States interpolating between number and coherent states

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September 20, 1999

Abstract

Using a ladder-operator approach to binomial states we construct new interpolating number-coherent states which reduce to number and coherent states in two different limits. We reveal the connection of this new state with photon-added coherent states and investigate their non-classical properties and quasi-probability distributions in detail. It is of interest to note that these new states, which interpolate between coherent states and number states, neither of which exhibit squeezing, are nevertheless squeezing states. A scheme to produce these states is also proposed.

PACS: 03.65.-w 02.20.-a,42.50.-p

1 Introduction

Since Stoler, Saleh and Teich proposed the binomial states (BS) in 1985 [1], so-called intermediate states between some fundamental states such as number states, coherent and squeezed states and phase states have attracted much attention [2]. The BS are finite linear combination of number states

\[ |\eta, M\rangle = \sum_{n=0}^{M} \left[ \binom{M}{n} \eta^n (1 - \eta)^{M-n} \right]^{1/2} |n\rangle, \]

where \( M \) is a non-negative integer, \( \eta \) is a real probability (0 < \( \eta \) < 1) and \( |n\rangle \) is a number state of the radiation field. The photon number distribution is clearly the binomial
distribution, whence the name binomial state. The BS are intermediate number-coherent states in the sense that they reduce to number and coherent states in different limits

$$|\eta, M\rangle \rightarrow \begin{cases} 
|M\rangle, & \eta \rightarrow 1, \\
|0\rangle, & \eta \rightarrow 0, \\
|\alpha\rangle, & \eta \rightarrow 0, \ M \rightarrow \infty, \eta M = \alpha^2.
\end{cases} \tag{1.2}$$

It was shown in a previous paper [3] that the BS also admit the ladder-operator form

$$\left(\sqrt{\eta}N + \sqrt{1 - \eta}\sqrt{M - Na}\right)|\eta, M\rangle = \sqrt{\eta}M|\eta, M\rangle, \tag{1.3}$$

where $a$, $a^\dagger$ and $N$ are the annihilation, creation and the number operators, respectively. The algebra involved is the $su(2)$ algebra (Holstein-Primakoff realization [4])

$$\begin{align*}
J^+ &= \sqrt{M - Na}, & J^- &= a^\dagger\sqrt{M - N}, & J^3 &= \frac{M}{2} - N, \tag{1.4}
\end{align*}$$

and in the present case the limit of coherent states is essentially the contraction of $su(2)$ to the Heisenberg-Weyl algebra generated by $a^\dagger, a$ and 1.

However, we know that number and the coherent states are eigenstates of the number operator $N$ and the annihilation operator $a$, respectively. So to define states interpolating between number and coherent states, it is more natural to consider the eigenvalue equation of a linear combination of $N$ and $a$ itself (not $J^+$), namely,

$$\left(\sqrt{\eta}N + \sqrt{1 - \eta}a\right)|\eta, \beta\rangle = \beta|\eta, \beta\rangle, \tag{1.5}$$

where $0 < \eta < 1$ as before and $\beta$ is the eigenvalue to be determined, not only by the eigenvalue equation (1.5) but also by a physical requirement (see Sec. 2).

In this paper we study the states $|\eta, \beta\rangle$ and their various properties. We find that for $\beta = \sqrt{\eta}M$ ($M$ a non-negative integer), the solutions to Eq. (1.5) are indeed intermediate states which interpolate between number and coherent states. We also find that these states are closely related to the photon-added coherent states proposed by Agarwal and Tara [5]. The properties of this new state, such as their sub-Poissonian statistics, antibunching effects and squeezing effects, as well as their quasi-probability distributions (the Q and Wigner functions), are studied in detail. Although coherent and number states are not squeezed, the new interpolating states are squeezed. It will be shown that they exhibit highly nonclassical behavior. Finally, we propose a scheme to produce these interpolating states in a cavity.

Physically, the interpolating number-coherent states are of particular interest. In the JC model, the atomic population inversion exhibits two completely different phenomena:
Rabi oscillation and periodic collapse and revival when the field is initially prepared in
a number state and a coherent state, respectively. The states proposed in this paper
present a method of interpolating between both phenomena, given that the initial state
of the field is in an intermediate state [6].

2 New interpolating number-coherent states

In this section we solve the eigenvalue equation (1.5), and reveal the relation of the states
(1.5) to photon-added coherent states and study the limit to number and coherent states.

2.1 Solutions

To solve the eigenvalue equation (1.5), we expand the state $|\eta, \beta\rangle$ in number states

$$|\eta, \beta\rangle = \sum_{n=0}^{\infty} C_n |n\rangle$$ \hspace{1cm} (2.1)

Inserting Eq. (2.1) into Eq. (1.5) and comparing the two sides of the equation, we find

$$C_n = \frac{[\beta - \sqrt{\eta}(n-1)][\beta - \sqrt{\eta}(n-2)]\cdots \beta}{(\sqrt{1-\eta})^n \sqrt{n!}} C_0.$$ \hspace{1cm} (2.2)

Substituting Eq. (2.2) into Eq. (2.1), we finally have

$$|\eta, \beta\rangle = C_0 \sum_{n=0}^{\infty} \frac{[\beta - \sqrt{\eta}(n-1)][\beta - \sqrt{\eta}(n-2)]\cdots \beta}{(\sqrt{1-\eta})^n \sqrt{n!}} |n\rangle,$$ \hspace{1cm} (2.3)

where $C_0$ is determined by the normalization condition

$$C_0 = \left[ \sum_{n=0}^{\infty} \frac{[\beta - \sqrt{\eta}(n-1)][\beta - \sqrt{\eta}(n-2)]\cdots \beta}{(1-\eta)^n n!} \right]^{-\frac{1}{2}}.$$ \hspace{1cm} (2.4)

The above solution is valid for any complex eigenvalue $\beta$ and it indeed reduces to a
coherent state in the limit $\eta \to 0$, as expected,

$$|\eta, \beta\rangle \xrightarrow{\eta \to 0} e^{-|\beta|^2/2} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle.$$ \hspace{1cm} (2.5)

However, it does not have a number state limit for arbitrary $\beta$ since number states are
eigenstates of $N$ with non-negative integer eigenvalues. In order to obtain non-negative
integer eigenvalues, we must choose $\beta = \sqrt{\eta} M$, where $M$ is a non-negative integer. In
fact, we shall see that this special choice of $\beta$ enables us to get not only the number
state limit but the coherent state limit, so it is enough to consider this special choice of
for the purposes of this paper. In fact, it is easy to see that in this case the coefficients $C_n$ are truncated

$$C_n = \begin{cases} 
0, & \text{when } n > M, \\
\left(\frac{\eta}{1 - \eta}\right)^n \frac{M!}{(M-n)!\sqrt{n!}} C_0, & \text{when } n \leq M.
\end{cases}$$ \hspace{1cm} (2.6)

Here the normalization constant $C_0(\eta, M)$ is obtained as

$$C_0(\eta, M) = \left[ \sum_{n=0}^{M} \left( \frac{\eta}{1 - \eta}\right)^n \frac{(M!)^2}{[(M-n)!]^2 n!} \right]^{-\frac{1}{2}} = \frac{\lambda^M}{\sqrt{M!L_M(-\lambda^2)}},$$ \hspace{1cm} (2.7)

where $\lambda \equiv \sqrt{(1 - \eta)/\eta}$ and $L_M(x)$ is the Laguerre polynomial [7]

$$L_M(x) = \sum_{n=0}^{M} \frac{1}{n!} \left( \frac{M}{M-n} \right) (-1)^n x^n.$$ \hspace{1cm} (2.8)

Inserting Eq. (2.6) and Eq. (2.7) into Eq. (2.1), we obtain the desired solution $\|\eta, (\beta = \sqrt{\eta M})\rangle \equiv \|\eta, M\rangle$

$$\|\eta, M\rangle = \frac{1}{\sqrt{M!L_M(-\lambda^2)}} \sum_{n=0}^{M} \lambda^{M-n} \frac{M!}{(M-n)!\sqrt{n!}} |n\rangle,$$ \hspace{1cm} (2.9)

which is a finite linear superposition of number states. However, similar to the binomial state case, the limit to coherent states corresponds to $M \to \infty$.

### 2.2 Connection with photon-added coherent states

The states (2.9) can be written in more elegant form. By making use of $|n\rangle = \frac{a^n}{\sqrt{n!}} |0\rangle$, we can write (2.9) as

$$\|\eta, M\rangle = \frac{1}{\sqrt{M!L_M(-\lambda^2)}} \sum_{n=0}^{M} \lambda^{M-n} \frac{M!}{(M-n)!\sqrt{n!}} |n\rangle = \frac{1}{\sqrt{M!L_M(-\lambda^2)}} (a^\dagger + \lambda)^M |0\rangle,$$ \hspace{1cm} (2.10)

where we have used the binomial formula.

Furthermore, thanks to the following equation (real $\lambda$ in our case)

$$D(-\lambda) a^\dagger D(\lambda) = a^\dagger + \lambda,$$ \hspace{1cm} (2.11)

where $D(\lambda)$ is the displacement operator

$$D(\lambda) = \exp \left[ \lambda (a^\dagger - a) \right],$$ \hspace{1cm} (2.12)
we can rewrite Eq. (2.10) in the following form

\[ \| \eta, M \rangle = \frac{1}{\sqrt{M!L_M(-\lambda^2)}} D(-\lambda) a^M D(\lambda) |0\rangle \]

\[ = \frac{1}{\sqrt{M!L_M(-\lambda^2)}} D(-\lambda) a^M |\lambda\rangle \]

\[ \equiv D(-\lambda)|\lambda, M\rangle \]  

(2.13)

where \(|\lambda\rangle = D(\lambda)|0\rangle\) is a coherent state and

\[ |\lambda, M\rangle \equiv \frac{1}{\sqrt{M!L_M(-\lambda^2)}} a^M |\lambda\rangle \]  

(2.14)

is a so-called photon-added coherent state or the excited coherent state \([5]\). So from Eq. (2.14) we conclude that the new interpolating number-coherent states are displaced excited coherent states.

However, we would like to point out that our states are very different from the photon-added coherent states. The photon-added states are an infinite superposition of number states from \(M\) to infinity, while our states are a finite superposition of number states from 0 to \(M\). So in this sense we can say that two states are complementary.

### 2.3 The limiting cases of number and coherent states

We first consider the reduction to number states. Let us consider the limit \(\eta \rightarrow 1\). At first, directly from Eq.(1.5) with \(\beta = \sqrt{\eta M}\), we see that it reduces to \(N|1, M\rangle = M|1, M\rangle\), this means \(|1, M\rangle = |M\rangle\). Then, from the number-state expansion (2.9), it follows that

\[ C_n = \frac{\lambda^{M-n} M!}{\sqrt{M!M!(M-n)!}} \rightarrow \delta_{M,n}, \]  

(2.15)

namely, \(|\eta, M\rangle \rightarrow |M\rangle\). Finally, the reduction to the number state \(|M\rangle\) of form Eq. (2.10) is obvious in the limit \(\lambda \rightarrow 0\). So we reach the same conclusion from different forms of the state \(|\eta, M\rangle\).

In the different limit \(\eta \rightarrow 0, M \rightarrow \infty\) with \(\sqrt{\eta M} = \alpha\) a real constant, \(|\eta, M\rangle\) reduces to the coherent state \(|\alpha\rangle\). Firstly, in this limit, the eigenvalue equation (1.5) reduces to the ladder operator definition of the coherent states \(a|0, \infty\rangle = \alpha|0, \infty\rangle\), namely, \(|0, \infty\rangle = |\alpha\rangle\). On the other hand, noticing that

\[ \frac{M!}{(M-n)!} \rightarrow M^n, \quad \lambda^{-n} M^n \rightarrow \alpha^n, \quad C_0 \rightarrow \exp(-\alpha^2/2). \]  

(2.16)
Eq. (2.9) reduces to the coherent state $|\alpha\rangle$. From Eq. (2.10) we can reach the same conclusion if we write

$$\|\eta, M\rangle = C_0 \left(1 + \frac{a^\dagger}{\lambda}\right)^M |0\rangle$$

and use the limit formula

$$\lim_{\lambda \to \infty} \left(1 + \frac{A}{\lambda}\right)^\lambda = e^A. \quad (2.18)$$

The above discussion shows that the state $\|\eta, M\rangle$ may be considered as an interpolating state which interpolates between a number state and a coherent state.

### 2.4 As generally-deformed oscillator coherent states

We now present an approach to $\|\eta, M\rangle$ in the context of the generally-deformed oscillator [8]. Define the box function

$$[n]_\beta = \frac{\beta^2 n}{(\beta - \sqrt{\eta}(n - 1))^2}, \quad (2.19)$$

and its factorial

$$[n]_\beta! \equiv [n]_\beta[n - 1]_\beta \cdots [1]_\beta, \quad [0]_\beta! \equiv 1. \quad (2.20)$$

Note that $[n]_\beta \to n$ when $\eta \to 0$. The state $\|\eta, \beta\rangle$ can be rewritten as

$$\|\eta, \beta\rangle = C_0 \sum_{n=0}^\infty \frac{\beta^n}{\sqrt{[n]_\beta!}} |n\rangle, \quad (2.21)$$

which clearly reduces to the coherent state $|\beta\rangle$ in the limit $\eta \to 0$. From $[n]_\beta$, we can construct a lowering operator $A_\beta$ and a raising operator $A^+_{\beta}$

$$A_\beta = \sqrt{\frac{[N + 1]_\beta}{N + 1}} a = \frac{\beta}{\beta - \sqrt{\eta}N} a, \quad A^+_{\beta} = a^\dagger \sqrt{\frac{[N + 1]_\beta}{N + 1}} = a^\dagger \frac{\beta}{\beta - \sqrt{\eta}N}. \quad (2.22)$$

The operators $A_\beta$, $A^+_{\beta}$ and $N$ satisfy the following algebraic relations

$$A^+_{\beta} A_\beta = [N]_\beta, \quad A_\beta A^+_{\beta} = [N + 1]_\beta, \quad (2.23)$$

and generate an associative algebra $A_{\beta}$, which is a generally deformed oscillator algebra with $[n]_\beta$ as the structure or characteristic function. Then in terms of the ladder (lowering and raising) operators, we conclude that state $\|\eta, \beta\rangle$ is an eigenstate of the lowering operator $A_\beta$

$$A_\beta \|\eta, \beta\rangle = \beta \|\eta, \beta\rangle \quad (2.24)$$
or, in other words, the state $\|\eta, \beta\rangle$ is the ladder-operator coherent state of the generally deformed oscillator algebra $A_\beta$.

Here we note that $A_\beta^+\equiv (A_\beta)^\dagger$ is not the hermitian conjugate of the lowering operator $A_\beta$ for complex number $\beta$. This is compatible with the fact that $[n]_\beta$ is not real for complex number $\beta$. Of course, when $\beta$ is real, $A_\beta^+\equiv (A_\beta)^\dagger$ and $[N]_\beta$ is hermitian.

In the above discussion, we assume that $\beta \neq \sqrt{\eta}M$.

When $\beta = \sqrt{\eta}M$, the box function becomes $([n]_{\sqrt{\eta}M} \equiv [n]_M)
\begin{equation}
[n]_M = \frac{M^2n}{(M-n+1)^2};
\end{equation}
which reduces to $n$ in the limit $M \to \infty$. In terms of the box function, the state (2.9) can be rewritten as
\begin{equation}
\|\eta, M\rangle = C_0 \sum_{n=0}^{M} \frac{(\lambda^{-1}M)^n}{\sqrt{[n]_M!}} |n\rangle.
\end{equation}

It is obvious that the above state reduces to the ordinary coherent state $|\alpha\rangle$ in the limit $M \to \infty$, $\lambda \to \infty$ keeping $\lambda^{-1}M = \alpha$. The lowering operator becomes
\begin{equation}
A_M = \frac{M}{M-N}a,
\end{equation}
which is well-defined in the subspace spanned by $\{|0\rangle, |1\rangle, \cdots, |M\rangle\}$. However, unfortunately, $A_M^+\equiv (A_M)^\dagger$ is not well-defined in the state $|M\rangle$. So we do not have a generally-deformed algebra.

Furthermore, although $A_M$ is well-defined, the state $\|\eta, M\rangle$, being finite, is not an eigenstate of $A_M$. In fact, $A_M$ is a nilpotent operator satisfying $(A_M)^M = 0$ and it has only one eigenstate, that is the vacuum state $|0\rangle$, which can be verified directly. This is similar to the fact that we cannot define the eigenstates of the ladder operator of the $su(2)$ algebra as coherent states. In fact, our state $\|\eta, M\rangle$ satisfies the following equation
\begin{equation}
a\|\eta, M\rangle = \lambda^{-1}(M-N)|\eta, M\rangle.
\end{equation}
However, we cannot move the operator $M-N$ to the left side since the inverse of $M-N$ does not exist in the subspace $\{|0\rangle, \cdots, |M\rangle\}$ ($\det(M-N)=0$).

3 Nonclassical Properties

In this section we shall investigate the statistical and squeezing properties of $\|\eta, M\rangle$, with a special emphasis on comparison with those of the BS.
3.1 Photon statistics

It is easy to derive the following relation

\[ a_k \|\eta, M \rangle = \left[ \frac{M(M-1) \cdots (M-k+1) L_{M-k}(-\lambda^2)}{L_M(-\lambda^2)} \right]^{1/2} \|\eta, M-k \rangle, \] (3.1)

for \( k \leq M \) and \( a_k \|\eta, M \rangle = 0 \) for \( k > M \). Then from \( N = a^\dagger a \) and \( N^2 = a^{12}a^2 + N \) we obtain the mean value of \( N \) and \( N^2 \)

\[ \langle N \rangle = \frac{ML_{M-1}(-\lambda^2)}{L_M(-\lambda^2)}, \] (3.2)

\[ \langle N^2 \rangle = \frac{M(M-1)L_{M-2}(-\lambda^2) + ML_{M-1}(-\lambda^2)}{L_M(-\lambda^2)}. \] (3.3)

The Mandel Q-parameter [9] is obtained as

\[ Q(\eta, M) = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} - 1 = (M-1) \frac{L_{M-2}(-\lambda^2)}{L_{M-1}(-\lambda^2)} - M \frac{L_{M-1}(-\lambda^2)}{L_M(-\lambda^2)}. \] (3.4)

If \( Q(\eta, M) < 0 \) (or \( Q(\eta, M) > 0 \)), the field in the state \( \|\eta, M \rangle \) is sub-Poissonian (super-Poissonian). \( Q(\eta, M) = 0 \) corresponds to Poissonian statistics.

For a fixed \( M \), there are two extreme cases, \( \eta = 0 \) (or \( \lambda = \infty \)) and \( \eta = 1 \) (or \( \lambda = 0 \)). It is easy to see that

\[ Q(\eta, M) \rightarrow \begin{cases} -1 & \lambda = 0, \\ 0 & \lambda \rightarrow \infty, \end{cases} \] (3.5)

which agrees with the Q-parameter of the number states and the vacuum state, as it should. Here we have used the fact \( L_M(0) = 1 \) and \( L_m(x)/L_n(x) \rightarrow 0 \) for \( m < n \) and \( x \rightarrow \infty \).

Fig. 1 is a plot of \( Q(\eta, M) \) with respect to \( \eta \) for \( M = 2, 50, 100 \). The Q-parameter of the binomial states is also presented in the figure \((-\eta \text{ for any } M)\). From this figure we find that the field in \( \|\eta, M \rangle \) is sub-Poissonian except for the case \( \eta = 0 \).

We say that a field is antibunched if the second-order correlation function \( g^{(2)}(0) = \langle a^\dagger a a a^\dagger \rangle/ \langle a^\dagger a \rangle^2 \) < 1 [10]. In fact, the occurrence of antibunching effects and sub-Poissonian statistics coincides for single mode and time-independent fields such as the state \( \|\eta, M \rangle \) of this paper. So the field \( \|\eta, M \rangle \) is antibunched except at the point \( \eta = 0 \).

3.2 Squeezing properties

Define two quadratures \( x \) (coordinate) and \( p \) (momentum)

\[ x = \frac{1}{\sqrt{2}} (a + a^\dagger), \quad p = \frac{1}{\sqrt{2i}} (a - a^\dagger). \] (3.6)
Then we can easily calculate the mean values of $a$ and $a^2$

\[
\langle a \rangle = \langle a^1 \rangle = \frac{M!}{L_M(-\lambda^2)} \sum_{n=0}^{M-1} \frac{\lambda^{2M-2n-1}}{(M-n)!(M-1-n)!n!} = \frac{\lambda L_{M-1}^{(1)}(-\lambda^2)}{L_M(-\lambda^2)}, \tag{3.7}
\]

\[
\langle a^2 \rangle = \langle a^1a^1 \rangle = \frac{M!}{L_M(-\lambda^2)} \sum_{n=0}^{M-2} \frac{\lambda^{2M-2n-2}}{(M-n)!(M-2-n)!n!} = \frac{\lambda^2 L_{M-2}^{(2)}(-\lambda^2)}{L_M(-\lambda^2)}, \tag{3.8}
\]

where $L_m^{(k)}(x)$ is the associated Laguerre polynomial defined by [7]

\[
L_m^{(k)}(x) = \sum_{n=0}^{m} \frac{(m+k)!}{(m-n)!n!(k+n)!} (-x)^n, \quad (k > -1). \tag{3.9}
\]

The variances $(\Delta x)^2 \equiv \langle x^2 \rangle - \langle x \rangle^2$ and $(\Delta p)^2 \equiv \langle p^2 \rangle - \langle p \rangle^2$ are obtained as

\[
(\Delta x)^2 = \frac{1}{2} + \frac{ML_{M-1}(-\lambda^2)}{L_M(-\lambda^2)} + \frac{\lambda^2 L_{M-2}^{(2)}(-\lambda^2)}{L_M(-\lambda^2)} - 2 \left[ \frac{\lambda L_{M-1}^{(1)}(-\lambda^2)}{L_M(-\lambda^2)} \right]^2, \tag{3.10}
\]

\[
(\Delta p)^2 = \frac{1}{2} + \frac{ML_{M-1}(-\lambda^2)}{L_M(-\lambda^2)} - \frac{\lambda^2 L_{M-2}^{(2)}(-\lambda^2)}{L_M(-\lambda^2)}. \tag{3.11}
\]

If $(\Delta x)^2 < 1/2$ (or $(\Delta p)^2 < 1/2$), we say the state is squeezed in the quadrature $x$ (or $p$).

Fig.2 is a plot showing how the variance $(\Delta x)^2$ depends on the parameters $\eta$ and $M$. When $\eta = 0$, $(\Delta x)^2 = 1/2$ since the state is just the vacuum state and in this case the field is not squeezed. Then, as $\eta$ increases the field becomes squeezed until maximum squeezing is reached; then the squeezing decreases until it disappears at a point $\eta_0$ depending on $M$. We note that $\eta_0 < 1$ when $M > 0$ since $(\Delta x^2) = M + 1/2 > 1/2$ when $\eta \to 1$.

We also find from Fig.2 that the larger $M$, the stronger the squeezing, and the wider the squeezing range.

### 3.3 Optimum signal-to-quantum noise ratio

It is known that the optimum signal-to-quantum noise ratio for an arbitrary quantum state

\[
\rho = \frac{\langle x \rangle^2}{(\Delta x)^2} \tag{3.12}
\]

has the value $4N_s(N_s + 1)$ which is attainable for the usual coherent squeezed state and that the optimum ratio for the coherent state is $4N_s$ ($N_s$ is the mean value of the number operator $N$ for the quantum state) [11].

For the interpolating number-coherent state $|\eta, M\rangle$, the signal-to-quantum noise ratios for different parameters $\eta$ and $M$ are shown in Fig. 3. The ratio for $\eta = 0$ and
\( \eta = 1 \), which correspond to the vacuum state and number state respectively, is zero. For other \( \eta \), we find from Fig. 3 (a) that the larger \( M \), the larger the ratio. Fig. 3 (b) gives plots of \( 4\langle N \rangle(\langle N \rangle + 1) \) \( \langle N \rangle \) is given by Eq. (3.2)), 4\( \langle N \rangle \) and the ratio for the state \( \| \eta, M \rangle \) with \( M = 10 \). We find that

1) the ratio for \( \| \eta, M \rangle \) is always smaller than the value \( 4\langle N \rangle(\langle N \rangle + 1) \), which is in accord with the general result[11];

2) for some values of \( \eta \) the ratio is larger than \( 4\langle N \rangle \). All the states with ratio larger than \( 4\langle N \rangle \) correspond to squeezed states (see Fig. 2.).

4 Quasi-probability distributions

Quasi-probability distributions [12] in the coherent state basis turn out to be useful measures for studying the nonclassical features of radiation fields. In this section we shall study the Q-function and Wigner functions of the state \( \| \eta, M \rangle \). We find that the Q-function and Wigner functions of \( \| \eta, M \rangle \) are simply a displacement of those of the photon-added coherent states.

In fact, we can prove a more general theorem: if two states \( |\psi\rangle_\alpha \) and \( |\psi\rangle \) satisfy

\[ |\psi\rangle_\alpha = D(\alpha)|\psi\rangle, \]

where \( D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \) is the displacement operator, the Q and Wigner function of \( |\psi\rangle_\alpha \) are simply a displacement of those of \( |\psi\rangle \), namely,

\[ Q(\beta)|\psi\rangle_\alpha = Q(\beta - \alpha)|\psi\rangle, \quad W(\beta)|\psi\rangle_\alpha = W(\beta - \alpha)|\psi\rangle. \tag{4.1} \]

For the Q function, we prove it as following:

\[
Q(\beta)|\psi\rangle_\alpha = |\langle \beta | D(\alpha)|\psi\rangle|^2 = |\langle 0 | D(-\beta) D(\alpha)|\psi\rangle|^2 = |\langle \beta - \alpha |\psi\rangle|^2 = Q(\beta - \alpha)|\psi\rangle, \tag{4.2}
\]

where we have used the relation

\[ D(\delta) D(\gamma) = D(\delta + \gamma) e^{\frac{i}{2}(\delta \gamma^* - \gamma \delta^*)} = D(\delta + \gamma) e^{\text{Im}(\delta \gamma^*)}, \tag{4.3} \]

for arbitrary complex numbers \( \delta \) and \( \gamma \). From the following definition of the Wigner function [13]

\[ W(\beta) = \frac{1}{\pi} \sum_{k=0}^{\infty} \langle \beta, k | \rho | \beta, k \rangle, \tag{4.4} \]

where \( |\beta, k \rangle = D(\beta)|k\rangle = e^{\beta a^\dagger - \beta^* a}|k\rangle \) is the displaced number state (\( |k\rangle \) is the number state) and \( \rho = |\eta, M\rangle \langle \eta, M | \) is the density matrix of the states considered, we can prove the second relation in (4.1) in the same way as in the Q function case.
For the current case, we can easily obtain the Q-function and the Wigner function of the state $|\eta, M\rangle$ from those of the photon-added coherent states given in [5], using Eq. (2.13)

$$Q(\beta) = |\langle \beta | M, \eta \rangle|^2 = e^{-|\beta|^2} \frac{\lambda + |\beta|^2 M}{M! L_M(-\lambda^2)}.$$  

$$W(\beta) = \frac{2(-1)^M L_M(|2\beta + \lambda|^2)}{\pi L_M(-\lambda^2)} \exp\left(-2|\beta|^2\right).$$

We now investigate numerically the behaviour of the Q-function and the Wigner function for different $\eta$ and $M$.

Fig.4 gives the Q-function for fixed $M = 10$ and different $\eta = 0, 0.2, 0.4, 0.6$. We can clearly see increasing deformation of the Q function with $\eta$.

From the Q-function we can also study squeezing properties by examining the deformation of its contours. Fig.5 is the contour plot of Q functions for $M = 10$ and different $\eta$. We see that, when we increase $\eta$, the contour is squeezed in the $x$ direction until a maximum squeezing is reached. Then the contour deforms to the shape of a banana, which occupies a wider range in the $x$ direction and the squeezing is reduced. Finally, we obtain a circular contour for larger $\eta$ corresponding to no squeezing (c.f. Fig. 2).

Fig.6 are plots of the Wigner function of $|\eta, M\rangle$ for $M = 3$ and different $\eta$. The case $\eta = 0$ corresponds to the vacuum state and its Wigner function is simply a Gaussian centered at the origin. As $\eta$ increases from 0, this Gaussian distribution continuously deforms to the Wigner function of the number state $|3\rangle$. From about $\eta = 0.4$ onwards, the negative parts of the Wigner functions are very clearly visible and this signifies nonclassical behaviour.

## 5 Generation of interpolating states

The main difference between our states and the photon-added coherent states is that ours are a finite superposition of number states. This suggests the possibility of an experimental generation of these states using the method proposed in [14].

We can also generate the state $|\eta, M\rangle$ by the interaction of a photon and a two-level atom with an external classical driving field in a cavity. In the rotating wave approximation, the Hamiltonian $(\hbar = 1)$ is

$$H = H_0 + V,$$

$$H_0 = \omega N + A(a^\dagger + a) + \frac{1}{2} \omega' \sigma_3,$$

$$V = g(a^\dagger \sigma_- + a \sigma_+).$$

(5.1)
where $\sigma_3 = |e\rangle \langle e| - |g\rangle \langle g|$, $\sigma_+ = |e\rangle \langle g|$ and $\sigma_- = |g\rangle \langle e|$ are atomic operators, $g$ is coupling constant, $\omega$ is the cavity frequency, $\omega'$ is the atomic frequency, and $A$ is the driving field frequency. Then, in the interaction picture, the interaction Hamiltonian is

$$H_I(t) = U_0^{-1}(t)VU_0(t), \quad U_0(t) = e^{-iH_0t} = e^{-i\omega tN - iAt(a^+ + a)} e^{-i\omega't\sigma_3}. \quad (5.2)$$

Using the following relation (see Appendix)

$$U_0^{-1}(t)aU_0(t) = e^{-i\omega t} D(-A/\omega)aD(A/\omega), \quad (5.3)$$

where $D(A/\omega)$ is the displacement operator, we have

$$H_I(t) = gD(-A/\omega)(a^+ \sigma_- + a \sigma_+)D(A/\omega). \quad (5.4)$$

Now we consider the on-resonance case, $\omega = \omega'$. Then the interaction Hamiltonian is time-independent

$$H_I = gD(-A/\omega)(a^+ \sigma_- + a \sigma_+)D(A/\omega) \quad (5.5)$$

and therefore its time evolution operator is

$$U_I(t) = e^{-iH_1t} = D(-A/\omega)e^{-igt(a^+ \sigma_- + a \sigma_+)}D(A/\omega). \quad (5.6)$$

Suppose that the field is initially prepared in the vacuum state $|0\rangle$ and the atom in the excited state $|e\rangle$; namely, at $t = 0$, the system is in the state $|0\rangle \otimes |e\rangle$. Then at time $t$, we have

$$U_I(t)|0\rangle \otimes |e\rangle = D(-A/\omega)e^{-igt(a^+ \sigma_- + a \sigma_+)}D(A/\omega)|0\rangle \otimes |e\rangle. \quad (5.7)$$

When $gt \ll 1$, we have

$$U_I(t)|0\rangle \otimes |e\rangle = |0\rangle \otimes |e\rangle - igt \left[D(-A/\omega)a^+D(A/\omega)|0\rangle \otimes |g\rangle\right]. \quad (5.8)$$

If the atom is detected in the ground state $|g\rangle$, the field is reduced to the state $|\eta, 1\rangle$ with $\eta = \omega^2/(A^2 + \omega^2)$.

The state $|\eta, M\rangle$ ($M > 1$) can be generated by a multiphoton generalization of the Hamiltonian (5.1), namely, $V = g(a^{+M} \sigma_- + a^M \sigma_+)$.

Note that the parameter $A$ depends on the external driving field and is a tunable parameter. In particular, for large enough $M$, we can control the output state to be either a number or a coherent state by tuning the parameter $A$. However, for photon-added coherent states, which correspond to $A = 0$ and the initial state of the field the coherent state $|\alpha\rangle$ [5], we cannot obtain the coherent state limit by changing the parameter $\alpha$.  

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for a fixed $M \neq 0$ ($M$ is not a tunable parameter). So in this sense the photon-added coherent state of [5] is not an interpolating number-coherent state although it includes them as special cases.

Finally we may infer the presence of these new interpolating states in an idealized non-linear optics experiment. Consider a nonlinear Mach-Zehnder interferometer with a Kerr medium in one arm. The output state is the displaced Kerr state [15]

$$D(\xi)U_K(\gamma)|\lambda\rangle, \quad U_K(\gamma) \equiv \exp \left( \frac{i}{2} \gamma a^\dagger a^2 \right),$$

(5.9)

where $D(\xi)$ is the displacement operator and $\gamma \equiv 2\chi L/v$, $L$ is the length of the Kerr medium, $v$ the appropriate phase velocity inside the medium and $\chi$ the third-order susceptibility. When $\xi = -\lambda$, and $\gamma$ is small enough, the above states can be approximated as

$$|0\rangle + \frac{i}{2} \gamma \lambda^2 |2, \lambda\rangle.$$  

(5.10)

If we use a $(2S + 1)$th-order nonlinear Kerr medium modelled in the interaction picture by [16]

$$H_{Kerr} = \frac{\hbar \gamma_S}{(S + 1)!} (a^\dagger)^{S+1} a^{(S+1)} = \frac{\hbar \gamma_S}{(S + 1)!} N(N - 1) \cdots (N - S),$$

(5.11)

we can find $|\eta, M\rangle$ when $\gamma_S$ is small enough.

6 Conclusion

We have described new states which interpolate between number and coherent states and have investigated their various properties. The relationship to photon-added coherent states was clarified and the limits to coherent and number states were analyzed. Salient statistical properties of $|\eta, M\rangle$ such as the sub-Poissonian distribution, the anti-bunching effect and the squeezing effects were investigated for a wide range of parameters. The non-classical features of the $|\eta, M\rangle$ for certain parameter ranges were demonstrated in terms of the quasiprobability distributions, the Q and Wigner functions. Finally, we proposed an experiment to generate these states, inferring their presence in certain non-linear systems. These remarkable properties leads us to believe that the states found in this paper may play an important role in quantum optics.

Acknowledgement

H. Fu is supported in part by the National Natural Science Foundation of China. We would like to thank Dr.A. Greentree of the Quantum Processes Group of the Open Uni-
versity for valuable discussions of the production of these states, and the referee for raising this point.

References


Appendix: Proof of Formula (5.9)

In this appendix we give a proof of Eq.(5.3). We use the following formula

\[ e^{-F}Ge^{F} = \sum_{n=0}^{\infty} \left( \frac{-1}{n!} \right)^n \left[ F, \underbrace{[F, \cdots, [F, G]]}_{n \text{ copies}} \right]. \] (6.1)

For the case in hand

\[ F = -i\omega tN - itA(a^\dagger + a), \quad G = a. \] (6.2)

It is easy to see that

\[ [F, G] = i\omega ta + iAt, \]
\[ [F, [F, G]] = i\omega t[F, G], \]
\[ [F, [F, [F, G]]] = i\omega t[F, [F, G]] = (i\omega t)^2[F, G], \]
\[ \ldots \ldots \]
\[ \underbrace{[F, [F, \cdots, [F, G]]}_{n \text{ copies}} = (i\omega t)^n[F, G] = (i\omega t)^n a + (i\omega t)^n A/\omega = (i\omega t)^n D(-A/\omega)aD(A/\omega) \] (6.3)

where \( D(A/\omega) \) is the displaced operator. Substituting Eq. (6.3) into Eq. (6.1) we obtain the formula (5.3).
Figure 1: Mandel’s Q parameter for \( M = 2, 50, 100 \).

Figure 2: Variance \((\Delta x)^2\) of \( \|\eta, M\rangle\) as a function of \( \eta \) for \( M = 2, 20, 50, \) and 200.

Figure 3: The signal-to-quantum noise ratio for \( \|\eta, M\rangle\): (a) The ratio for different \( M \); (b) Comparison of \( \rho, 4\langle N\rangle(\langle N\rangle + 1) \) and \( 4\langle N\rangle \) for \( M = 10 \).
Figure 4: $Q$ function of $\|\eta, M\|$. Here $\alpha = x + iy$. 

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\[ \eta = 0 \]
\[ \eta = 0.1 \]
\[ \eta = 0.4 \]
\[ \eta = 0.8 \]
Figure 5: Contours of Q-function of $\|\eta, M\rangle$. In all cases $M = 10$. $\alpha = x + iy$. 
Figure 6: Wigner function of $\eta, M\rangle$ for $M = 3$ and $\eta = 0.1, 0.4, 0.7$ and 1. $\alpha = x + iy$. 

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