Interaction of Magnetic Monopoles and Domain Walls

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We study the interaction of magnetic monopoles and domain walls in a model with SU(5)×Z₂ symmetry by numerically evolving the field equations. We find that the monopoles unwind and dissipate their magnetic energy on collision with domain walls within which the full SU(5) symmetry is restored.

The interactions of topological defects can have a profound effect on the outcome of phase transitions. The scaling of a network of domain walls and strings, and a distribution of magnetic monopoles crucially depends on how the defects interact among themselves and with each other. Thus far attention has focussed on the interactions of walls with walls, strings with strings, and monopoles with monopoles. The cosmological importance of the interactions of walls and monopoles was highlighted in Ref. [1] and it is this problem that we study in the present paper.

Earlier work on the interaction of solitons and domain walls (phase boundaries) has been carried out in the following contexts: (i) mutual interaction of domain walls [2], (ii) He³ A-B phase boundaries and vortices [3], (iii) Skyrmions and domain walls [4], and (iv) global SU(2) monopoles and embedded domain walls [5]. Here we will numerically study the interaction of gauged SU(5) monopoles with a Z₂ domain wall. This is quite distinct from the earlier work since it looks at magnetic monopoles which necessarily include gauge fields. It is also the most relevant problem for the cosmological consequences of Grand Unified theories [1].

The SU(5) model we consider is given by the Lagrangian:

\[ L = -\frac{1}{4} X_{\mu\nu}^a X^{a\mu\nu} + \frac{1}{2} |D_\mu \Phi|^2 - V(\Phi), \]

where \( \Phi \) is an SU(5) adjoint scalar field, \( X_{\mu\nu}^a \) (\( a = 1, \ldots, 24 \)) are the gauge field strengths and the covariant derivative is defined by:

\[ D_\mu \Phi = \partial_\mu \Phi - ie [X_\mu, \Phi]. \]

The potential \( V(\Phi) \) is the most general quartic potential but we exclude the cubic term in \( \Phi \) so as to obtain the extra \( Z_2 \) symmetry under \( \Phi \to -\Phi \):

\[ V(\Phi) = -\frac{m^2}{2} \text{Tr} \Phi^2 + \frac{h}{4} (\text{Tr} \Phi^2)^2 + \frac{\lambda}{4} \text{Tr} \Phi^4. \]
The parameters of the potential are chosen so that $\langle \Phi \rangle = v \text{diag}(2, -3, 2, 2, -3)/\sqrt{30}$ with $v = m/\sqrt{\lambda'}$ and $\lambda' = h + 7\lambda/30$. With this vacuum expectation value (VEV), the SU(5) symmetry is spontaneously broken to SU(3)×SU(2)×U(1). The desired constraints on the parameters are: $\lambda, \lambda' > 0$. The magnetic monopoles in this model were discussed by Dokos and Tomaras [6] except that also included the effects of a scalar field in the fundamental representation of SU(5). Here we do not have such a field. Yet the basic construction of [6] goes through and the fundamental monopole is essentially an SU(2) monopole embedded in the full theory. In the case when the potential vanishes (the Bogomolnyi-Prasad-Sommerfield (BPS) case [7,8]), the monopole solution is [9]:

$$\Phi_M \equiv \sum_{a=1}^{3} \Phi^a T^a + \Phi^4 T^4 , ~ (4)$$

where the subscript $M$ denotes the monopole field configuration,

$$T^a = \frac{1}{2} \text{diag}(\sigma^a, 0, 0, 0) , \quad T^4 = \frac{1}{2\sqrt{35}}(1, 1, -4, -4, 6) ,$$

$\sigma^a$ being the Pauli spin matrices,

$$\Phi^a = P(r)x^a, ~ (5)$$

$$\Phi^4 = \sqrt{\frac{7}{5}} C e, ~ (6)$$

where

$$P(r) = \frac{1}{er^2} \left( \frac{Cr}{\tanh(Cr)} - 1 \right), \quad (7)$$

and $r = \sqrt{x^2 + y^2 + z^2}$ is the spherical radial coordinate. The constant $C$ is arbitrary and corresponds to the choice of boundary condition at infinity. The gauge fields for the BPS monopole are:

$$W_i^a = \epsilon_{ij} \frac{x^j}{er^2}(1 - K(r)) , \quad (a = 1, 2, 3)$$

$$W_i^4 = 0 , \quad (8)$$

where

$$K(r) = \frac{Cr}{\sinh(Cr)} . \quad (9)$$

In the non-BPS case, the monopole profile functions $P(r)$ and $K(r)$ need to be found numerically but the field orientations are the same as in eq. (4).

Depending on the parameters in the potential, it is possible to have different stable domain wall solutions. The domain wall across which $\Phi \rightarrow -\Phi$ is stable provided...
\(-3/20 > h/\lambda > -7/30\) [1]. At the center of this wall, \(\Phi\) must necessarily vanish and so the full SU(5) symmetry is restored at the center of this wall. Certain components of \(\Phi\) do not vanish at the center of the domain wall solutions in this model for other values of parameters. In these walls, only a subgroup of the full SU(5) symmetry is restored in the center. We will only study the interaction of monopoles with walls in which \(\Phi = 0\) at the center in this paper. The interactions of other types of walls and monopoles will be discussed separately.

The solution for the domain wall located in the xy-plane is

\[
\Phi_{DW} = \frac{v}{\sqrt{30}} \tanh(\sigma z)(2, -3, 2, 2, -3),
\]

where \(\sigma = v \sqrt{\lambda/8}\).

If the monopole and the domain wall are initially \((t = 0)\) very far from each other, the joint field configuration is given by the product ansatz:

\[
\Phi = \tanh(\gamma \sigma(z - z_0 - vt))\Phi_M,
\]

where \(v\) is the velocity of the domain wall in the \(+z\)-direction, \(\gamma = 1/\sqrt{1 - v^2}\) is the Lorentz factor and \(z_0\) is the position of the wall at \(t = 0\). Here \(\Phi_M\) denotes the monopole solution in eq. (4) but with a non-BPS profile function. The gauge fields are unaffected by the presence of the wall and are still given by eq. (8).

Eq. (11) specifies the initial conditions for the scalar field for a wall approaching a monopole with velocity \(v\). The initial scalar and gauge field profiles are taken to the BPS profiles as this is more convenient to implement. We have checked that the results are not affected if we replace the BPS profiles with those found by numerical relaxation. The field dynamics is described by the equations of motion following from the Lagrangian in (1). At first sight, there are 24 components of \(\Phi\) and 96 components of the gauge fields that need to be evolved. However, it is not hard to check that all the dynamics occurs in an SU(2) subgroup of the original SU(5). This then reduces the dynamical fields to a triplet and a singlet of SU(2) (i.e. 4 fields) and \(3 \times 4 = 12\) gauge field components. Choosing the temporal gauge \((W^0_a = 0)\) reduces the number of gauge field components to 9.

Further reduction of the problem occurs since the initial conditions are axially symmetric and the evolution equations preserve this symmetry. The angular dependence in cylindrical coordinates can easily be imposed on the scalar field. For the gauge fields it can be extracted by using the fact that the covariant derivatives of the scalar field must vanish at large distances from the monopole. This then leads to the following ansatz for the 4 scalar and 9 gauge fields:

\[
\begin{align*}
\Phi_1 &= f_1 x, & \Phi_2 &= f_1 y, & \Phi_3 &= f_2 z, & \Phi_4 &= f_3 \\
W^1_x &= f_4 xy, & W^1_y &= f_4 y^2 - f_5, & W^1_z &= f_6 y
\end{align*}
\]
\[ W_x^2 = -f_4 x^2 + f_5 , \quad W_y^2 = -f_4 xy , \quad W_z^2 = -f_6 x , \]
\[ W_x^3 = -f_7 y , \quad W_y^3 = f_7 x , \quad W_z^3 = 0 , \]

where the \( f_i \) (\( i = 1, \ldots, 7 \)) are functions only of \( t, \rho = \sqrt{x^2 + y^2} \) and \( z \). We have explicitly checked that this ansatz is preserved by the evolution equations. So now the problem is reduced to one in 7 real functions of time and two spatial coordinates.

An attempt to numerically solve the 7 equations of motion directly in cylindrical coordinates failed due to numerical instabilities that developed within the time scale of the simulation. An analysis showed that the problem was due to large numerical errors in evaluating the equations of motion in cylindrical coordinates. This shortcoming of using cylindrical (and spherical) coordinates in numerical work is well recognized and the authors of [10] have proposed a solution that we have successfully implemented. The idea is to solve the problem, not in two spatial dimensions like the \( \rho z \)-plane, but to solve it in a thin three dimensional slab whose central slice is taken to lie in the \( xz \)-plane and with only 3 lattice spacings along the \( y \) direction. Then Cartesian coordinates can be used to solve the equations of motion in the \( y = 0 \) plane, thus minimizing numerical errors. On the \( y \neq 0 \) lattice sites the fields are evaluated by using the axial symmetry of the problem. This scheme improved the numerical stability of our staggered leapfrog code dramatically and allowed us to observe the monopole and wall for a sufficiently long duration without the development of numerical instabilities.

We have evolved the initial wall and monopole configuration with several velocities and for \( h = -\lambda/5 \) (\( c \) can be rescaled out of the problem and we chose \( \lambda = 1, \quad m = v \sqrt{\lambda} \) with \( v = 1 \)). The numerical results are given in the figures and clearly show that the energy of the monopole dissipates after the passage of the wall. The final snapshot shows that the magnetic energy has escaped the lattice and that the energy in the scalar field is located entirely along the wall. An examination of the scalar fields themselves shows the mechanism for the disappearance of the topology of the monopole. As is clear from the third panel in Fig. 3, the final state of the wall is different from that of the initial wall since the field \( \Phi_3 \) does not vanish on it. The destruction of the topology of the monopole has left a residue of scalar excitations on the domain wall. In other words, the topology of the monopole has disappeared on the wall, while its magnetic energy has been converted into radiation which has propagated off to infinity behind the wall.
FIG. 1. The first panel shows the potential energy density in the xz-plane for the magnetic monopole with $h = -\lambda/5$ and domain wall with velocity 0.8c. The second panel shows the corresponding magnetic energy density (proportional to $B^2$). The third panel shows the orientation of the Higgs field in the $\Phi_1 - \Phi_3$ plane.
FIG. 2. As in Fig. 1 at an intermediate time step.
We have checked that these results continue to hold for velocities in the range $v \in (0, 0.99c)$. Very low velocities require placing the monopole and wall quite close to each other initially, otherwise we would need our numerical code to run for a time longer than the time for which
our numerical technique is stable. Walls moving with yet higher velocities get thinner due to Lorentz contraction, falling below the resolution of our finest lattice. We have explored a few values of the parameters appearing in the SU(5) Lagrangian and have never seen a qualitative departure from the evolution shown in Figures 1, 2 and 3.

The dissolution of magnetic monopoles by domain walls implies that the number density of magnetic monopoles will fall off faster than if there were no domain walls. The cosmology of such a system of walls and monopoles has been discussed in [1] where it was argued that such interactions might resolve the cosmological monopole over-abundance problem. Similar interactions between strings and domain walls would affect the cosmological implications of cosmic strings. The numerical techniques presented here can also be used to study the interactions of walls and (global) monopoles or vortices in other systems.

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