$ and $LS^{(-)} + LS^{(-)}\sigma$ contributions in the simplest approximation. The signs of these two contributions

\(^{10}\) When we set $\lambda = 1$ in Eq. (52), we also neglect the mass difference of baryons; i.e., $\xi = 1$. 20
Table 7
The nuclear-matter density dependence of the Scheerbaum factors $S_B$ for $N$, $\Lambda$ and $\Sigma$, predicted by quark-model $G$-matrices in the continuous prescription. The model is FSS. The unit is MeV·fm$^5$.

<table>
<thead>
<tr>
<th>$k_F$ (fm$^{-1}$)</th>
<th>1.07</th>
<th>1.20</th>
<th>1.35</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>-43.0</td>
<td>-42.3</td>
<td>-41.3</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>-2.0</td>
<td>-2.7</td>
<td>-3.5</td>
</tr>
<tr>
<td>$\Sigma$</td>
<td>-21.5</td>
<td>-22.0</td>
<td>-21.8</td>
</tr>
</tbody>
</table>

are opposite to each other, and they largely cancel; namely, the half of the $LS$ contribution is cancelled by the $LS^{(-)} + LS^{(-)}\sigma$ contribution. Because of this cancellation, the strong $\lambda$-dependence in $S_\Lambda$ is even enhanced.

Table 6 shows the predictions of $S_B$ by FSS, calculated in the various prescriptions. The first column with $G_W(0)$ implies the simplest $q = 0$ prescription, the second with $G_W$ the Scheerbaum approximation with $q = 0.7$ fm$^{-1}$, the third with the $P$-wave approximation of Eq. (43). In the last case, we have assumed $q_f = q_i = Q$ with $Q = \bar{q}/2 = 0.35$ fm$^{-1}$. We have examined the $\bar{q}$ or $Q$ dependence in Eq. (37) or Eq. (43). Actually, the averaged spatial function $\bar{f}_L S^T(\theta)$ has some momentum dependence, so does $S_B$. However, this weak momentum dependence almost disappears if we take the ratio $S_B/S_N$. After all, we have found that $S_B/S_N$ ratios in the Born approximation are approximately given by

$$
\frac{S_\Lambda}{S_N} \sim \frac{1}{5}, \quad \frac{S_\Sigma}{S_N} \sim \frac{1}{2}, \quad \frac{S_\Xi}{S_N} \sim -\frac{1}{4},
$$

(53)

independently of whichever approximation of the spatial functions is used. Table 6 also shows the results of the realistic calculation using $G$-matrix solutions in the $QTQ$ and continuous prescriptions for intermediate spectra. Here the $q_1$ value in $S_B(q_1)$ (see Eq. (50)) is assumed to be $q_1 = 0$. We find that $S_\Lambda$ receives a strong effect due to the short-range correlation and $S_\Lambda/S_N$ is further reduced to almost 1/12. On the other hand, $S_\Lambda$ and $S_\Sigma$ do not change so much, except for the increase of $|S_\Sigma|$ in the $QTQ$ prescription. The ratio, $S_\Sigma/S_N \sim 1/2$, does not seem to change very much even in the $G$-matrix calculation.

Table 7 shows the $k_F$ dependence of $S_B(q_1 = 0)$ for $N$, $\Lambda$ and $\Sigma$, which are calculated from the FSS $G$-matrices with the continuous choice.$^{11}$ The three

$^{11}$In [25] we assumed $U_B(q_1) = U_B(q_1 = 3.8$ fm$^{-1}$) for $q_1 \geq 3.8$ fm$^{-1}$, in order to avoid the unrealistic behavior [24] of the s.p. potentials in the high momentum region. The results in Tables 7 and 8 are obtained with this prescription, while those in Table 6 are without this prescription. The difference of $S_B$ between these two prescriptions is very small, as is seen for $k_F = 1.35$ fm$^{-1}$. 

21
Decomposition of $S_{\Lambda} = -3.5 \text{ MeV} \cdot \text{fm}^5$ and $S_{\Sigma} = -22.3 \text{ MeV} \cdot \text{fm}^5$ at $k_F = 1.35 \text{ fm}^{-1}$ into various contributions. The model is FSS. The unit is MeV · fm$^5$.

<table>
<thead>
<tr>
<th></th>
<th>$I = 1/2$</th>
<th></th>
<th>$I = 3/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>odd</td>
<td>even</td>
<td>odd</td>
</tr>
<tr>
<td>$S_{\Lambda}$</td>
<td>$LS$</td>
<td>17.1</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>$LS(-)$</td>
<td>12.7</td>
<td>0.3</td>
</tr>
<tr>
<td>$S_{\Sigma}$</td>
<td>$LS$</td>
<td>2.7</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>$LS(-)$</td>
<td>-10.5</td>
<td>-0.6</td>
</tr>
</tbody>
</table>

values of the Fermi momentum, $k_F = 1.07$, 1.2 and 1.35 fm$^{-1}$, correspond to the three densities of $\rho = 0.5\rho_0$, $0.7\rho_0$ and $\rho_0$, respectively. Here $\rho_0 = 0.17$ fm$^{-3}$ is the normal density. We find that $S_{\Lambda}/S_N$ becomes even smaller for lower densities, while $S_{\Sigma}/S_N$ does not change much. Each contribution from the $LS$ and the $LS(-)$ components in even and odd states as well as $I = 1/2$ and $I = 3/2$ channels is shown in Table 8 for $k_F = 1.35$ fm$^{-1}$. It is clear that in the case of $S_{\Lambda}$ the $LS(-)$ contribution almost cancels the $LS$ one just as in the Born approximation, which makes the ratio $S_{\Lambda}/S_N$ to be less than $1/10$ for $\rho < \rho_0$. For the $\Sigma$ hyperon, the contribution from the $LS(-)$ force has an opposite sign to that for the $\Lambda$ hyperon, and the ratio $S_{\Sigma}/S_N$ turns out to be about $1/2$ even in the realistic calculation, which is very much independent of the precise value of $k_F$.

4 Summary

Since the spin-orbit force is the simplest momentum-dependent short-range force in the baryon-baryon interaction, it is sometimes discussed that the quark substructure of baryons might play an essential role as the microscopic origin of this very important non-central force [11,13,18]. In the hyperon-nucleon ($YN$) interaction, the spin-orbit force has very rich contents, consisting of three different types; $LS$, $LS(-)$, and $LS(-)\sigma$ [23]. These $LS$ forces predicted by the color-analogue of the Fermi-Breit (FB) interaction in the $(3q)-(3q)$ resonating-group method have correct spin and flavor dependence, which is very similar to that predicted by traditional meson-exchange models [12]. As to the magnitude of these $LS$ forces, we have pointed out [11] that the inclusion of the Galilean non-invariant $aLS$ term of the FB interaction is important, since it gives almost one-third of the Galilean invariant $sLS$ term with the same sign. The choice of the harmonic oscillator constant $b$ is also crucial to obtain enough strength of the $LS$ forces. In order to confirm that these $LS$ forces are consistently described with the short-range repulsion, we have proposed sev-
eral unified models of the $NN$ and $YN$ interactions [3–7], in which a realistic
description of these interactions is achieved not only for the $LS$ forces but
also for many other components of the central and non-central forces. In these
models, the short-range interaction composed of the strongly repulsive central
force and the $LS$ forces is mainly described by the quark-exchange kernel of
the FB interaction, and the medium- and long-range interaction composed of
the attractive central force and the long-range tensor force is described by
meson-exchange processes acting between quarks.

In this paper we have developed a formulation of the single-particle (s.p.)
spin-orbit ($\ell s$) potentials for the nucleon and hyperons, following the idea pre-
sented by Scheerbaum [16]. The quark-exchange kernel from the color-analogue
of the FB interaction is directly employed to calculate the strength factor $S_B$
for the s.p. $\ell s$ potentials in the Born approximation. In the simplest treat-
ment, $S_B$ is concisely expressed in terms of quark parameters, among which
the parameter $\lambda = (m_s/m_{ud})$, representing the flavor-$SU_3$ symmetry breaking
(FSB) at the quark level, plays an important role. Such expressions are very
useful for examining the characteristic structure of the s.p. $\ell s$ potentials. The
ratio of $S_B$ to the nucleon strength $S_N$ for the spin-saturated $Z = N$ nuclei
is found to be $S_\Lambda/S_N \sim 1/5$, $S_\Sigma/S_N \sim 1/2$ and $S_\Xi/S_N \sim -1/4$ in the Born
approximation with the full FSB, irrespective of various versions of our quark
model. This result is consistent with the estimation by Morimatsu et al. [13],
$U_N : U_\Lambda : U_\Sigma = 1 : 0.21 : 0.55$, although they used only the Galilean invariant
$sLS$ term in the FB interaction. This ratio is also preserved by the Galilean
non-invariant $aLS$ term in the FB interaction, but the inclusion of this term
makes the magnitude of $S_B$ reasonable for the realistic description, in contrast
to the large value presented in [13]. It is interesting to note that Dover and Gal
[27] also predicted $V_\Lambda^{SO}/V_\Sigma^{SO} \sim 0.2$ and $V_\Sigma^{SO}/V_N^{SO} \sim 0.6$, by using the coupling
constants of the Nijmegen model F potential.

We have also developed a formulation to evaluate the $S_B$ factor from the
$G$-matrix solution of our quark-model potential. Here we first calculated $NN$,
$\Lambda N$ and $\Sigma N$ $G$-matrices in symmetric nuclear matter by solving the Bethe-
Goldstone equation for the exchange kernel of our quark model FSS [5,6].
These $G$-matrices are then used to calculate $S_B$ for spin-saturated symmetric
nuclear matter, in the same way as the calculation of the s.p. potentials. In the
limit of the zero-momentum hyperons, we have found a fairly large reduction
of $S_\Lambda$, resulting in the ratio $S_\Lambda/S_N \sim 1/12$. For $S_N$ and $S_\Sigma$, the effect pro-
duced by solving the $G$-matrix equation is comparatively weak against usual
phenomenological potentials with a short-range repulsive core. In particular,
we have found $S_N \sim -40$ MeV $\cdot$ fm$^5$ both in the Born approximation and in
the $G$-matrix calculation. This implies that the effect of the shot-range cor-
relation is rather moderate in the quark-model description of the short-range
repulsion.
In the hyperon s.p. $\ell s$ potentials, the antisymmetric $LS$ ($LS^{(-)}$) force originating from the FB spin-orbit interaction (both from the $sLS$ and $aLS$ pieces) plays a characteristic role. If we neglect the FSB, the $S_\Lambda/S_N$ ratio is already around $1/3$. This is because the half of the $LS$ contribution is cancelled by the $LS^{(-)}$ contribution. The ratio is further reduced to $1/5$ by the FSB, originating from the strange to up-down quark-mass difference and the reduction factor of the $LS$ operator due to the difference of $N$ and $\Lambda$ baryon masses.

The former feature of the FSB at the quark level is a special situation of the $\Lambda$ hyperon, which results from the structure of its spin-flavor $SU_6$ wave function. Finally, the short-range correlation by solving the $G$-matrix equation further reduces the ratio to less than $1/10$. It may be argued that the $\ell s$ potential is relevant at the surface region in finite nuclei, where the nucleon density is rather low. We have checked that a small deviation of the Fermi-momentum from the value of ordinary symmetric nuclear matter, $k_F = 1.35$ fm$^{-1}$, does not change this small ratio. Experimental confirmation of the small s.p. $\ell s$ potentials for the $\Lambda$ hyperon is highly desirable [15].

A Isospin factors $C_I^I(B)$ and spatial integrals $f_{LS}^I(\theta)$

In this appendix we give explicit expressions of the isospin factors $C_I^I(B)$ in Eq. (13) and the spatial integrals $f_{LS}^I(\theta)$ in Eq. (9). The explicit values of $C_I^I(B)$ are

\[
\begin{align*}
C_p^I(p) &= C_n^I(n) = C_p^I(\Xi^0) = C_n^I(\Xi^-) = \delta_{I,1} ,
C_p^I(n) &= C_n^I(p) = C_p^I(\Xi^-) = C_n^I(\Xi^0) = \frac{1}{2} \quad \text{for both } I = 0 \text{ and } I = 1 ,
C_p^\frac{1}{2}(\Lambda) &= C_n^\frac{1}{2}(\Lambda) = 1 ,
C_p^I(\Sigma^+) &= C_n^I(\Sigma^-) = \delta_{I,\frac{3}{2}} ,
C_p^I(\Sigma^-) &= C_n^I(\Sigma^+) = \delta_{I,\frac{1}{2}} \frac{2}{3} + \delta_{I,\frac{3}{2}} \frac{1}{3} ,
C_p^I(\Sigma^0) &= C_n^I(\Sigma^0) = \delta_{I,\frac{1}{2}} \frac{1}{3} + \delta_{I,\frac{3}{2}} \frac{2}{3} ,
\end{align*}
\]

(A.1)

and a sum rule

\[
\sum_\tau C_I^I(B) = \frac{2I + 1}{2I_B + 1} \quad \text{(A.2)}
\]

is satisfied for each $B$. The spatial functions $f_{LS}^I(\theta)$ are given by\(^{12}\)

\(^{12}\)See Appendix B of [24].
\[ f_T^{LS}(\theta) = (-2\pi)\alpha_s x^3 m_{ud} c^2 b^5 \]

\[
\times \begin{cases} 
\left( \frac{12}{11} \right)^{\frac{3}{2}} \exp \left\{ -\frac{2}{11} b^2 \left[ \frac{4}{3}(q^2 + k^2) - k \cdot q \right] \right\} \tilde{h}_1 \left( \frac{1}{\sqrt{11}} |b| q + k \right) \quad \text{for } T = \begin{cases} S \\ S' \\ D_+ \\ D_- \end{cases} \\
\left( \frac{3}{4} \right)^{\frac{3}{2}} \exp \left\{ -\frac{1}{3} b^2 (q^2 + k^2) \right\} \tilde{h}_1 \left( \frac{1}{2} |b| k \right) \quad \text{for } T = \begin{cases} S \\ S' \\ D_+ \\ D_- \end{cases} \\
\left( \frac{2}{3} \right)^{\frac{3}{2}} \exp \left\{ -\frac{1}{3} b^2 k^2 \right\} \tilde{h}_1 \left( \frac{1}{3} |b| k \right) \quad \text{for } T = \begin{cases} S \\ S' \\ D_+ \\ D_- \end{cases} \\
\end{cases}
\]

where \( k = q_f - q_i, \ q = (1/2)(q_f + q_i), \ \cos \theta = \hat{q}_f \cdot \hat{q}_i, \) and \( \tilde{h}_1(x) \) is defined as

\[
\tilde{h}_1(x) = 3e^{-x^2} \frac{1}{\sqrt{\pi}} \int_0^1 e^{tx^2} t^2 dt = 1 + 3 \sum_{n=1}^{\infty} \frac{(-1)^n (2x^2)^n}{(2n + 3)!} .
\]

Here \( \tilde{h}_1(x) \) is normalized as \( \tilde{h}_1(0) = 1. \) If we set \( q = 0 \) in Eq. (A.3), it is simplified to

\[
\tilde{f}_T^{LS}(\theta) \sim f_T^{LS}(\theta)|_{q=0} = (-2\pi)\alpha_s x^3 m_{ud} c^2 b^5 \]

\[
\times \begin{cases} 
\left( \frac{12}{11} \right)^{\frac{3}{2}} \exp \left\{ -\frac{8}{33} (bk)^2 \right\} \tilde{h}_1 \left( \frac{1}{\sqrt{11}} bk \right) \quad \text{for } T = \begin{cases} S \\ S' \\ D_+ \\ D_- \end{cases} \\
\left( \frac{3}{4} \right)^{\frac{3}{2}} \exp \left\{ -\frac{1}{12} (bk)^2 \right\} \tilde{h}_1 \left( \frac{1}{2} bk \right) \quad \text{for } T = \begin{cases} S \\ S' \\ D_+ \\ D_- \end{cases} \\
\left( \frac{2}{3} \right)^{\frac{3}{2}} \exp \left\{ -\frac{1}{3} (bk)^2 \right\} \tilde{h}_1 \left( \frac{1}{3} bk \right) \quad \text{for } T = \begin{cases} S \\ S' \\ D_+ \\ D_- \end{cases} \\
\end{cases}
\]

where we assume \( k = \bar{q} \sim (2\pi/4t) \sim 0.7 \text{ fm}^{-1}. \) The analytic expression of \( S_B \) in Eq. (52) is easily derived, if we further set \( \bar{q} = 0. \)

References

