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On Average Properties of Inhomogeneous Fluids in General Relativity I: Dust Cosmologies

by

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Summary: For general relativistic spacetimes filled with irrotational 'dust' a generalized form of Friedmann's equations for an 'effective' expansion factor $a_{\mathcal{D}}(t)$ of inhomogeneous cosmologies is derived. Contrary to the standard Friedmann equations, which hold for homogeneous–isotropic cosmologies, the new equations include the 'backreaction effect' of inhomogeneities on the average expansion of the model. A universal relation between 'backreaction' and average scalar curvature is also given. For cosmologies whose averaged spatial scalar curvature is proportional to $a_{\mathcal{D}}^{-2}$, the expansion law governing a generic domain can be found. However, as the general equations show, 'backreaction' acts as to produce average curvature in the course of structure formation, even when starting with space sections that are spatially flat on average.

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1. The Averaging Problem

A longstanding question in cosmology is, how to average a general inhomogeneous model (Ellis 1984). Also one would like to know under which assumptions, if any, the average variables obey Friedmann's equations lying at the basis of any theory for structure formation in the Universe. An answer for cosmologies containing a pressure–free fluid ('dust') has been given recently in the framework of Newtonian cosmology (Buchert & Ehlers 1997; see the references therein for the formulation of the averaging problem and earlier attempts to solve it). Some results relevant to the present work may be briefly summarized as follows: Consider any simply–connected, compact spatial domain in Euclidean space $\mathcal{D} \subset \mathbb{E}^3$ with volume $|\mathcal{D}| = V_{\mathcal{D}}$. Then, the spatial average of Raychaudhuri's equation for the evolution of the expansion rate, under the assumption of conservation of the domain's mass, yields an equation for the scale factor $a_{\mathcal{D}} \propto V_{\mathcal{D}}^{1/3}$ (which depends on content, shape and position of the spatial domain). This equation contains as source terms, besides the average mass density, averages over fluctuations of the shear, vorticity and expansion scalars due to the presence of inhomogeneities. These 'backreaction' terms vanish, if the average is performed over the whole universe having topologically closed space sections. For vanishing 'backreaction' this equation is equivalent to the standard Friedmann equation for homogeneous–isotropic universes.

In the present paper I provide the corresponding answer in the framework of general relativity. A geometrical relation, having no Newtonian analogue, delivers additional information that allows to obtain a single equation relating the 'backreaction' and average curvature terms. This equation holds for any spatial domain and for a large class of inhomogeneous cosmologies with curved space sections without perturbative assumptions. The solution of the 'backreaction problem' for scalar characteristics can be found in the case of space sections whose Ricci scalar averages out to zero, or, displays a dependence $\propto a_{\mathcal{D}}^{-2}$ as in the standard model, which implies that the averaged scalar curvature decouples from the 'backreaction' term. However, in general, 'backreaction' due to the presence of inhomogeneities impacts on the averaged Ricci scalar in the course of structure formation. A general solution to this problem would incorporate a scale–dependent description of the density, the expansion and other scalar variables of any structure formation model.

In Section 2 we proceed by prescribing the basic equations and the averaging procedure; then we give the general equations governing the domain–dependent scale factor $a_{\mathcal{D}}(t)$ in a *Theorem*. An equivalent set of these equations presented in *Corollary 1* demonstrates the surprising result that, inspite of the non–commutativity of averaging and time evolution, the averaged quantities obey the same equations as the local ones. Inferred from the averaged equations *Corollary 2* defines a set of average characteristics in analogy to the cosmological parameters of the standard model. A compact form of the previous results is presented in *Corollary 3*, which displays a universal relation between average scalar curvature and 'backreaction'. Finally, in Section 3, we discuss some immediate consequences of this result, and comment on related work. In Appendix A we present the Newtonian analogues, while Appendix B gives some illustrative examples of expansion laws. Appendix C is dedicated to an alternative derivation of the averaged equations.

2. Averaging Einstein's Equations for Scalars

2.1. Choice of foliation and basic equations

We shall restrict ourselves to the case of irrotational fluid motion with the simplest matter model 'dust' (i.e. vanishing pressure). In this case the flow is geodesic and space–like hypersurfaces can be constructed that are flow–orthogonal at every spacetime event in a $3+1$ representation of Einstein's equations.

We start with Einstein's equations[1]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \varrho u_\mu u_\nu - \Lambda g_{\mu\nu} \quad , \tag{1a}$$

with the Ricci tensor $R_{\mu\nu}$, its trace R, the fluid's 4−velocity u^μ ($u^\mu u_\mu = -1$), the cosmological constant Λ, and the rest mass density ϱ obeying the conservation law

$$(\varrho u^\mu u^\nu)_{;\mu} \; = \; 0 \quad . \tag{1b}$$

We choose a flow–orthogonal coordinate system $x^\mu = (t, X^k)$ (i.e., Gaussian or normal coordinates which are comoving with the fluid). Writing $x^\mu = f^\mu(X^k, t)$ we have $u^\mu = \dot{f}^\mu = (1,0,0,0)$ and $u_\mu = \dot{f}_\mu = (-1,0,0,0)$, where the dot denotes partial derivative with respect to proper time t.

These coordinates are defined such as to label geodesics in spacetime, i.e., $u^\nu u^\mu{}_{;\nu} = 0$. With the choice of vanishing 3−velocity the coordinates are in addition chosen to be *comoving*. Thus, in a $3+1$–splitting of spacetime, the spatial set of Gaussian coordinates also labels fluid elements or trajectories in 3–space, $\dot{X}^k = 0$; we are entitled to call X^k *Lagrangian coordinates*, because they are identical to those in classical fluid dynamics. It should be emphasized, however, that the final result will be covariant with respect to the given foliation of spacetime and thus not dependent on this particular choice of coordinates.

Let (t, X^k) be the independent variables. As dependent variables we may choose the spatial 3−metric g_{ij} (the first fundamental form of the hypersurfaces of constant t) in the line element

$$ds^2 = -dt^2 + g_{ij} dX^i dX^j \quad , \tag{1c}$$

the extrinsic curvature tensor $K_{ij} := -h^\alpha{}_i h^\beta{}_j u_{\alpha;\beta}$ (the second fundamental form of the hypersurfaces of constant t) with the projection tensor into the hypersurfaces orthogonal to u_α, $h_{\alpha\beta} := g_{\alpha\beta} + u_\alpha u_\beta$, and the rest mass density ϱ. Einstein's equations (1a) together with the continuity equation (1b) (contracted with u_ν) then are equivalent to the following

[1] Greek indices run through 0...3, while latin indices run through 1...3; summation over repeated indices is understood. A semicolon will denote covariant derivative with respect to the 4–metric with signature $(-,+,+,+)$; the units are such that $c = 1$.

system of equations (see, e.g., Arnowitt et al. 1962, York 1979) consisting of the *constraint equations*[2]

$$\frac{1}{2}\left(\mathcal{R}+K^2-K^i{}_jK^j{}_i\right)=8\pi G\varrho+\Lambda\ \ , \tag{2a}$$

$$K^i{}_{j||i}-K_{|j}=0\ \ , \tag{2b}$$

and the *evolution equations* for the density and the two fundamental forms:

$$\dot{\varrho}=K\varrho\ \ , \tag{2c}$$

$$\left(g_{ij}\right)^{\bullet}=-2\ g_{ik}K^k{}_j\ \ , \tag{2d}$$

$$\left(K^i{}_j\right)^{\bullet}=KK^i{}_j+\mathcal{R}^i{}_j-(4\pi G\varrho+\Lambda)\delta^i{}_j\ \ . \tag{2e}$$

$\mathcal{R}:=\mathcal{R}^i{}_i$ and $K:=K^i{}_i$ denote the traces of the spatial Ricci tensor $\mathcal{R}_{ij}$ and the extrinsic curvature tensor K_{ij}, respectively. Expressing the latter in terms of kinematical quantities,

$$-K_{ij}=\Theta_{ij}=\sigma_{ij}+\frac{1}{3}\theta g_{ij}\quad;\quad -K=\theta\quad , \tag{3}$$

with the expansion tensor Θ_{ij}, the trace–free symmetric shear tensor σ_{ij} and the expansion rate θ, we may write Eqs. (2) in the form

$$\frac{1}{2}\mathcal{R}+\frac{1}{3}\theta^2-\sigma^2=8\pi G\varrho+\Lambda\ \ , \tag{4a}$$

$$\sigma^i{}_{j||i}=\frac{2}{3}\theta_{|j}\ \ , \tag{4b}$$

$$\dot{\varrho}=-\theta\varrho\ \ , \tag{4c}$$

$$\left(g_{ij}\right)^{\bullet}=2\ g_{ik}\sigma^k{}_j+\frac{2}{3}\theta g_{ik}\delta^k{}_j\ \ , \tag{4d}$$

$$\left(\sigma^i{}_j\right)^{\bullet}=-\theta\sigma^i{}_j-\mathcal{R}^i{}_j+\frac{2}{3}\delta^i{}_j\left[\sigma^2-\frac{1}{3}\theta^2+8\pi G\varrho+\Lambda\right]\ \ , \tag{4e}$$

where we have introduced the rate of shear $\sigma^2:=\frac{1}{2}\sigma^i{}_j\sigma^j{}_i$. To derive this last equation we have used Raychaudhuri's equation

$$\dot{\theta}+\frac{1}{3}\theta^2+2\sigma^2+4\pi G\varrho-\Lambda\ =\ 0\ \ , \tag{5a}$$

which follows from the trace of Eq. (4e) combined with the constraint (4a). Using again the constraint (4a), we may cast the trace–free part (4e) into the form

$$\left(\sigma^i{}_j\right)^{\bullet}+\theta\sigma^i{}_j=-\left(\mathcal{R}^i{}_j-\frac{1}{3}\delta^i{}_j\mathcal{R}\right)\ \ . \tag{5b}$$

[2]In Eq. (2b) $_{||}$ denotes covariant derivative with respect to the 3−metric g_{ij}, while a single vertical slash denotes partial derivative with respect to the Lagrangian coordinates X^i; note that in the present case the covariant spatial derivative of a scalar reduces to the partial derivative. The overdot denotes partial time–derivative (at constant X^i) as before.

This set of equations has recently been discussed in connection with perturbation theory by Kasai (1995), Matarrese (1996, and ref. therein) and by Matarrese & Terranova (1996), as well as in the papers by Russ et al. (1996, 1997). Here, we proceed without perturbation theory.

Taking the trace of Eq. (2d), written in the form

$$K^i{}_j = -\frac{1}{2} g^{ik} \left(g_{kj} \right)^{\bullet} \quad ,$$

and defining

$$J(t, X^i) := \sqrt{\det(g_{ij})} \quad , \tag{6a}$$

we obtain with $\frac{1}{2} g^{ik} \left(g_{ki} \right)^{\bullet} = (\ln J)^{\bullet}$ the identity

$$\dot{J} = -KJ = \theta J \quad . \tag{6b}$$

Using it we can integrate the continuity equation for the rest mass density (2c) along the flow lines:

$$\varrho(t, X^i) = (\varrho(t_0, X^i) J(t_0, X^i)) J^{-1} \quad . \tag{7}$$

Both, Raychaudhuri's equation (5a) and the integral of the continuity equation (7) are identical to their Newtonian counterparts (Buchert & Ehlers 1997). Below we shall make explicit use of the constraint (4a), which has no Newtonian analogue. This equation will provide a key element for the understanding of the 'backreaction' problem.

For later discussion it is convenient to also introduce the abbreviations **I** and **II** for two of the scalar invariants of the expansion tensor, its trace,

$$\mathbf{I} := \Theta^{\ell}{}_{\ell} = \theta \quad , \tag{8a}$$

and the dispersion of its diagonal components,

$$\mathbf{II} := \frac{1}{2} \left(\theta^2 - \Theta^{\ell}{}_k \Theta^k{}_{\ell} \right) = \frac{1}{3} \theta^2 - \sigma^2 \quad . \tag{8b}$$

2.2. Averaging the traces of Einstein's equations

Spatially averaging equations for scalar fields is a covariant operation given a foliation of spacetime. Therefore, as we already pointed out in the paper on averaging the Newtonian equations (Buchert & Ehlers 1997), we may average, e.g., Raychaudhuri's equation (5a) in full formal analogy to the Newtonian case.

Let us define the averaging operation. Spatial averaging of a scalar field Ψ as a function of Lagrangian coordinates and time on an arbitrary compact portion of the fluid $\mathcal{D}$ is straighforward[3] and is defined by the volume integral

$$\langle \Psi(t, X^i) \rangle_{\mathcal{D}} := \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} J d^3X \;\; \Psi(t, X^i) \quad , \tag{9a}$$

with the volume element $dV := Jd^3X$ of the spatial hypersurfaces of constant time. The volume itself is given by[4]

$$V_{\mathcal{D}}(t) := \int_{\mathcal{D}} J d^3X \quad . \tag{9b}$$

We also introduce a dimensionless ('effective') scale factor via the volume (normalized by the volume of the initial domain $V_{\mathcal{D}_o}$),

$$a_{\mathcal{D}}(t) := \left(\frac{V_{\mathcal{D}}(t)}{V_{\mathcal{D}_o}} \right)^{1/3} \quad . \tag{9c}$$

Thus, the averaged expansion rate may be written in terms of the scale factor:

$$\langle \theta \rangle_{\mathcal{D}} = \frac{\dot{V}_{\mathcal{D}}}{V_{\mathcal{D}}} = 3 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \quad . \tag{9d}$$

The integral (7) states the conservation of the total rest mass $M_{\mathcal{D}}$ within a portion of the fluid $\mathcal{D}$ as it is transported along the flow lines,

$$M_{\mathcal{D}} = \int_{\mathcal{D}} J d^3X \, \varrho \;\; = \;\; const. \;\; \Leftrightarrow \;\; \langle \varrho \rangle_{\mathcal{D}} = \frac{M_{\mathcal{D}}}{V_{\mathcal{D}_o} a_{\mathcal{D}}^3} \quad . \tag{9e}$$

Employing this averaging procedure we may easily prove many statements found in (Buchert & Ehlers 1997) which also hold in general relativity. From these results we are going to use the 'Commutation rule' (here written for an arbitrary scalar field Ψ):

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[3]This averaging method functionally depends on content, shape and position of the spatial domain of averaging, which we consider as being given (see: Stoeger et al. 1999 for alternative averagers). It should be stressed that we do not attempt to average the spacetime geometry as a whole; useful information for cosmology may be already obtained by averaging the scalar parts.

[4]In comparison with the Newtonian definition $V_{\mathcal{D}} = \int_{\mathcal{D}(t)} d^3x = \int_{\mathcal{D}(t_0)} J d^3X$, where x_i are Eulerian and X_i Lagrangian coordinates, our domain $\mathcal{D}$ corresponds to a Lagrangian domain, because it is transported along geodesics; however, in contrast to the Newtonian case, it is implicitly time–dependent due to the evolution of the 3–metric.

Lemma (*Commutation rule*)

$$\langle\Psi\rangle^{\bullet}_{\mathcal{D}} - \langle\dot{\Psi}\rangle_{\mathcal{D}} = \langle\Psi\theta\rangle_{\mathcal{D}} - \langle\Psi\rangle_{\mathcal{D}}\langle\theta\rangle_{\mathcal{D}} \quad . \tag{9f}$$

Averaging the Hamiltonian constraint (4a) and Raychaudhuri's equation (5a) with the help of the prescribed procedure, we end up with the following two equations, which we may summarize in the form of a theorem.

Theorem (*Equations for the effective scale factor*)

The spatially averaged equations for the scale factor $a_{\mathcal{D}}$, respecting mass conservation, read:

averaged Raychaudhuri equation:

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G\frac{M_{\mathcal{D}}}{V_{\mathcal{D}_o}a_{\mathcal{D}}^3} - \Lambda \;=\; \mathcal{Q}_{\mathcal{D}} \quad ; \tag{10a}$$

averaged Hamiltonian constraint:

$$3\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 - 8\pi G\frac{M_{\mathcal{D}}}{V_{\mathcal{D}_o}a_{\mathcal{D}}^3} + \frac{1}{2}\langle\mathcal{R}\rangle_{\mathcal{D}} - \Lambda = -\frac{\mathcal{Q}_{\mathcal{D}}}{2} \quad , \tag{10b}$$

where the mass $M_{\mathcal{D}}$, the averaged spatial Ricci scalar $\langle\mathcal{R}\rangle_{\mathcal{D}}$ and the 'backreaction' $\mathcal{Q}_{\mathcal{D}}$ are domain–dependent spatial constants and, except the mass, time–dependent functions. In particular, the 'backreaction' source term is given by

$$\mathcal{Q}_{\mathcal{D}} := 2\langle\mathbf{II}\rangle_{\mathcal{D}} - \frac{2}{3}\langle\mathbf{I}\rangle_{\mathcal{D}}^2 \;=\; \frac{2}{3}\langle(\theta - \langle\theta\rangle_{\mathcal{D}})^2\rangle_{\mathcal{D}} - 2\langle\sigma^2\rangle_{\mathcal{D}} \quad . \tag{10c}$$

We also note the following surprising property of the averaged equations compared with their local forms: in spite of the non–commutativity of the averaging procedure and the dynamical evolution, which is expressed by the Commutation rule (9f), we find that the same equations hold for the averaged and the local quantities provided we express them in terms of the invariants (8). This establishes the following

Corollary 1 (*Averaged equations*)

The spatial averages of the Hamiltonian constraint (4a), the continuity equation (4c) and Raychaudhuri's equation (5a) read:

$$\frac{1}{2}\langle\mathcal{R}\rangle_{\mathcal{D}} = 8\pi G\langle\varrho\rangle_{\mathcal{D}} + \Lambda - \langle\mathbf{II}\rangle_{\mathcal{D}} \quad , \tag{11a}$$

$$\langle\varrho\rangle^{\bullet}_{\mathcal{D}} = -\langle\theta\rangle_{\mathcal{D}}\langle\varrho\rangle_{\mathcal{D}} \quad , \tag{11b}$$

$$\langle\theta\rangle^{\bullet}_{\mathcal{D}} = -\langle\theta\rangle^2_{\mathcal{D}} + \Lambda - 4\pi G\langle\varrho\rangle_{\mathcal{D}} + 2\langle\mathbf{II}\rangle_{\mathcal{D}} \quad , \tag{11c}$$

i.e., the averages $\langle\varrho\rangle_{\mathcal{D}}$, $\langle\theta\rangle_{\mathcal{D}}$, $\langle\mathcal{R}\rangle_{\mathcal{D}}$ and $\langle\mathbf{II}\rangle_{\mathcal{D}}$ obey the same equations as the local fields ϱ, θ, $\mathcal{R}$ and $\mathbf{II}$. (The reason for this nontrivial property is the special type of nonlinearities featured by the gravitational system, e.g. the nonlinearity in θ contained in Raychaudhuri's equation.)

Corollary 2 (*Dimensionless characteristics of inhomogeneous cosmologies*)

As in the standard homogeneous–isotropic cosmologies we may introduce a domain–dependent Hubble function $H_{\mathcal{D}} := \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}$, and dimensionless average characteristics as follows:

$$\Omega_m := \frac{8\pi G M_{\mathcal{D}}}{3 V_{\mathcal{D}_o} a_{\mathcal{D}}^3 H_{\mathcal{D}}^2} \quad , \tag{12a}$$

$$\Omega_\Lambda := \frac{\Lambda}{3 H_{\mathcal{D}}^2} \quad , \tag{12b}$$

$$\Omega_k := -\frac{\langle \mathcal{R} \rangle_{\mathcal{D}}}{6 H_{\mathcal{D}}^2} \quad , \tag{12c}$$

$$\Omega_{\mathcal{Q}} := -\frac{\mathcal{Q}_{\mathcal{D}}}{6 H_{\mathcal{D}}^2} \quad , \tag{12d}$$

which, in view of (10b), obey

$$\Omega_m \; + \; \Omega_\Lambda \; + \; \Omega_k \; + \; \Omega_{\mathcal{Q}} \; = 1 \quad . \tag{12e}$$

All these dimensionless "cosmological parameters" actually depend on the spatial scale of averaging including the dimensionless cosmological constant, which depends on scale through $H_{\mathcal{D}}$.

The equations (10a,b) form a system of two equations for the three unknown variables $a_{\mathcal{D}}$, $\langle \mathcal{R} \rangle_{\mathcal{D}}$ and $\mathcal{Q}_{\mathcal{D}}$. Therefore, we cannot solve the 'backreaction' problem for scalars based on this system. We may eliminate the 'backreaction' term from (10a) and insert it into (10b). This results in an equation for the scale factor $a_{\mathcal{D}}$ and the average Ricci scalar of the domain. Alternatively we may proceed as follows: we calculate the time–derivative of Eq. (10b) and insert into the resulting equation (10a) and (10b). This yields a universal relation between the averaged Ricci scalar and the 'backreaction' term:

Corollary 3 (*Relation between average scalar curvature and 'backreaction'*)

A necessary condition of integrability of Eq. (10a) to yield Eq. (10b) is provided by the relation:

$$\mathcal{Q}_{\mathcal{D}}^{\bullet} + 6 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \mathcal{Q}_{\mathcal{D}} + \langle \mathcal{R} \rangle_{\mathcal{D}}^{\bullet} + 2 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \langle \mathcal{R} \rangle_{\mathcal{D}} \; = \; 0 \quad , \tag{13a}$$

or, equivalently,

$$\left(a_{\mathcal{D}}^6 \mathcal{Q}_{\mathcal{D}} \right)^{\bullet} + a_{\mathcal{D}}^4 \left(a_{\mathcal{D}}^2 \langle \mathcal{R} \rangle_{\mathcal{D}} \right)^{\bullet} \; = \; 0 \quad . \tag{13b}$$

In Appendix B we give some examples of expansion laws that can be derived from this relation.

Notes:

Eq. (10a) has been already confirmed as a valid equation in general relativity in the work on averaging the Newtonian equations (Buchert & Ehlers 1997); an equation analogous to Eq. (10b) has been derived by Carfora & Piotrkowska (1995) in connection with manifold deformations at one instant using the constraint equations of general relativity. In our derivation of Eq. (10b) we have inserted the 'backreaction' term $\mathcal{Q}_{\mathcal{D}}$ back into the constraint equation (4a) and have used (9d).

Russ et al. (1997) have also used the equations (10a,b) (in a truncated form and using a reference background solution) for the purpose of perturbative calculations of the 'backreaction' effect. Note that, contrary to their derivation, we have not performed a conformal rescaling of the metric, nor have we used the splitting into a background reference solution and deviations thereof. We postpone further comments on their work to Section 3.

3. Discussion and Perspectives

3.1. Summary of results

We have derived a generalized form of Friedmann's differential equations including 'backreaction terms' due to the presence of inhomogeneities. One of these equations was obtained on the basis of averaging Raychaudhuri's equation on spatial domains whose mass content is preserved in time. It is formally identical to the equation derived in the framework of Newtonian cosmology (Appendix A). The other equation arises by averaging the Hamiltonian constraint (having no Newtonian analogue). It delivers an additional relation between the averaged spatial Ricci curvature scalar and the 'backreaction term'. We have shown that there exist special solutions which describe the evolution of the average curvature and 'backreaction' terms (Appendix B).

Let us list some immediate conclusions which may be drawn:

1. The average expansion of inhomogeneous cosmologies does, in general, not follow the expansion law of the standard FRW cosmologies. There are, however, generalized expansion laws which govern the motion of arbitrary spatial domains, provided assumptions on the relation between the averaged Ricci scalar and the 'backreaction term' are made. Here, perturbation theory would be useful to establish such relations.

2. The general expansion law shows that 'backreaction' due to the presence of inhomogeneities implies the existence of a non–vanishing average Ricci scalar in general situations. This is true even if we consider domains which are on average Ricci flat at some initial instant.

Some comments about these conclusions are in order.

*ad*1. We may stipulate that the assumption of vanishing average Ricci scalar could be a sensible one, if we consider *typical* portions of the Universe (which itself may have on average flat space sections to a good approximation). As demonstrated in Appendix B a solution can be obtained in this case and the evolution of the average expansion is then exactly known. Consideration of the general case reveals, however, that this point of view is too naive: Looking at Eq. (10b) we must expect that the dimensionless contribution to the averaged scalar curvature $\langle\mathcal{R}\rangle_{\mathcal{D}}/G\langle\varrho\rangle_{\mathcal{D}}$ might be of the same order as the dimensionless contribution to the 'backreaction' $\mathcal{Q}_{\mathcal{D}}/G\langle\varrho\rangle_{\mathcal{D}}$. Approximating the average curvature by zero relies on a similar prejudice as saying that 'backreaction' may be neglected. *Both approximations imply restrictions on general inhomogeneous models.*

*ad*2. Here, we may imagine the likely situation that an initially critical universe in the sense of the Einstein–de Sitter model may evolve into an under– or overcritical universe, respectively, in the course of structure formation. Hence, it is possible that a Ricci flat universe develops into a universe with on average negative/positive spatial curvature at the present epoch. From the point of view of the standard inflationary paradigm the former situation is favoured, when the theoretical expectation of an on average Ricci flat universe at the exit epoch is compared with measurements of the density parameter at the present epoch.

For the general case a solution seems to lie beyond the scope of this article. Let us illustrate why the system of averaged equations (13a,b) cannot be closed on the level of scalars and also, how we may achieve closure by additional assumptions.

3.2. Attempting closure of the averaged equations

In order to obtain a more general result, we would like to find an independent evolution equation for the spatial Ricci scalar. Kasai (1995, appendix) has derived an evolution equation for the spatial Ricci tensor. His equation reads[5]:

$$\left(\mathcal{R}^i{}_j\right)^{\bullet} - 2K^i{}_\ell\mathcal{R}^\ell{}_j \;=\; -K^i{}_{\ell||j}{}^{||\ell} - K^\ell{}_j{}^{||i}{}_{||\ell} + K^i{}_j{}^{||\ell}{}_{||\ell} + K^\ell{}_\ell{}^{||i}{}_{||j} \quad . \tag{14a}$$

This relation is purely geometrical and makes no use of the field equations. Taking the trace of Eq. (14a) we first obtain

$$\frac{1}{2}\dot{\mathcal{R}} - K^\sigma{}_\ell\mathcal{R}^\ell{}_\sigma = -\left(K^\ell{}_{\sigma||\ell} - K_{|\sigma}\right)^{||\sigma} \quad . \tag{14b}$$

The r.–h.–s. of this equation vanishes according to the momentum constraints (2b). Using these constraints, the field equation (2e) to eliminate $\mathcal{R}_{ij}$ in favour of K_{ij} and the definition (8b) we obtain with $K_{ij} = -\Theta_{ij}$:

$$\frac{1}{2}\dot{\mathcal{R}} = \dot{\theta}\theta + \theta^3 - \dot{\mathbf{II}} - 2\theta\mathbf{II} - \theta(4\pi G\varrho + \Lambda) \;\; . \tag{14c}$$

Inserting Raychaudhuri's equation (5a) we find that Eq. (14c) is just the time–derivative of the Hamiltonian constraint (4a) combined with (4c). Hence, the trace of the evolution

[5]Note that we have used the canonical definition of the extrinsic curvature in this paper: $K_{ij} = -\Theta_{ij}$.

equation (14a) for the spatial Ricci tensor cannot be used to close the system of equations and to solve the 'backreaction' problem.

We may try to use Eq. (5b), which is an equation for the trace–free parts, and contract this equation into a relation among scalar quantities. Indeed, if we contract Eq. (5b) with ${\sigma^j}_i$ and eliminate the expression ${\mathcal{R}^i}_j {\sigma^j}_i$ from Eq. (14b) we obtain

$$\frac{1}{2}\dot{\mathcal{R}} + \frac{1}{3}\theta\mathcal{R} = (\sigma^2)^{\bullet} + 2\theta\sigma^2 \quad . \tag{14d}$$

Using the Hamiltonian constraint (4a), Eq. (14d) can be written as

$$\dot{\mathcal{R}} + \theta\mathcal{R} - 2\theta\Lambda + 2\dot{\mathbf{II}} + 2\theta\mathbf{II} \;=\; 0 \quad , \tag{14e}$$

which, however, can also be obtained by inserting the Hamiltonian constraint into its time–derivative.

These examples show that the use of any scalar part of Einstein's equations will not give a closed system of *ordinary* differential equations (see also Kofman & Pogosyan 1995). For the averaged variables we can also not expect this in view of *Corollary 1*: it states the equivalence of the averaged dynamics to the dynamics for the local field quantities. If this could be achieved in full generality, then this would be equivalent to having solved the full Einstein dynamics for the scalar parts, since the size of the domains could be arbitrarily chosen.

3.3. Expansion law for closed universe models ?

As discussed above an effort beyond the scope of ordinary differential equations for scalars is needed to close the system of equations for the average dynamics on any spatial domain.

Notwithstanding, such an effort can be successful, if further constraints on the average dynamics are considered, most notably integral constraints which restrict the morphological characteristics of domains. Among them the integrated curvatures and, in particular, topological constraints that arise by restricting the Euler–characteristic of the surfaces bounding the domains. We already implied the topological constraint that the domains over which we average have to be simply–connected. In Newtonian cosmology (Buchert & Ehlers 1997) we have established a global criterion: if we extend this simply–connected domain to the whole Universe having topologically closed space sections (e.g., toroidal models), then this results in $\mathcal{Q}_{\mathcal{D}} = 0$ on the closure scale. It is therefore to be expected that such a constraint may also close the system of averaged equations in general relativity. We do not necessarily have the simple Newtonian condition. *Corollary 3* states a general connection between $\mathcal{Q}_{\mathcal{D}}$ and $\langle\mathcal{R}\rangle_{\mathcal{D}}$: a vanishing 'backreaction' would imply that all the contributions of the local curvatures that are produced by the inhomogeneities obey the "conspiracy" to sum up to the standard value $\propto a(t)^{-2}$ (where $a(t)$ is a solution of a standard FRW cosmology).

In curved spacetimes it is not straightforward to establish such a constraint and the line of arguments given in the Newtonian treatment is not conclusive in the present context. To illustrate this statement let us look at the extrinsic curvature tensor according to its definition as the (4−dimensional) covariant spatial derivative of the 4–velocity. Invariants built from K_{ij} and, consequently, the expression $\mathcal{Q}_{\mathcal{D}}$, cannot be written as total covariant divergences of vector fields *in* the hypersurfaces. As an example we look at the trace of K_{ij} , $K = -u^{\alpha}{}_{;\alpha} = -\theta$; the value of K on the hypersurfaces is covariantly defined, but the vector field u^i vanishes according to our spacetime foliation. A similar problem arises in the case of the second invariant.

Forthcoming efforts should be directed towards finding a topological closure condition for the hypersurfaces in order to determine the global average properties of the world models. This problem is more involved, since we cannot expect that the domains remain simply–connected. Working in a 4–dimensional tube of spacetime that is bounded by space–like hypersurfaces and considering the limit of vanishing distance between these hypersurfaces, Yodzis (1974) attempted to derive average properties of *closed* space sections. His argument is reviewed in Appendix C, where it is shown that topological restrictions do not enter and his result holds for arbitrary compact and simply–connected domains.

We conclude:

3. We were not able to produce an argument analoguous to the Newtonian treatment stating that the 'backreaction term' vanishes for topologically closed space sections, if integrated over the whole space. Without such an argument averaged inhomogeneous cosmologies cannot be identified with the standard FRW cosmologies on any spatial scale. To justify this identification as an approximation there is presently no sufficiently general quantitative result as to whether the 'backreaction' term could be neglected on some scale or, in words suggested by *Corollary 3*, whether the averaged curvature decouples from the inhomogeneities.

3.4. Remarks on perturbation theory

I stated above that until present we don't know any quantitative calculation which may justify neglection of the 'backreaction' term on some scale. The reader may object that there exist several approximate calculations of the 'backreaction' effect in perturbation theory. However, there are severe obstacles for perturbative calculations which we are going to discuss now.

As an example I would like to comment on a recent calculation by Russ et al. (1997; see also the references therein): based on the system of equations (10a), (10b) (using a rescaling of the metric and a split into background and perturbations) the 'backreaction' term was calculated within a second–order perturbation approach. In order to make the calculations concrete, Russ et al. have assumed periodic boundary conditions on some (very large) domain. Looking at their expression for the 'backreaction' term (B4) it is evident from their Eqs. (B7) and (B8) that 'backreaction' (in the sense defined in the present paper) has to vanish identically: together with mass conservation (their Eq. (B10)) the introduction of periodic boundary conditions already leads to the result that the scale factor $a_{\mathcal{D}}$ obeys the standard Friedmann equations. (It should be noticed that $\mathcal{Q}_{\mathcal{D}} = 0$ is already *sufficient* to have $a_{\mathcal{D}}(t) = a(t)$.) Hence, according to *Corollary 3*, it is no surprise that the average Ricci scalar has to obey the expansion law of models with spatial Ricci scalar $\propto a^{-2}$ in any consistent treatment of periodic perturbations on spatially flat space sections. Therefore, Eq. (3.1) of Russ et al. (loc. cit.) cannot give any quantitative result about the *global* value of the 'backreaction' term: it vanishes by assumption and the scale factor is given by the standard FRW cosmologies. Note also that the introduction of a Fourier transform, or a decomposition into plane waves, respectively, is only meaningful in the case of spatially flat space sections, i.e., also the averaged Ricci scalar *has to* vanish. It must be noted that the notion of 'backreaction' as implied by Russ et al. (loc. cit.) is slightly different from that in the present work. Any departure from the *flat* FRW cosmology in *curvature* (quantified by perturbation theory) may already be interpreted as 'backreaction' (Kasai, priv. comm.).

This attempt illustrates the possibly cyclic nature of calculations of the 'backreaction' term: if we start with spatially flat space sections and a model for the inhomogeneous deformation tensor, the standard methods of treating the perturbations as periodic on some scale already constrain the cosmology to one without 'backreaction' (measuring the deviations from the family of FRW cosmologies). Note that in perturbation theory the first–order perturbations are sources of higher–order perturbations and, as demonstrated by Russ et al. (loc. cit.), a large class of periodic first–order and, in turn, a large class of second–order perturbations on a flat hypertorus average $\mathcal{Q}_{\mathcal{D}}$ to zero.

In a realistic situation the domains on which one averages are not on average Ricci flat. Large domains (e.g. of the size of the Universe) may not be easily compactified to make global statements about the evolution of the Universe: non–trivial topological spaceforms have to be considered and simple periodic boundary conditions are no longer useful. For some further remarks see (Buchert 1997).

3.5. Global structure versus local models

The insight gained from the set of generalized Friedmann equations (10) may be focussed in two ways: first, we are interested in the global structure of inhomogeneous cosmologies and, second, we would like to know more about average properties of spatial portions of the Universe without severely restricting the dynamical model.

As for the first point, a *globally* non–vanishing 'backreaction' that may be small at early epochs of the Universe' evolution could, on the scale of the size of the Universe, slowly build up due to an amplification of inhomogeneities. The more large–scale structure develops, the more the whole Universe might undergo global changes in morphology including the possibility of topology change. In order to analyze global changes during inflationary stages the present matter model 'dust' has to be generalized (which is the subject of a forthcoming work, Paper II). At later epochs constraints from the microwave background anisotropies can be used to give upper limits on the 'backreaction' characteristic $\Omega_{\mathcal{Q}}$ (Eq. 12d): if we accept that the microwave background dipol is only due to our proper motion against an isotropic CMB reference frame, then limits on the global shear parameter

$$\Sigma^2 := \frac{\langle \sigma^2 \rangle_{\rm CMB}}{3H^2_{\rm CMB}} \tag{15a}$$

may be related by the assumption that, on the CMB scale,

$$\theta \approx \langle \theta \rangle_{\rm CMB} \;\; ; \tag{15b}$$

then, on the this scale,

$$\Omega_{\mathcal{Q}} \approx \Sigma^2 \quad . \tag{15c}$$

Maartens et al. (1995) have given upper limits on the shear parameter for a Bianchi–type universe, in which case $\Sigma < 10^{-4}$ is a tight constraint on the global magnitude of the 'backreaction' characteristic at the epoch of last scattering (see also: Wainwright & Ellis 1997). As a disclaimer we note that an average over inhomogeneous models is performed in Eq. (15a), and the average model must not necessarily isotropize as the Bianchi–type models (except type IX) do. Therefore, care must be taken in using such constraints at times after last scattering.

As for the second point, the expansion laws discussed in Appendix B and similar relations (calculated, e.g., from perturbation theory, or hybrid models employing perturbation theory on large scales, but including the full nonlinearities on small scales, Takada & Futamase 1999) provide a more general architecture for the study of hierarchical cosmologies, understood in the sense of models which do not single out the non–generic case of scale–independent mean density, as the standard model does. The focus here is on the *effective* dynamics of portions of the Universe on some spatial scale including the possibility of statistically studying ensembles of spatial domains (see: Buchert et al. 1999 for an investigation within Newtonian cosmology). With regard to the old ideas of hierarchical cosmologies the suggestion by Wertz (1971) may be put into perspective. It relies on spherically symmetric domains which depend on their own parameters of a standard FRW cosmology. In this line the expansion laws (B.3) (and more general relations) can also be applied to finite domains having their individual parameters. These parameters belong to generalized FRW cosmologies that include averages over inhomogeneities encoded in an additional 'backreaction' parameter (Eq. (12d)). The relevance of the characteristics (12) on a finite domain for the interpretation of volume–limited observational data, where we cannot a priori assume that the surveyed volume is a portion of a standard Hubble flow, is obvious.

A similar view applies to so–called collapse models like the spherical "top–hat" model (e.g., Peebles 1980): on smaller spatial scales expansion fluctuations may become dominant in an overdense domain leading to collapse. The transition when $\mathcal{Q}_{\mathcal{D}}$ moves through zero as we come from large scales can be used to mark the scale of "decoupling of inhomogeneities from the global expansion". In this context 'backreaction' models provide straightforward generalizations of the top–hat model. While Birkhoff's theorem lies at the basis of the spherical model, the average dynamics including 'backreaction' is not restricted by symmetry assumptions. Thus, the averaged equations furnish a general framework with which one can describe the effective dynamics of individual collapsing or expanding domains.

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Appendix A: Newtonian Analogues

The general expansion law in Newtonian cosmology reads (Buchert & Ehlers 1997):

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G\frac{M_{\mathcal{D}}}{V_{\mathcal{D}_o} a_{\mathcal{D}}^3} - \Lambda \;=\; \mathcal{Q}_{\mathcal{D}} \quad . \tag{A.1}$$

As in the main text, M_D denotes the total (conserved) rest mass contained in $\mathcal{D}$, and $\mathcal{Q}_{\mathcal{D}}$ is the same expression as Eq. (13c). This equation is identical to (10a), as we already pointed out in (Buchert, Ehlers 1997).

The integral of Equation (A.1) with respect to time can be performed and yields the generalized form of Friedmann's differential equation for the first derivative of the scale factor (Buchert 1996 – with a different sign convention for $\mathcal{Q}_{\mathcal{D}}$):

$$\frac{\dot{a}_{\mathcal{D}}^2 + k_{\mathcal{D}}}{a_{\mathcal{D}}^2} - \frac{8\pi G M_D}{3V_{\mathcal{D}_o} a_{\mathcal{D}}^3} - \frac{\Lambda}{3} \;=\; \frac{1}{3a_{\mathcal{D}}^2}\int_{t_0}^{t} dt' \; \mathcal{Q}_{\mathcal{D}} \; \frac{d}{dt'} a_{\mathcal{D}}^2(t') \quad , \tag{A.2}$$

where $k_{\mathcal{D}}$ is a (domain–dependent) integration constant.

Comparing with the general relativistic equation (10b) we find the analogy:

$$\frac{k_{\mathcal{D}}}{a_{\mathcal{D}}^2} - \frac{1}{3a_{\mathcal{D}}^2}\int_{t_0}^{t} dt' \; \mathcal{Q}_{\mathcal{D}} \; \frac{d}{dt'} a_{\mathcal{D}}^2(t') = \frac{1}{6}\left(\langle\mathcal{R}\rangle_{\mathcal{D}} + \mathcal{Q}_{\mathcal{D}}\right) \quad . \tag{A.3}$$

The time–derivative of Eq. (A3) is equivalent to the integrability condition (13) in *Corollary 3.* We may view Eq. (A3) as an integral of (13b). Eliminating the average curvature from this integral, Eq. (A3), and inserting it into the integrability condition (13b) formally results in a differential equation for $\mathcal{Q}_{\mathcal{D}}$ alone, which, however, reduces to an identity.

Notice that we cannot separately identify the integration constant $k_{\mathcal{D}}$ with the average Ricci scalar, since this would determine the evolution of the average curvature and the 'backreaction' term to the special solution (B.3). We might be able to show that, e.g., the solution (B.3) could also be found within the Newtonian framework for a special type of deformation of the domain's boundary. However, we cannot conclude that for the subclass of Newtonian cosmologies, which can be obtained from the Newtonian limit of the corresponding class of GR solutions, the solutions (B.3) would be the correct limit; the limit ($c \to \infty$; $\mathcal{R} \to 0$) is not obvious in the expression $c^2\mathcal{R}$.

Appendix B: Examples of Expansion Laws

From Equation (13b) we conclude that,
first, for on average spatially flat domains the 'backreaction' can be integrated to give

$$\mathcal{Q}_{\mathcal{D}}(t) = \mathcal{Q}^0_{\mathcal{D}}\; a_{\mathcal{D}}^{-6} \quad ; \quad \mathcal{Q}^0_{\mathcal{D}} := \mathcal{Q}_{\mathcal{D}}(t_0) \quad . \tag{B.1a}$$

Using the integral (B.1a) we write down a closed equation for the scale factor in this case:

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G \frac{M_{\mathcal{D}}}{V_{\mathcal{D}_o} a_{\mathcal{D}}^3} - \Lambda \;=\; \frac{\mathcal{Q}^0_{\mathcal{D}}}{a_{\mathcal{D}}^6} \quad . \tag{B.1b}$$

In view of Eq. (10b) the integral of (B.1b) is given by

$$3\left(\frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}}\right)^2 - 8\pi G \frac{M_{\mathcal{D}}}{V_{\mathcal{D}_o} a_{\mathcal{D}}^3} - \Lambda = -\frac{\mathcal{Q}^0_{\mathcal{D}}}{2a_{\mathcal{D}}^6} \quad . \tag{B.1c}$$

Hence, the problem is reduced to a quadratur.

Second, for (on some spatial domain) vanishing 'backreaction' we obtain the special case of conformally constant curvature models; the average scalar curvature is inversely proportional to the square of the "radius of curvature" on the domain,

$$\langle \mathcal{R} \rangle_{\mathcal{D}}(t) = \langle \mathcal{R} \rangle^0_{\mathcal{D}}\; a_{\mathcal{D}}^{-2} \quad ; \quad \langle \mathcal{R} \rangle^0_{\mathcal{D}} := \langle \mathcal{R} \rangle_{\mathcal{D}}(t_0) \quad , \tag{B.2a}$$

where $a_{\mathcal{D}}(t) = a(t)$ is a solution of the standard FRW models. We are faced with the situation that the domain represents on average a small FRW universe with its own domain–dependent parameters. However, here, this result holds for any spatial domain on which the 'backreaction' vanishes, i.e. inhomogeneities are present and their fluctuations can even be large, but they have to compensate each other.

Third, one obvious solution of Eq. (13b) in the case of non–vanishing average scalar curvature *and* non–vanishing 'backreaction' is given by

$$\mathcal{Q}_{\mathcal{D}}(t) = \mathcal{Q}^0_{\mathcal{D}}\; a_{\mathcal{D}}^{-6} \quad ; \quad \langle \mathcal{R} \rangle_{\mathcal{D}}(t) = \langle \mathcal{R} \rangle^0_{\mathcal{D}}\; a_{\mathcal{D}}^{-2} \quad ; \tag{B.3a,b}$$

here, $\langle \mathcal{R} \rangle_{\mathcal{D}}$ is proportional to an "effective radius of curvature" of the domain, and $a_{\mathcal{D}}(t) \neq a(t)$. This solution features *independent* evolution of average Ricci scalar and 'backreaction': the spatial domain behaves like a small "almost FRW" universe, still being characterized by its own parameters, which are exclusively determined by the values of the fields *inside the domain.* Although special, this case offers the possibility of understanding some properties of the 'backreaction' effect.

At first glance, it might look wrong that the dynamics of any patch of matter can be described independently of the environment; the non–local gravitational influence from the matter outside the domain seems not to have impact on the dynamics of the domain. This interpretation is, however, misleading: although we have to specify only initial data within the domain in, e.g., the solutions to (B.3), we still have to solve the constraint equations for these initial data which is a non–local operation and involves also the fields outside the domain under consideration. Still, solutions to (B.3) uniquely describe the averaged dynamics of the domain for all times, once the initial data are specified, without solving the constraints at later times.

All of our examples restrict the generality: solution (B.1) is found by the requirement of on the domain vanishing average Ricci scalar. Hence, in view of the Hamiltonian constraint (4a) the inhomogeneities have to obey a special relation between the rest mass density, the cosmological constant and the second scalar invariant of the expansion tensor. Eqs. (B.3) together with the generalized Friedmann equations determine a more general class of motions. Still, in general, we expect that the evolution of 'backreaction' is coupled to the evolution of the averaged spatial curvature in a more complex way. Indeed, as Eq. (13a) shows, even for initially vanishing average curvature, there is generation of curvature in the course of structure formation, since amplification of inhomogeneities builds up the 'backreaction' term.

We finish the discussion of special expansion laws by giving a useful formula for the dynamical relation of the average characteristics (12). Combining (12) with *Corollary 3* and using the *Theorem* we obtain:

$$\dot{\Omega}_{\mathcal{Q}} + 6H_{\mathcal{D}}\Omega_{\mathcal{Q}}(1-\Omega_k-\Omega_{\mathcal{Q}}) + \dot{\Omega}_k + 2H_{\mathcal{D}}\Omega_k(1-\Omega_k-\Omega_{\mathcal{Q}})$$
$$-3H_{\mathcal{D}}(1-\Omega_\Lambda-\Omega_k-\Omega_{\mathcal{Q}})(\Omega_k+\Omega_{\mathcal{Q}}) \;=\; 0 \;\; . \qquad (B.4)$$

One example may express a warning that the average characteristics in the present case are dynamically tightly related and should not be treated independently: let us assume that the cosmological term vanishes (which remains an independent parameter), and that the restmass density characteristic remains constant in time mimicking the situation in an Einstein–de Sitter universe. Then, in the simplest case $\Omega_m = 1$, the curvature and 'backreaction' characteristics have to compensate each other, $\Omega_k + \Omega_{\mathcal{Q}} = 0$, and Eq. (B.4) reduces to

$$\dot{\Omega}_{\mathcal{Q}} + 6H_{\mathcal{D}}\Omega_{\mathcal{Q}} + \dot{\Omega}_k + 2H_{\mathcal{D}}\Omega_k = 0 \;\; .$$

Eliminating one of the characteristics from this equation shows that the other has to vanish identically, reducing the average model to the standard Einstein–de Sitter universe. There exists no generic inhomogeneous cosmology with $\Omega_m = 1$ throughout its evolution.

Appendix C: Yodzis' Argument

Let us view the averaging problem within a 4–dimensional tube of spacetime that is swept out by a compact 3–dimensional domain $\mathcal{D}$ and bounded from above and below by space–like hypersurfaces $t_1 = const.$, $t_2 = const.$ From the Ricci identity

$$(u_{\alpha;\beta})_{;\gamma} - (u_{\alpha;\gamma})_{;\beta} = -R_{\alpha\delta\beta\gamma}u^{\delta} \tag{C.1a}$$

we obtain after contraction in α and β and multiplication with u^{γ}:

$$[u^{\alpha}u^{\beta}{}_{;\beta} - u^{\beta}u^{\alpha}{}_{;\beta}]_{;\alpha} = -R_{\alpha\beta}u^{\alpha}u^{\beta} + (u^{\gamma}{}_{;\gamma})^2 - u^{\gamma}{}_{;\delta}u^{\delta}{}_{;\gamma} \quad . \tag{C.1b}$$

Recalling the definition of the second invariant of K_{ij} (8b), we rewrite Eq. (C.1b):

$$2\mathbf{II} = [u^{\alpha}u^{\beta}{}_{;\beta} - u^{\beta}u^{\alpha}{}_{;\beta}]_{;\alpha} + R_{\alpha\beta}u^{\alpha}u^{\beta} \quad . \tag{C.1c}$$

After averaging we obtain for the 'backreaction' term:

$$\mathcal{Q}_{\mathcal{D}} = 2\langle\mathbf{II}\rangle_{\mathcal{D}} - \frac{2}{3}\langle\mathbf{I}\rangle^2_{\mathcal{D}} =$$

$$\frac{2}{V_{\mathcal{D}}}\int_{\mathcal{D}} Jd^3X\;[u^{\alpha}u^{\beta}{}_{;\beta} - u^{\beta}u^{\alpha}{}_{;\beta}]_{;\alpha} - \frac{2}{3V^2_{\mathcal{D}}}\left(\int_{\mathcal{D}} Jd^3X\;u^{\beta}{}_{;\beta}\right)^2 + \langle R_{\alpha\beta}u^{\alpha}u^{\beta}\rangle_{\mathcal{D}} \quad . \tag{C.2}$$

We now need to evaluate the total 4–divergences in Eq. (C.2) on the hypersurfaces. Yodzis (1974) gave the answer using the following argument: perform the volume integral over the 4–dimensional tube of spacetime Γ and let then the distance between the hypersurfaces tend to zero, $t_2 - t_1 = \epsilon \to 0$. Applying Green's theorem on the integrals of the invariants,

$$\int_{\Gamma(t_1,t_2)} d\Gamma\;u^{\alpha}{}_{;\alpha} =$$

$$-\int_{\mathcal{D}_{t_2}} Jd^3X\;u^{\alpha}u_{\alpha} + \int_{\mathcal{D}_{t_1}} Jd^3X\;u^{\alpha}u_{\alpha} = V_{\mathcal{D}}(t_2) - V_{\mathcal{D}}(t_1) \quad ; \tag{C.3a}$$

$$\int_{\Gamma(t_1,t_2)} d\Gamma\;[u^{\alpha}u^{\beta}{}_{;\beta} - u^{\beta}u^{\alpha}{}_{;\beta}]_{;\alpha} =$$

$$-\int_{\mathcal{D}_{t_2}} Jd^3X\;[u^{\alpha}u^{\beta}{}_{;\beta} - u^{\beta}u^{\alpha}{}_{;\beta}]u_{\alpha} + \int_{\mathcal{D}_{t_1}} Jd^3X\;[u^{\alpha}u^{\beta}{}_{;\beta} - u^{\beta}u^{\alpha}{}_{;\beta}]u_{\alpha} \quad , \tag{C.3b}$$

he arrived at the following result by deviding Eqs. (C.3a,b) by ϵ and taking the limit $\epsilon \to 0$ (in our notations and conventions): first, he obtains the familiar equation (9d) for the average of the first invariant $\mathbf{I} = \theta$:

$$\int_{\mathcal{D}} Jd^3X\;\theta = \langle\theta\rangle_{\mathcal{D}}V_{\mathcal{D}} \quad ; \tag{C.3c}$$

for the average of the second invariant **II** he derives:

$$\int_{\mathcal{D}} Jd^3X \,[-R_{\alpha\beta}u^\alpha u^\beta + 2\mathbf{II}] = (\langle\theta\rangle_{\mathcal{D}} V_{\mathcal{D}})^{\bullet} \quad . \qquad (C.3d)$$

Although Yodzis (loc. cit.) seems to imply that this result only holds for closed 3–spaces, we can immediately see that these equations and especially Eq. (C.3d) hold for any compact and simply–connected domain in the hypersurfaces: from Einstein's equations (1a) we have $R_{\alpha\beta}u^\alpha u^\beta = 4\pi G\varrho - \Lambda$. Thus, Eq. (C.3d) is equivalent to the averaged Raychaudhuri equation in *Corollary 1*, Eq. (11c). Calculating $\mathcal{Q}_{\mathcal{D}}$ from Eq. (C.2) we accordingly obtain Eq. (10a) of the main text.

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