The commutativity principle and lagrangian symmetries

R. Banerjee*
S.N. Bose National Centre For Basic Sciences
She commutativity principle Lake City, Block JD, Sector III
Calcutta-700 091, INDIA

Using the commutativity of a general variation with the time differentiation we discuss both global and local (gauge) symmetries of a lagrangian from a unified point of view. The Noether considerations are thereby applicable for both cases. A complete equivalence between the hamiltonian and lagrangian formulations is established.

An important problem is the study of various symmetries of a given action. Thus, for example, global symmetries, or gauge invariances of the first kind are crucial for condensed matter systems, whereas local symmetries, or gauge invariances of the second kind pervade the whole of gauge theories. Symmetry transformations are those transformations that keep the invariance of the action without using the equations of motion. The quantum mechanical implementation of these symmetry principles is naturally carried out in the lagrangian formalism since the equations of motion assume the form of a variational principle. This is how the global symmetries are usually studied, leading to conservation laws using Noether’s theorem.

Local (gauge) symmetries, on the other hand, are best understood in the hamiltonian formalism by using the procedure of Dirac [1] to identify the constraints, which are a consequence of the gauge freedom of the theory. The generator is constructed as a linear combination of these constraints. For this generator to act as a symmetry of the action, there have to be certain conditions [2–4] among the parameters entering in the definition of the generator. Alternatively, there exist purely lagrangian methods [5–7] of extracting the gauge symmetry, but the connection with the hamiltonian approach remains obscure, just as the meaning of Noether’s theorem in this context remains unclear.

In this paper we present a unified approach to the implementation of either global or local symmetries. It is based on the commutativity of a general variation with the time differentiation operation, which was used by us [3] recently to discuss certain aspects of local symmetries. An analogue of Noether’s theorem is obtained. A complete equivalence between the lagrangian and hamiltonian formalisms is shown. With this in mind we will consider first order systems since here both the lagrangian and hamiltonian formalisms can be applied rightaway. This is not a serious restriction since any second order lagrangian can always be brought to a first order form by a suitable extension of the configuration space.

Consider the following lagrangian,

\[ L = a^\alpha(q) \dot{q}_\alpha - V(q) \]  \hspace{1cm} (1)

where the first and second terms denote the kinetic and potential pieces, respectively. Note that the \( a_\alpha(q)(\alpha = 1, \ldots, N) \) includes the extra variables that might be needed to recast the lagrangian into a first order form. Under an infinitesimal variation \( \delta q_\alpha \), the lagrangian changes as,

\[ \delta L = \left( f_{\alpha \beta} \dot{q}_\beta - \frac{\partial V}{\partial \dot{q}_\alpha} \right) \delta q^\alpha \]  \hspace{1cm} (2)

where the symplectic two form is given by,

\[ f_{\alpha \beta} = \frac{\partial a_\beta}{\partial q^\alpha} - \frac{\partial a_\alpha}{\partial q^\beta} \]  \hspace{1cm} (3)

If the above matrix is invertible, then the equations of motion for all the coordinates can be determined in the usual way from the invariance of the action,

\[ \dot{q}_\alpha = f^{\alpha \beta} \frac{\partial V}{\partial \dot{q}_\beta} \]  \hspace{1cm} (4)

where \( f^{\alpha \beta} \) is the inverse of \( f_{\alpha \beta} \) defined as \( f_{\alpha \beta} f^{\beta \gamma} = \delta_\alpha^\gamma \).

If the matrix \( f_{\alpha \beta} \) is not invertible, then the equations of motion cannot be determined for all the coordinates. In other words the number of equations of motion is less than the number of variables so that there is a degeneracy in the Cauchy problem. This corresponds to the case of a gauge theory where local symmetries play an important role. The global symmetries can be discussed without this additional complication and so we turn to the case where the symplectic matrix is invertible.

Just to show the equivalence between the lagrangian and hamiltonian formalisms, recall that the basic brackets are given by the inverse of the symplectic matrix,

\[ \{ a^\alpha, a^\beta \} = \frac{\partial a^\alpha}{\partial q^\gamma} f^{\gamma \sigma} \frac{\partial a^\beta}{\partial q^\sigma} \]  \hspace{1cm} (5)

The equation of motion (4) is now expressed as,

\[ \dot{q}_\alpha = \{ q_\alpha, V \} \]  \hspace{1cm} (6)

thereby yielding the conventional hamiltonian form of the equation of motion.

An important step in obtaining (2) from (1) has been to use the commutativity of the general variation with the time derivative operation,

\[ \delta \left( \frac{dq_\alpha}{dt} \right) = \frac{d}{dt} \left( \delta q_\alpha \right) \]  \hspace{1cm} (7)
This relation is crucial for deriving the lagrange’s equation. In the hamiltonian context, it leads to nontrivial consequences. Under a global transformation, the variation $\delta q_a$ is given by,

$$\delta q_a = \{q_a, G\}$$  \hspace{1cm} (8)

Independently computing both sides of (7) by using (6) and (8), and then exploiting the Jacobi identity yields,

$$\{q_a, \{G, V\}\} = 0$$

Since the result is true for all $q_a$, we can make the stronger statement that,

$$\{G, V\} = 0$$  \hspace{1cm} (9)

It is easy to see that this condition leads to the off-shell invariance of the action, since using (8) in (2), the expression for $\delta L$ reduces, modulo a total time derivative, to $\{G, V\}$. The conservation laws following from the global symmetries require the explicit use of the equations of motion. Hence (9) may be reexpressed as,

$$\frac{dG}{dt} = 0$$  \hspace{1cm} (10)

thereby reproducing the usual statement of Noether’s theorem regarding the conservation of the generator.

Let us next discuss the case of local symmetries. As stated before the symplectic matrix is not invertible and there are constraints in the system related to these non-invertible velocities. As shown by Dirac, the action principle for a constrained system follows from a lagrangian with a general structure,

$$L = a^i(q)\dot{q}_i - \lambda^a \phi_a(q) - V(q)$$  \hspace{1cm} (11)

The coordinates $q_i (i = 1, ... n)$ constitute the nonsingular part of the original lagrangian (1) while the constraints $\phi_a (a = 1, ... N - n)$ are implemented by the lagrange multipliers $\lambda_a$. For standard (i.e. second order) lagrangians the momenta corresponding to the noninvertible velocities are defined to be the primary constraints of the theory. The other (secondary, tertiary, ... ) constraints are obtained from the successive time consistency of these constraints till the iterative process terminates. This is the Dirac algorithm in the hamiltonian approach. In passing to the first order form the primary constraints are naturally eliminated and the variables (say $\lambda_{a1}$) conjugate to these constraints impose the secondary constraints ($\phi_{a1}$). All the other constraints (labelled by the index $(a_2)$) are put in by hand through their corresponding (unknown) lagrange multipliers $\lambda_{a2}$. The complete set of constraints is then labelled by the index $a$, which is a sum of $a_1$ and $a_2$.

Under an arbitrary variation, the lagrangian (11) changes as,

$$\delta L = (f\dot{q} - \lambda^a \frac{\partial \phi_a}{\partial q} - \frac{\partial V}{\partial q}) \delta q^i - \phi_a \delta \lambda^a$$

The symplectic matrix is now invertible and the invariance of the action leads to the following equations of motion,

$$\dot{q}^i = f^{ij}(\lambda^b \frac{\partial \phi_a}{\partial q^j} + \frac{\partial V}{\partial q})$$ \hspace{1cm} (12)

$$\phi_a = 0$$

which can also be put in the hamiltonian form by using the brackets,

$$\dot{q}_i = \{q_i, \phi_a\} \lambda^a + \{q_i, V\}$$  \hspace{1cm} (13)

Since the full set of constraints has been found, consistency with the equations of motion demands that they satisfy the algebra,

$$\{\phi_a, \phi_b\} = C^c_{ab} \phi_c \hspace{2cm} \{V, \phi_a\} = V^b_c \phi_b$$  \hspace{1cm} (14)

The structure of the above algebra shows that the constraints act as the generator in the hamiltonian framework in the sense that the hamiltonian has vanishing brackets with these constraints. Indeed following Dirac, the usual form of this generator is taken as,

$$G = e^a \phi_a$$  \hspace{1cm} (15)

where $e^a$ are the corresponding parameters and the infinitesimal transformations are given by,

$$\delta q_i = \{q_i, \phi_a\} e^a$$  \hspace{1cm} (16)

We are now in a position to discuss the local gauge invariance of (11). Computing the l.h.s. of (7) from (13) by using (16), we obtain,

$$\delta \dot{q}_i = \{\{q_i, \phi_a\}, \phi_b\} e^c \lambda^a + \{\{q_i, V\}, \phi_b\} e^b + \{q_i, \phi_a\} \delta \lambda^a$$  \hspace{1cm} (17)

Note that the expression $\delta \lambda_a$ is only formal since in general we do not know the lagrange multipliers. Its precise meaning will be abstracted from the commutativity law (7). Next, the r.h.s. of (7) is computed independently and equated with (17). Exploiting the Jacobi identity we find,

$$\{q_i, \phi_a\} \left( \delta \lambda^a - e^a + C^a_{ab} \lambda^b \right) e^b = 0$$

Since the result is valid for all $q_i$, we get the following transformation law for the multipliers,

$$\delta \lambda^a - e^a + C^a_{ab} \lambda^b = 0$$  \hspace{1cm} (18)

It is simple to show that with this expression, the action obtained from (11) remains invariant under the local transformation (16). This explicit check has been discussed earlier in the literature [8].
We now illustrate how the fundamental relation (10), which was derived from general arguments based on the commutativity property, plays a role for the local symmetries. Since the generator is given by (15) it might appear that we get a trivial relation $0 = 0$, which just follows from the time consistency of the constraints. Such a conclusion is, however, valid only when we pass to the constraint shell $\phi_a = 0$. This is not allowed in the present context because then the generator itself vanishes. A proper interpretation of (10) is needed meaning that the passage to constraint shell is disallowed. This also entails a slight modification in (13). The usual hamilton equation following from (11) actually contains an extra term which drops out once the constraint condition (12) is imposed. Since we do not want to impose this condition, the complete equation of motion is,

$$\dot{q}_i = \{q_i, \phi_a\} \lambda^a + \{q_i, \lambda^a\} \phi_a + \{q_i, V\}$$

Putting (15) in (10) and using the above equation of motion, along with the constraint algebra (14), to evaluate $\phi_a$, we obtain,

$$\left(\{\lambda^a, \phi_b\} \epsilon^{b} - \epsilon^{a} + C_{cb}^{a} \lambda^c b + V_a^{\epsilon b}\right) \phi_a = 0$$

The first term is just $\delta \lambda^a$ following from the definition (16), thereby reproducing the condition (18). Both global and local symmetries may thus be discussed from (10). Interpreted this way, it is possible to regard (10) as an analogue of Noether’s theorem for the local case.

After completing the hamiltonian analysis we discuss a purely lagrangian approach which reveals the equivalence of both methods. The first step is to identify the constraints within the lagrangian formalism. There is a standard method [9] of doing this thing. Going back to (2) we see that the term in the parentheses must vanish for the invariance of the action. If there are constraints there will be zero modes of the symplectic matrix. Computing these zero modes $\nu^a_\alpha$, and multiplying from the left leads to a set of constraints,

$$\phi^a = (\nu^a_\alpha)^T \frac{\partial V}{\partial \eta^\alpha} = 0$$

where $T$ stands for the transpose and $\alpha$ is the independent number of zero modes. These constraints are now inserted in the lagrangian by means of lagrangian multipliers, so they acquire a form similar to (11),

$$L = a^i(q) \dot{q}_i + \dot{\eta}^\alpha \phi_a(q) - V(q)$$

Note however that the constraints have been shifted from the potential to the kinetic part, implying that $\dot{\phi} = 0$ is being implemented in lieu of $\dot{\phi} = 0$. This ensures the time consistency of the constraints. The symplectic matrix with the basic variables $\chi^a_A = (q_i, \eta^\alpha)$ has the form,

$$F_{AB} = \left(\begin{array}{cc} f_{ij} & \frac{\partial \phi_a}{\partial q_i} \\ -\frac{\partial \phi_a}{\partial q_j} & 0 \end{array}\right)$$

where the first entry is the invertible two form corresponding to the coordinates $q_i$ and has exactly the same structure as (3). The zero modes of the above symplectic matrix are given by [10,11],

$$\nu^a_\alpha = \left(\begin{array}{c} f_{ij} \frac{\partial \phi_a}{\partial q_j} \\ \frac{\partial \phi_a}{\partial q_i} \end{array}\right)$$

where $A = i(b)$ for the top(bottom) entry. Multiplication of (22) with $\frac{\partial V}{\partial \eta^\alpha}$ to obtain fresh constraints in analogy with (19) just corresponds to the l.h.s. of (14). If this turns out to be a combination of the constraints, the process terminates; else it continues. This is the exact parallel of the hamiltonian way of extracting the constraints. We assume that the process has terminated and the lagrangian incorporating all the constraints is given by (20).

The variation of the lagrangian is given by,

$$\delta L = \left( f_{ij} \dot{q}^i + \dot{\eta}^\alpha \frac{\partial \phi_a}{\partial q_i} - \frac{\partial V}{\partial q^j} \right) \delta q^i + \delta \eta^a \phi^a = \left( F_{AB} \chi^B - \frac{\partial V}{\partial \chi^A} \right) \delta \chi^A \tag{23}$$

where $F$ is defined in (21), and the passage to the second line from the first has been done by using the commutativity principle. Let us next discuss the invariance properties.

As emphasised in this approach [11] the zero modes generate the infinitesimal transformations,

$$\delta \chi_A = -\epsilon_\alpha (\nu^a_\alpha)^T$$

which, in components, has the form,

$$\delta q^i = -\epsilon_a \frac{\partial \phi_a}{\partial q^i} f^{ji}; \delta \eta^a = -\epsilon_a$$

Under these transformations the variation (23) is given by,

$$\delta L = \left( C_{cb}^{a} \dot{\eta}^c - V^a_{\epsilon b}\right) \phi_a \epsilon^b \tag{25}$$

which vanishes only if the structure functions $C$ and $V$ vanish. For nonvanishing structure functions, however, the variation of the multipliers $\eta$ in (24) can be modified such that the r.h.s. of (25) vanishes. This is possible since the variation (25) is proportional to the constraints while the first line of (23) involves a piece $\delta \eta^a \phi_a$. Hence the complete transformation of $\eta$ to achieve the off-constraint shell invariance of the lagrangian is given by,

$$\delta \eta^a = -\epsilon^a - C_{bc}^{a} \dot{\eta}^c \epsilon^b + V^b_{\epsilon} \epsilon^b$$

Making the necessary identifications ($\lambda = -\dot{\eta}$) this relation is identical with (18). This completes the demonstration of the equivalence between the lagrangian and hamiltonian approaches.
It should be stressed that what has been achieved is the invariance of the lagrangian (11). However this is not the original lagrangian with which one starts. The latter contains some variables which appear as lagrange multipliers (say \(\lambda_{a_1}\)) and implement the constraints \(\phi_{a_1}\) when written in the first order form. A typical example is the \(A_0\) field implementing the Gauss constraint in Maxwell’s theory. The other multipliers which occur in (11) are put in by hand to enforce the remaining constraints. Therefore to get the invariance of the original lagrangian, it is essential to set the remaining multipliers (say \(\lambda_{a_2}\)) to zero. Using (18) this leads to a restriction among the gauge parameters,

\[
\epsilon^{a_2} = C^{a_2}_{c_1 b} \lambda^{c_1} \epsilon^b + V^{a_2}_{b} \epsilon^b
\]  

(26)

This equation determines only \(a_2\) gauge parameters. Thus the number of independent free gauge parameters is just \(a_1\), namely the number of original multipliers. The other relation in (18) yields the variation of the multipliers (the cyclic variables) in the original lagrangian,

\[
\delta \lambda^{a_1} = \dot{e}^{a_1} - C^{a_1}_{c_1 b} \lambda^{c_1} \epsilon^b - V^{a_1}_{b} \epsilon^b
\]  

(27)

Together with (16) the above relation yields the symmetry variations on all the variables in the lagrangian.

It might be mentioned that relations connecting gauge parameters were also obtained by purely hamiltonian methods [2,3] using Dirac’s classification of constraints. However the invariance shown there was for the total action which is the original action modified by the inclusion of the primary constraints; hence those relations involved the undetermined multipliers associated with the primary constraints. Since these constraints never occur here the undetermined multipliers also dont occur in our relations. Also, the invariance shown here is directly with regard to the original action.

We end this section by providing an example with a lagrangian,

\[
L = \frac{1}{2} \left[ (\dot{q}_2 - e^{a_1})^2 + (\dot{q}_3 - e^{a_2})^2 \right]
\]  

(28)

Its first order form is given by,

\[
L = p_2 \dot{q}_2 + p_3 \dot{q}_3 - p_2 e^{a_1} - p_3 e^{a_2} - \frac{1}{2} (p_2^2 + p_3^2)
\]  

(29)

where we use a notation to easily identify the canonical pairs following from the symplectic brackets. The constraints, using either the hamiltonian or the lagrangian version, are found to be: \(\phi_1 = p_2 e^{a_1}\); \(\phi_2 = p_1 e^{(q_1 + q_2)}\). Note that the third term in the r.h.s. of (29) is a constraint and hence it is dropped when we actually implement the constraints through the lagrange multipliers. Moreover since \(q_1\) is a cyclic variable, \(e^{a_1}\) is absorbed in the multipliers and the final lagrangian incorporating the constraints is expressed as,

\[
L = p_2 \dot{q}_2 + p_3 \dot{q}_3 - \eta p_2 - \lambda p_2 e^{a_2} - p_3 e^{a_2} - \frac{1}{2} (p_2^2 + p_3^2)
\]

It is simple to check that the modified constraints satisfy the consistency algorithm. The variations of the coordinates are given by, \(\delta q_2 = \epsilon_2\); \(\delta q_3 = e^{a_1} \epsilon_3\) where \(\epsilon_2\) and \(\epsilon_3\) are the parameters associated with the two constraints. To get the invariance of the original action, we have to set \(\eta = e^{a_1}\) and \(\lambda = 0\). Using (26) this yields a relation \(\epsilon_2 = \epsilon_3 + e q \dot{q}_2\) connecting the two parameters. Also, the variation of the cyclic variable \(q_1\) can be obtained from (27). Using all this information, the final transformations turn out to be,

\[
\delta q_1 = e^{-(q_1 + q_2)} (\dot{\lambda} - \dot{\lambda} \dot{q}_2); \delta q_2 = e^{-q_2} \dot{\lambda}; \delta q_3 = \lambda
\]

where we have redefined \(\epsilon_3 e^{a_2} = \lambda\). It is simple to check that (28) is invariant under these local transformations.

To conclude, based on the principle of commutativity of a general variation with the time differentiation operation, it was possible to discuss global and local symmetries simultaneously. The Noether result concerning the time conservation of the generator is therefore applicable for gauge invariances of either kind. Just as the variational principle plays a key role in the lagrangian formulation of symmetries, the most natural way of understanding the hamiltonian formulation is the commutativity property mentioned earlier. Since this property is an essential ingredient in deriving the variational principles, a direct contact between the lagrangian and hamiltonian formulations was feasible and a complete equivalence between the two was established.