State Vector Collapse Probabilities and Separability of Independent Systems in Hughston’s Stochastic Extension of the Schrödinger Equation

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ABSTRACT

We give a general proof that Hughston’s stochastic extension of the Schrödinger equation leads to state vector collapse to energy eigenstates, with collapse probabilities given by the quantum mechanical probabilities computed from the initial state. We also show that for a system composed of independent subsystems, Hughston’s equation separates into similar independent equations for each of the subsystems, correlated only through the common Wiener process that drives the state reduction.
A substantial body of work [1] has addressed the problem of state vector collapse by proposing that the Schrödinger equation be modified to include a stochastic process, presumably arising from physics at a deeper level, that drives the collapse process. Although interesting models have been constructed, there so far has been no demonstration that for a generic Hamiltonian, one can construct a stochastic dynamics that collapses the state vector with the correct quantum mechanical probabilities. Part of the problem has been that most earlier work has used stochastic equations that do not preserve state vector normalization, requiring additional ad hoc assumptions to give a consistent physical interpretation.

Recently, a number of authors [2] have proposed rewriting the Schrödinger equation as an equivalent dynamics on projective Hilbert space, i.e., on the space of rays, a formulation in which the imposition of a state vector normalization condition is not needed. Within this framework, Hughston [3] has proposed a simple stochastic extension of the Schrödinger equation, constructed solely from the Hamiltonian function, and has shown that his equation leads to state vector reduction to an energy eigenstate, with energy conservation in the mean throughout the reduction process. In the simplest spin-1/2 case, Hughston exhibits an explicit solution that shows that his equation leads to collapse with the correct quantum mechanical probabilities, but the issue of collapse probabilities in the general case has remained open. In this Letter, we shall give a general proof that Hughston’s equation leads to state vector collapse to energy eigenstates with the correct quantum mechanical probabilities, using the martingale or “gambler’s ruin” argument pioneered by Pearle [4]. We shall also show that Hughston’s equation separates into independent equations of similar structure for a wave function constructed as the product of independent subsystem wave functions.

We begin by explaining the basic elements needed to understand Hughston’s equa-
tion, working in an \( n + 1 \) dimensional Hilbert space. We denote the general state vector in this space by \( |z\rangle \), with \( z \) a shorthand for the complex projections \( z^1, z^2, ..., z^{n+1} \) of the state vector on an arbitrary fixed basis. Letting \( F \) be an arbitrary Hermitian operator, and using the summation convention that repeated indices are summed over their range, we define

\[
(F) \equiv \frac{\langle z | F | z \rangle}{\langle z | z \rangle} = \frac{\bar{z}^\alpha F_{\alpha\beta} z^\beta}{\bar{z}^\gamma z^\gamma},
\]

so that \( (F) \) is the expectation of the operator \( F \) in the state \( |z\rangle \), independent of the ray representation and normalization chosen for this state. Note that in this notation \((F^2)\) and \((F)^2\) are not the same; their difference is in fact the variance \( \Delta F \),

\[
\Delta F = (F^2) - (F)^2.
\]

We shall use two other parameterizations for the state \( |z\rangle \) in what follows. Since \( (F) \) is homogeneous of degree zero in both \( z^\alpha \) and \( \bar{z}^\alpha \), let us define new complex coordinates \( t^j \) by

\[
t^j = z^j / z^0, \quad \bar{t}^j = \bar{z}^j / \bar{z}^0, \quad j = 1, ..., n.
\]

Next, it is convenient to split each of the complex numbers \( t^j \) into its real and imaginary part \( t^j_R, t^j_I \), and to introduce a \( 2n \) component real vector \( x^a, \) \( a = 1, ..., 2n \) defined by \( x^1 = t^1_R, x^2 = t^1_I, x^3 = t^2_R, x^4 = t^2_I, ..., x^{2n-1} = t^n_R, x^{2n} = t^n_I \). Clearly, specifying the projective coordinates \( t^j \) or \( x^a \) uniquely determines the unit ray containing the unnormalized state \( |z\rangle \), while leaving the normalization and ray representative of the state \( |z\rangle \) unspecified.

As discussed in Refs. [2], projective Hilbert space is also a Riemannian space with respect to the Fubini-Study metric \( g_{\alpha\beta} \), defined by the line element

\[
4 ds^2 = g_{\alpha\beta} d\bar{z}^\alpha dz^\beta = 4 \left( 1 - \frac{\langle (z + dz) | z + dz \rangle^2}{\langle z | z \rangle \langle z + dz | z + dz \rangle} \right).
\]
Abbreviating $\bar{z}z^\gamma \equiv \bar{z} \cdot z$, a simple calculation gives

$$g_{\alpha\beta} = 4(\delta_{\alpha\beta}\bar{z} \cdot z - z^\alpha \bar{z}^\beta)/(\bar{z} \cdot z)^2 = 4 \frac{\partial}{\partial\bar{z}^\alpha} \frac{\partial}{\partial z^\beta} \log \bar{z} \cdot z . \quad (3b)$$

Because of the homogeneity conditions $\bar{z}^\alpha g_{\alpha\beta} = z^\beta g_{\alpha\beta} = 0$, the metric $g_{\alpha\beta}$ is not invertible, but if we hold the coordinates $\bar{z}^0$, $z^0$ fixed in the variation of Eq. (3a) and go over to the projective coordinates $t^i$, we can rewrite the line element of Eq. (3a) as

$$ds^2 = g_{jk}dt^jdt^k \quad , \quad (4a)$$

with the invertible metric [5]

$$g_{jk} = \frac{4[(1 + T^j T^k)\delta_{jk} - t^j T^k]}{(1 + T^m T^m)^2} \quad , \quad (4b)$$

with inverse

$$g^{jk} = \frac{1}{4}(1 + T^m T^m)(\delta_{jk} + t^j T^k) \quad . \quad (4c)$$

Reexpressing the complex projective coordinates $t^i$ in terms of the real coordinates $x^a$, the line element can be rewritten as

$$ds^2 = g_{ab}dx^a dx^b \quad , \quad (4d)$$

$$g_{ab} = \frac{4[(1 + x^d x^d)\delta_{ab} - (x^a x^b + \omega_{ac} x^c \omega_{bd} x^d)]}{(1 + x^e x^e)^2} \quad , \quad (4d)$$

$$g^{ab} = \frac{1}{4}(1 + x^e x^e)(\delta_{ab} + x^a x^b + \omega_{ac} x^c \omega_{bd} x^d) \quad . \quad (4d)$$

Here $\omega_{ab}$ is a numerical tensor whose only nonvanishing elements are

$$\omega_{a=2j-1 \ b=2j} = 1 \ \text{and} \ \omega_{a=2j \ b=2j-1} = -1 \ \text{for} \ j = 1, \ldots, n. \ \text{As discussed by Hughston,}$$

one can define a complex structure $J^b_a$ over the entire projective Hilbert space for which

$$J^c_a J^d_b g_{cd} = g_{ab}, \ J^b_a J^c_b = -\delta^c_a, \ \text{such that} \ \Omega_{ab} = g_{bc} J^c_a \ \text{and} \ \Omega^{ab} = g^{ac} J^b_c \ \text{are antisymmetric tensors. At} \ x = 0, \ \text{the metric and complex structure take the values}$$

$$g_{ab} = 4\delta_{ab}, \ g^{ab} = \frac{1}{4}\delta_{ab} \quad , \quad \Omega_{ab} = 4\omega_{ab}, \ \Omega^{ab} = \frac{1}{4}\omega_{ab} . \quad (5)$$
Returning to Eq. (1a), we shall now derive some identities that are central to what follows. Differentiating Eq. (1a) with respect to $z^\alpha$, with respect to $z^\beta$, and with respect to both $z^\alpha$ and $z^\beta$, we get

$$
\langle z | z \rangle \frac{\partial (F)}{\partial z^\alpha} = F_{\alpha \beta} z^\beta - (F) z^\alpha,
$$

$$
\langle z | z \rangle \frac{\partial (F)}{\partial z^\beta} = \pi^\alpha F_{\alpha \beta} - (F) z^\beta,
$$

$$
\langle z | z \rangle \frac{\partial^2 (F)}{\partial \pi^\alpha \partial z^\beta} = \langle z | z \rangle \left[ F_{\alpha \beta} - \delta_{\alpha \beta} (F) \right] + 2 z^\alpha \pi^\beta (F) - \pi^\gamma F_{\gamma \beta} z^\alpha - \pi^\beta F_{\alpha \gamma} z^\gamma.
$$

Writing similar expressions for a second operator expectation $(G)$, contracting in various combinations with the relations of Eq. (6a), and using the homogeneity conditions

$$
\langle \pi \rangle \frac{\partial (F)}{\partial \pi^\alpha} = \pi^\beta \frac{\partial (F)}{\partial z^\beta} = \pi^\alpha \frac{\partial^2 (F)}{\partial \pi^\alpha \partial \pi^\beta} = \pi^\beta \frac{\partial^2 (F)}{\partial \pi^\alpha \partial z^\beta} = 0
$$

(6b)

to eliminate derivatives with respect to $\pi^\alpha$, $\pi^\beta$, we get the following identities,

$$
-i(FG - GF) = -i\langle z | z \rangle \left( \frac{\partial (F)}{\partial \pi^\alpha} - \frac{\partial (G)}{\partial z^\alpha} \frac{\partial (F)}{\partial \pi^\beta} \right) = 2\Omega^{ab} \nabla_a (F) \nabla_b (G),
$$

$$
(FG + GF) - 2(F)G = \langle z | z \rangle \left( \frac{\partial (F)}{\partial \pi^\alpha} + \frac{\partial (G)}{\partial z^\alpha} \frac{\partial (F)}{\partial \pi^\beta} \right) = 2g^{ab} \nabla_a (F) \nabla_b (G),
$$

$$
(FGF) - (F^2)G - (FG + GF) + 2(F)^2 = \langle z | z \rangle \left( \frac{\partial^2 (F)}{\partial \pi^\alpha \partial \pi^\beta} - \frac{\partial^2 (G)}{\partial \pi^\alpha \partial \pi^\beta} \frac{\partial (F)}{\partial \pi^\beta} \right) = -2\nabla^a (F) \nabla^b (F) \nabla_a \nabla_b (G)
$$

(7a)

with $\nabla_a$ the covariant derivative with respect to the Fubini-Study metric. It is not necessary to use the detailed form of the affine connection to verify the right hand equalities in these identities, because since $(G)$ is a Riemannian scalar, $\nabla_a \nabla_b (G) = \nabla_a \partial_b (G)$, and since projective Hilbert space is a homogeneous manifold, it suffices to verify the identities at the single point $x = 0$, where the affine connection vanishes and thus $\nabla_a \partial_b (G) = \partial_a \partial_b (G)$. Using Eqs. (7a) and the chain rule we also find

$$
-\nabla^a [(F^2) - (F)^2] \nabla_a (G) = -\frac{1}{2} (F^2 G + GF^2) + (F^2) (G) + (F) (FG + GF) - 2(F)^2 (G)
$$

(7b)
which when combined with the final identity in Eq. (7a) gives

\[ \nabla^a(F)\nabla^b(F)\nabla_a\nabla_b(G) - \frac{1}{2}\nabla^a[(F^2) - (F)^2]\nabla_a(G) = \frac{1}{4}(2FGF - F^2G - GF^2) \quad , \quad (7c) \]

the right hand side of which vanishes when the operators \( F \) and \( G \) commute [6].

Let us now turn to Hughston’s stochastic differential equation, which in our notation is

\[ dx^a = [2\Omega^{ab}\nabla_b(H) - \frac{1}{4}\sigma^2\nabla^aV]dt + \sigma\nabla^a(H)dW_t \quad , \quad (8a) \]

with \( W_t \) a Brownian motion or Wiener process, with \( \sigma \) a parameter governing the strength of the stochastic terms, with \( H \) the Hamiltonian operator and \( \langle H \rangle \) its expectation, and with \( V \) the variance of the Hamiltonian,

\[ V = \Delta H = \langle H^2 \rangle - \langle H \rangle^2 \quad . \quad (8b) \]

When the parameter \( \sigma \) is zero, Eq. (8a) is just the transcription of the Schrödinger equation to projective Hilbert space. For the time evolution of a general function \( G[x] \), we get by Taylor expanding \( G[x + dx] \) and using the Itô stochastic calculus rules

\[ dW_t^2 = dt \quad , \quad dt^2 = dtdW_t = 0 \quad , \quad (9a) \]

the corresponding stochastic differential equation

\[ dG[x] = \mu dt + \sigma\nabla_aG[x]\nabla^a(H)dW_t \quad , \quad (9b) \]

with the drift term \( \mu \) given by

\[ \mu = 2\Omega^{ab}\nabla_aG[x]\nabla_b(H) - \frac{1}{4}\sigma^2\nabla^aV\nabla_aG[x] + \frac{1}{2}\sigma^2\nabla^a(H)\nabla^b(H)\nabla_a\nabla_bG[x] \quad . \quad (9c) \]

Hughston shows that with the \( \sigma^2 \) part of the drift term chosen as in Eq. (8a), the drift term \( \mu \) in Eq. (9c) vanishes for the special case \( G[x] = \langle H \rangle \), guaranteeing conservation of the
expectation of the energy with respect to the stochastic evolution of Eq. (8a). But referring to Eq. (7c) and the first identity in Eq. (7a), we see that in fact a much stronger result is also true, namely that $\mu$ vanishes [and thus the stochastic process of Eq. (9b) is a martingale] whenever $G[x] = (G)$, with $G$ any operator that commutes with the Hamiltonian $H$.

Let us now make two applications of this fact. First, taking $G[x] = V = (H^2) - (H)^2$, we see that the contribution from $(H^2)$ to $\mu$ vanishes, so the drift term comes entirely from $-(H)^2$. Substituting this into $\mu$ gives $-2(H)$ times the drift term produced by $(H)$, which is again zero, plus an extra term

$$-\sigma^2 \nabla_a(H) \nabla^b(H) \nabla_a(H) \nabla_b(H) = -\sigma^2 V^2,$$

where we have used the relation $V = \nabla_a(H) \nabla^a(H)$ which follows from the $F = G = H$ case of the middle identity of Eq. (7a). Thus the variance $V$ of the Hamiltonian satisfies the stochastic differential equation, derived by Hughston by a more complicated method,

$$dV = -\sigma^2 V^2 dt + \sigma \nabla_a V \nabla^a(H) dW_t .$$

This implies that the expectation $E[V]$ with respect to the stochastic process obeys

$$E[V_t] = E[V_0] - \sigma^2 \int_0^t ds E[V_s^2] ,$$

which using the inequality $0 \leq E[(V - E[V])^2] = E[V^2] - E[V]^2$ gives the inequality

$$E[V_t] \leq E[V_0] - \sigma^2 \int_0^t E[V_s]^2 .$$

Since $V$ is necessarily positive, Eq. (10d) implies that $E[V_\infty] = 0$, and again using positivity of $V$ this implies that $V_s$ vanishes as $s \to \infty$, apart from a set of outcomes of probability measure zero. Thus, as concluded by Hughston, the stochastic term in his equation drives the system, as $t \to \infty$, to an energy eigenstate.
As our second application of the vanishing of the drift term $\mu$ for expectations of operators that commute with $H$, let us consider the projectors $\Pi_e \equiv |e\rangle\langle e|$ on a complete set of energy eigenstates $|e\rangle$. By definition, these projectors all commute with $H$, and so the drift term $\mu$ vanishes in the stochastic differential equation for $G[x] = (\Pi_e)$, and consequently the expectations $E[(\Pi_e)]$ are time independent; additionally, by completeness of the states $|e\rangle$, we have $\sum_e (\Pi_e) = 1$. But these are just the conditions for Pearle’s [4] gambler’s ruin argument to apply. At time zero, $E[(\Pi_e)] = (\Pi_e) \equiv p_e$ is the absolute value squared of the quantum mechanical amplitude to find the initial state in energy eigenstate $|e\rangle$. At $t = \infty$, the system always evolves to an energy eigenstate, with the eigenstate $|f\rangle$ occurring with some probability $P_f$. The expectation $E[(\Pi_e)]$, evaluated at infinite time, is then

$$E[(\Pi_e)] = 1 \times P_e + \sum_{f \neq e} 0 \times P_f = P_e \ ;$$

hence $p_e = P_e$ for each $e$ and the state collapses into energy eigenstates at $t = \infty$ with probabilities given by the usual quantum mechanical rule applied to the initial wave function.

Let us now examine the structure of Hughston’s equation for a Hilbert space constructed as the direct product of independent subsystem Hilbert spaces, so that

$$|z\rangle = \prod_\ell |z_\ell\rangle \ ,$$

$$H = \sum_\ell H_\ell \ ,$$

(12a)

with $H_\ell$ acting as the unit operator on the states $|z_k\rangle$, $k \neq \ell$. Then a simple calculation shows that the expectation of the Hamiltonian $(H)$ and its variance $V$ are both additive over the subsystem Hilbert spaces,

$$(H) = \sum_\ell (H_\ell) = \sum_\ell |z_\ell\rangle \langle z_\ell| \ ,$$

$$V = \sum_\ell V_\ell = \sum_\ell \left[(H_\ell)^2 - (H_\ell)^2\right] \ ,$$

(12b)

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with \((F_\ell)_\ell\) the expectation of the operator \(F_\ell\) formed according to Eq. (1a) with respect to the subsystem wave function \(|z_\ell\rangle\). In addition, the Fubini-Study line element is also additive over the subsystem Hilbert spaces, since [7]

\[
1 - ds^2 / 4 = \frac{|\langle z | z + dz \rangle|^2}{\langle z | z \rangle \langle z + dz | z + dz \rangle} = \prod_\ell \frac{|\langle z_\ell | z_\ell + dz_\ell \rangle|^2}{\langle z_\ell | z_\ell \rangle \langle z_\ell + dz_\ell | z_\ell + dz_\ell \rangle}
\]

\[
= \prod_\ell [1 - ds^2_\ell / 4] = 1 - \left[ \sum_\ell \frac{ds^2_\ell}{4} \right] + O(ds^4). \tag{13}
\]

As a result of Eq. (13), the metric \(g^{ab}\) and complex structure \(\Omega^{ab}\) block diagonalize over the independent subsystem subspaces. Equation (12b) then implies that Hughston’s stochastic extension of the Schrödinger equation given in Eq. (8a) separates into similar equations for the subsystems, that do not refer to one another’s \(x^a\) coordinates, but are correlated only through the common Wiener process \(dW_\ell\) that appears in all of them. Under the assumption [8] that \(\sigma \sim M^{-1/2}_{\text{Planck}}\) in microscopic units with \(\hbar = c = 1\), these correlations will be very small; it will be important to analyze whether they can have observable physical consequences on laboratory or cosmological scales.

To summarize, we have shown that Hughston’s stochastic extension of the Schrödinger equations has properties that make it a viable physical model for state vector reduction. This opens the challenge of seeing whether it can be derived as a phenomenological approximation to a fundamental pre-quantum dynamics. Specifically, we suggest that since Adler and Millard [9] have argued that quantum mechanics can emerge as the thermodynamics of an underlying non-commutative operator dynamics, it may be possible to show that Hughston’s stochastic process is the leading statistical fluctuation correction to this thermodynamics.

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References

[1] For a representative, but not exhaustive, survey of the earlier literature, see the papers of Diósi, Ghirardi et. al., Gisin, Pearle, and Percival cited by Hughston, Ref. [3] below.


[5] What we have called \( z^0 \) could be any \( z^\alpha \neq 0 \). There is therefore a set of holomorphically overlapping patches, so that the metric of Eq. (4b) is globally defined. See, for example, S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. II, p. 159, Wiley Interscience, New York, 1969.
An alternative demonstration of this result uses the fact, noted by Hughston [3], that $\xi^a_F \equiv \Omega^{ab}_c \nabla_b(F)$ is a Killing vector obeying $\nabla_c \xi^a_F + \nabla^a \xi_F = 0$. First rewrite $\nabla^a(F)\nabla^b(F)\nabla_a \nabla_b(G)$ as $\Omega^{ca}_b \xi_F \Omega^{ca}_c \nabla_a \Omega^{ca}_b$. By the Killing vector property, this becomes $-\xi_F \Omega^{ca}_b \nabla^c \Omega^{ca}_b$, which can be rewritten as $-\nabla^c[\xi_F \Omega^{ca}_b \Omega^{ca}_b] + \nabla^c \xi_F \Omega^{ca}_b \Omega^{ca}_b + \xi_F \nabla^c \xi_F \Omega^{ca}_b$. When $F$ and $G$ commute, the first two terms vanish by the first identity in Eq. (7a), while using the Killing vector property for $\xi_F$ in the third term gives $-\nabla^c \xi_F \Omega^{ca}_b \Omega^{ca}_b = -(1/2) \nabla^c [\xi_F \Omega^{ca}_b \Omega^{ca}_b] J^b_a \nabla_b(G)$, which using $\Delta F = \xi^e_F \xi_F$ reduces to $(1/2) \nabla^c [\Delta F] \nabla^c(G)$.

An alternative way to see this is to use the identity $\log \sum \log \Delta F = \log \prod \Delta F z \cdot z = \sum z \cdot z = \sum \log \Delta F z \cdot z$ in Eq. (3b), along with a change of variable from $z$ to the $z_\ell$’s.

See L. P. Hughston, Ref. [3], Sec. 11 and earlier work of Diósi, Ghirardi et. al., and Penrose cited there; also D. I. Fivel, preprint quant-ph/9710042.