Electromagnetic corrections for the analysis of low energy $\pi^- p$ scattering data

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We calculate the electromagnetic corrections to the isospin invariant mixing angle and to the two eigenphases for the $s$-, $p_{1/2}$- and $p_{3/2}$-partial waves for $\pi^- p$ elastic and charge exchange scattering. These corrections have to be applied to the nuclear quantities in order to obtain the two hadronic phase shifts for each partial wave. The calculation uses relativised Schrödinger equations containing the sum of an electromagnetic potential and an effective hadronic potential. The mass differences between $\pi^-$ and $\pi^0$ as well as between $p$ and $n$ are taken into account. We compare our results with those of previous calculations and qualitatively estimate the uncertainties in the corrections.

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I. INTRODUCTION

In two earlier papers [1,2] we described an iterative procedure for calculating the electromagnetic corrections to the three hadronic phase shifts with \( l = 0, 1 \) for \( \pi^{+}p \) elastic scattering at pion laboratory kinetic energy \( T_{\pi} < 100 \text{ MeV} \) and at the same time determining those hadronic phase shifts by fitting the available experimental data in that energy region. The electromagnetic corrections were obtained using relativised Schrödinger equations (RSEs) containing the sum of an electromagnetic potential and an effective hadronic potential. In Sec.I of Ref.[1] we gave our reasons for using such a potential model rather than the dispersion theory model of the NORDITA group [3,4]. The chief reasons are that with the dispersion theory model it is not possible to calculate potentially important medium and short range contributions to the corrections and that only the potential model gives a well defined separation between hadronic quantities and their electromagnetic corrections and enables the uncertainties in the corrections to be estimated. In Ref.[1] we gave the results for the electromagnetic corrections and a detailed description of the model and the method of evaluation. Ref.[2] presented all the details of the phase shift analysis of the experimental data and gave the results for the hadronic phase shifts.

Our aim in this paper and the one that follows is to describe the corresponding calculations for the two-channel (\( \pi^{-}p, \pi^{0}n \)) system. As before, this paper will give the results for the electromagnetic corrections and all the necessary details of their calculation, while the subsequent paper will discuss the analysis of the experimental data. The energy region remains the same but now we need to consider data for both \( \pi^{-}p \) elastic scattering and the charge exchange reaction \( \pi^{-}p \rightarrow \pi^{0}n \). In a later paper we shall discuss the results from experiments on pionic hydrogen.

The analysis of the low energy \( \pi^{+}p \) elastic scattering data and the simultaneous calculation of the electromagnetic corrections was possible because, ignoring the minute inelasticities due to bremsstrahlung, only one real phase shift is needed for each partial wave. Such a programme is not possible for the low energy \( \pi^{-}p \) scattering data because three parameters (two eigenphases and a mixing angle), and therefore three electromagnetic corrections, need to be obtained for each partial wave. However, while the set of \( \pi^{-}p \) elastic scattering data is slightly smaller (roughly 300 points) than the set of \( \pi^{+}p \) data, there are only 53 published data points for charge exchange scattering in our energy range. It is quite out of the question to extract nine parameters and their electromagnetic corrections from the analysis of the available data without making further assumptions. The data can reliably yield only one or two parameters for each partial wave, as in the \( \pi^{+}p \) case.

That means that we are forced in the two-channel case to invoke the assumption of isospin invariance for the hadronic interaction in some form. The situation is further complicated by the presence of differences between the physical masses of the charged and neutral particles. There may even be a difference between the hadronic masses of \( p \) and \( n \). Since we keep to the philosophy adopted in Ref.[1,2], that we take account only of those electromagnetic effects that can be calculated with reasonable
confidence, we analyse the $\pi^-p$ data and calculate the electromagnetic corrections on the assumption that the two-channel system can be treated at the effective hadronic level (which is certainly not the true hadronic situation when the electromagnetic interaction is switched off) as an isospin invariant system with all the pions having the physical mass $\mu_c$ of $\pi^\pm$ and both $p$ and $n$ having the physical mass $m_p$. This implies that the hadronic phase shifts $\delta^h_{l\pm}$, obtained in Refs.[1,2] from the analysis of the $\pi^+ p$ data and simultaneous calculation of the electromagnetic corrections, are identified with phase shifts ($\delta^h_3$)_{l\pm} corresponding to total isospin $T = 3/2$. They can therefore be used as known input to the analysis of the $\pi^- p$ data. This analysis will then yield three new hadronic phase shifts ($\delta^h_1$)_{l\pm} corresponding to $T = 1/2$.

We therefore assume that there are real symmetric $2 \times 2$ matrices $t^{h}_{l\pm}$ which are the natural generalisation of the $\tan \delta^h_{l\pm}$ for the $\pi^+ p$ case,

$$t^{h}_{l\pm} = O(\phi^h) \begin{pmatrix} \tan(\delta^h_1)_{l\pm} & 0 \\ 0 & \tan(\delta^h_3)_{l\pm} \end{pmatrix} O(\phi^h)^t.$$

(1)

where the isospin invariant mixing angle is $\phi^h = \arcsin(1/\sqrt{3})$ and

$$O(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.$$

In order to analyse the low energy $\pi^- p$ scattering data it is necessary to calculate electromagnetic correction matrices $t^{em}_{l\pm}$ which, when added to the $t^{h}_{l\pm}$, give the real symmetric $2 \times 2$ nuclear matrices $t^{n}_{l\pm}$:

$$t^{n}_{l\pm} = t^{h}_{l\pm} + t^{em}_{l\pm}.$$

(2)

The experimental observables are related to the nuclear partial wave matrices $t^{n}_{l\pm}$ by formulae which are given in Sec.II and in Eqs.(1,2) of Ref.[3].

For the calculation of the $t^{em}_{l\pm}$ we use the potential model of Ref.[1] in a natural generalisation to the two-channel case. We emphasise that the potentials are introduced only in order to calculate the corrections. In addition to the potentials $(V^h_3)_{l\pm}$, which we identify with the effective hadronic potentials $V^h_{l\pm}$ for $\pi^+ p$ scattering, we have new potentials $(V^h_1)_{l\pm}$ which are constructed in order to reproduce the phase shifts ($\delta^h_1$)_{l\pm}, using the same RSEs containing the masses $m_p$ and $\mu_c$. We therefore have $2 \times 2$ effective hadronic potential matrices $V^h_{l\pm}$ which are isospin invariant:

$$V^h_{l\pm} = O(\phi^h) \begin{pmatrix} (V^h_1)_{l\pm} & 0 \\ 0 & (V^h_3)_{l\pm} \end{pmatrix} O(\phi^h)^t.$$

(3)

The electromagnetic correction matrices are then obtained by adding electromagnetic potential matrices $V^{em}_{l\pm}$ to the $V^h_{l\pm}$ and then using these total potential matrices in coupled RSEs that model the physical situation. Full details will be given in Sec.III.
As we remarked in Ref.[1], the hadronic potential matrices in the absence of the electromagnetic interaction would be different from the effective matrices in its presence. Therefore isospin invariance for the $V_{h \pm}^h$ (Eq.(3)) is a separate assumption. However, in the absence of any reliable model results for what happens when the electromagnetic interaction is switched on, in order to analyse the $\pi^- p$ scattering data we have no choice but to make this assumption and to test it by seeing if it is possible to obtain a statistically acceptable fit to the data.

We tested the assumption of isospin invariance at the effective hadronic level by first of all analysing only the $\pi^- p$ elastic scattering data. These data are far more extensive than the charge exchange data and the latter raise questions about the violation of isospin invariance (see Ref.[5] and the next paper). The basic ideas for the analysis of the $\pi^- p$ data and the calculation of the electromagnetic corrections that have been sketched in the preceding paragraphs will be fully developed in the rest of this paper and in its sequel. We shall concentrate in this paper on the calculations of the corrections and describe the analysis of the $\pi^- p$ data, including the difficulties arising from the charge exchange (and pionic hydrogen) data, in the next. Sec.II will set out the formalism for the scattering amplitude matrices in the two-channel case, while Sec.III will give the method of calculating the electromagnetic correction matrices for the $s$, $p_{1/2}$ and $p_{3/2}$-waves. The numerical results for these corrections will be given in Sec.IV.

II. SCATTERING FORMALISM

We begin by writing the $2 \times 2$ matrices of no-flip and spin-flip scattering amplitudes for the ($\pi^- p, \pi^0 n$) system in the form

$$f = f^{em} + \sum_{l=0}^{\infty} \{ (l + 1) f_{l+} + l f_{l-} \} P_l,$$

$$g = g^{em} + i \sum_{l=1}^{\infty} (f_{l+} - f_{l-}) P_{l}^1.$$

Eqs.(4,5) are the obvious generalisations of Eqs.(1,2) of Ref.[1]. The matrices $f^{em}$, $g^{em}$ are just

$$f^{em} = \begin{pmatrix} f^{em} & 0 \\ 0 & 0 \end{pmatrix}, \quad g^{em} = \begin{pmatrix} g^{em} & 0 \\ 0 & 0 \end{pmatrix}$$

where the electromagnetic amplitudes $f^{em}$, $g^{em}$ for $\pi^- p$ scattering have the same decomposition as for $\pi^+ p$ scattering, as given in Eqs.(5,6) of Ref.[1]. The expressions for the components of $f^{em}$, $g^{em}$ are those given in the equations following Eqs.(5,6) of Ref.[1], with the replacements $\alpha \rightarrow -\alpha, \eta \rightarrow -\eta$ throughout. The form factors are unchanged since $F_\pi$ is the same for $\pi^+$ and $\pi^-$. 
The $2 \times 2$ matrices $f_{l\pm}$ of partial wave amplitudes are written most conveniently in the form

$$f_{l\pm} = \left( \begin{array}{cc} \exp(i\Sigma_{l\pm}) & 0 \\
0 & 1 \end{array} \right) \left( \begin{array}{cc} q_c^{-1/2} & 0 \\
0 & q_0^{-1/2} \end{array} \right) t_{l\pm}^n (1_2 - it_{l\pm}^n)^{-1} \left( \begin{array}{cc} \exp(i\Sigma_{l\pm}) & 0 \\
0 & 1 \end{array} \right),$$

where $q_c$ is given by Eq.(3) of Ref.[1] and $q_0$ is the corresponding c.m. momentum for the $\pi^0 n$ channel,

$$q_0^2 = \frac{[W^2 - (m_n - \mu_0)^2][W^2 - (m_n + \mu_0)^2]}{4W^2},$$

$m_n$ and $\mu_0$ being the masses of the neutron and $\pi^0$ respectively. In Eq.(7), $\Sigma_{l\pm}$ has the same decomposition as in Ref.[1],

$$\Sigma_{l\pm} = (\sigma_l - \sigma_0) + \sigma_{l\pm}^{ext} + \sigma_{l\pm}^{rel},$$

but now the phase shifts on the right side, and the $\Sigma_{l\pm}$ themselves, have the opposite sign compared with the $\pi^+ p$ case. For example,

$$\sigma_l = \arg \Gamma(l + 1 - i\eta).$$

Once again the difference $(\sigma_l - \sigma_0)$ appears in $\Sigma_{l\pm}$ because the matrix factors

$$\left( \begin{array}{cc} \exp(i\sigma_0) & 0 \\
0 & 1 \end{array} \right)$$

are removed from each side of the full amplitude matrices $f$, $g$ of Eqs.(4,5). The nuclear matrices $t_{l\pm}^n$ in Eq.(7) are those introduced in Sec.I. We shall also use the matrices $T_{l\pm}^n$ defined by

$$T_{l\pm}^n = t_{l\pm}^n (1_2 - it_{l\pm}^n)^{-1}.$$

The expression (7) with real symmetric matrices $t_{l\pm}^n$ assumes two-channel unitarity, that is the absence of any competing channels. This is not quite true, since the $\gamma n$ channel introduces inelastic corrections to the $T_{l\pm}^n$ which need to be taken into account. Table IV of Ref.[3] gives quantities $\eta_1$, $\eta_3$ and $\eta_{13}$ for the $s$- and $p_{3/2}$-waves which we use for these corrections. We convert the quantities in Ref.[3] into corrections $\Delta T_{cc}^n$ and $\Delta T_{0c}^n$ to the matrix elements of $T^n$ that are needed to calculate the observables for $\pi^- p$ elastic and charge exchange scattering respectively. The necessary formulae are

$$2i \Delta T_{cc}^n = -\frac{2}{3} \eta_1 \exp(2i\delta_1^h) - \frac{1}{3} \eta_3 \exp(2i\delta_3^h) - \frac{8}{9} \eta_{13} \exp\{i(\delta_1^h + \delta_3^h)\},$$

$$2i \Delta T_{0c}^n = \sqrt{2} \eta_1 \exp(2i\delta_1^h) - \frac{\sqrt{2}}{3} \eta_3 \exp(2i\delta_3^h) - \frac{2\sqrt{2}}{9} \eta_{13} \exp\{i(\delta_1^h + \delta_3^h)\}.$$
Note that in writing the elements of $2 \times 2$ matrices we use the subscript $c$ to refer to the $\pi^-p$ channel and 0 to denote the $\pi^0n$ channel. To save writing we have temporarily dropped the subscript $l\pm$; we shall continue to do this when convenient.

The corrections $\Delta T_n^{cc}$ and $\Delta T_n^{0c}$ are added to the quantities $T_n^{cc}$ and $T_n^{0c}$ obtained from Eq.(9) and then the observables are calculated. Having explained how the $\gamma n$ channel is taken into account, we work in what follows with the equations already written.

The decomposition of the matrices $t_n^{l\pm}$ into a hadronic part $t_h^{l\pm}$ and its electromagnetic correction $t_{em}^{l\pm}$ is given in Eq.(2). As we have discussed, the matrices $t_h^{l\pm}$ refer to an effective hadronic situation which is isospin invariant and in which all the pions have the mass $\mu_c$ and the nucleons the mass $m_p$. The aim of our calculation is to obtain for the $s$, $p_{1/2}$- and $p_{3/2}$-waves the three independent elements of the symmetric correction matrix $t_{em}^{l\pm}$. However, the corrections in this form do not convey information in the way we would naturally like to have it. It is therefore customary to write the nuclear matrices in the form

$$t^n = O(\phi) \left( \begin{array}{cc} \tan \delta_1^n & 0 \\ 0 & \tan \delta_3^n \end{array} \right) O(\phi)^t$$  \hspace{1cm} (10)

and to define three new corrections $C_1$, $C_3$ and $\Delta \phi$ by

$$C_1 = \delta_1^n - \delta_1^h, \quad C_3 = \delta_3^n - \delta_3^h, \quad \Delta \phi = \phi - \phi^h.$$  \hspace{1cm} (11)

Here $\phi$ is the mixing angle, which we choose to lie between 0 and $\pi/2$. This convention then fixes the labelling of the eigenphases and ensures that $\tan \delta_i^n$ is close to $\tan \delta_i^h$ ($i = 1, 3$).

To proceed with the calculation of the $t_{em}^{l\pm}$ we need the potential matrices $(V^h)_{l\pm}$ and $(V^{em})_{l\pm}$ that appear in the coupled RSEs that lead to the nuclear matrices $t^n_{l\pm}$. The effective hadronic potential matrices have the isospin invariant form of Eq.(3) and the potentials $V_h^\alpha$ ($\alpha = 1, 3$) are constructed so as to reproduce the hadronic phase shifts $\delta_h^\alpha$ via the RSEs

$$\left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + q_c^2 - 2m_c f_c V_h^\alpha(r) \right) u_\alpha(r) = 0.$$  \hspace{1cm} (12)

The phase shifts are given by the asymptotic behaviour $\sin(q_c r - l\pi/2 + \delta_h^\alpha)$ of the regular wavefunctions. The quantities $q_c$, $m_c$ and $f_c$ are defined in Ref.[1] (Eqs.(3),(19) and the equation following Eq.(19)). The electromagnetic potential matrix $V^{em}$ is

$$V^{em} = \left( \begin{array}{cc} V^{em} & 0 \\ 0 & 0 \end{array} \right)$$  \hspace{1cm} (13)

and $V^{em}$ has the decomposition given in Eq.(16) of Ref.[1]:

$$V^{em} = V^{pc} + V^{ext} + V^{rel} + V^{vp}.$$  \hspace{1cm} (14)
The full potential matrix is
\[ V = V_{em} + V^h. \] (15)
The pieces of \( V_{em} \) in Eq.(14), and therefore \( V_{em} \) itself, have the opposite sign compared with the case of \( \pi^+p \) scattering. This is true even for \( V_{rel} \) since \( \sigma_{rel} \) is calculated only to order \( \alpha \). Full details of the parts of \( V_{em} \) were given in Sec.II of Ref.[1].

III. EVALUATION OF THE CORRECTIONS

The partial wave RSEs for the two-channel case, which we use in order to model the physical situation are given by the natural generalisation of Eq.(18) of Ref.[1]:
\[ \left\{ \left( \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right) I_2 + Q^2 - 2mfV_{l\pm}(r) \right\} u_{l\pm}(r) = 0. \] (16)
The full potential matrices \( V_{l\pm} \) have the form given in Eqs.(3,13-15). The matrices \( Q, m \) and \( f \) are
\[ Q = \begin{pmatrix} q_c & 0 \\ 0 & q_0 \end{pmatrix}, \quad m = \begin{pmatrix} m_c & 0 \\ 0 & m_0 \end{pmatrix}, \quad f = \begin{pmatrix} f_c & 0 \\ 0 & f_0 \end{pmatrix}, \] (17)
where \( q_0 \) is defined in Eq.(8) and
\[ m_0 = \frac{m_n\mu_0}{m_n + \mu_0}, \quad f_0 = \frac{W^2 - m_n^2 - \mu_0^2}{2m_0W}. \] (18)
The reasons for the inclusion of the relativistic amplification via the matrix \( f \), which boosts the full potentials, were given in Sec.III of Ref.[1].

The RSEs (16) are integrated outwards from \( r = 0 \), with the initial conditions
\[ u_{l\pm}(0) = 0, \quad u'_{l\pm}(0) = e, \quad e \neq 0. \] (19)
In fact we require two linearly independent solutions vectors which arise from any two choices \( e_1 \) and \( e_2 \) for \( e \) that are not multiples of each other. The integration proceeds to a distance \( R \) (around 1000 fm) where the only part of the potential matrix that is not negligible is \( V_{pc} \), which appears in \( V_{cc} \). The components \( V_{0c} \) and \( V_{00} \) of \( V \), which contain only linear combinations of the hadronic potentials, become negligible beyond distances of a few fm, so the integration as far as \( r = R \) is necessary only for the first component of \( u_{l\pm} \).

The matching at \( r = R \), which leads to the matrix \( t^n_{l\pm} \), requires particular attention. In the \( \pi^-p \) channel the matching is to a linear combination of the standard point charge Coulomb wavefunctions \( F_i(-\eta; q, r) \) and \( G_i(-\eta; q, r) \) and in the \( \pi^0 \) channel it is to a linear combination of the free particle wavefunctions \( q_0r^j_l(q_0r) \).
and \( q_0 n_l(q_0 r) \). To make the notation less complicated, we drop the subscripts \( l \pm \) and \( l \) for the moment and write the solutions for \( r > R \) in the form

\[
(u^{(i)}(r)) = \left( a_{1i}\{\cos(\Delta \sigma) F(-\eta; q_c r) + \sin(\Delta \sigma) G(-\eta; q_c r)\} + b_{1i}\{\cos(\Delta \sigma) G(-\eta; q_c r) - \sin(\Delta \sigma) F(-\eta; q_c r)\}\right)(m_c/q_c)^{1/2},
\]

(20)

\[
(u^{(i)}(r)) = \left\{ a_{2i}q_0 r j(q_0 r) + b_{2i}q_0 r n(q_0 r)\right\}(m_0/q_0)^{1/2},
\]

(21)

where \( u^{(i)}, i = 1, 2, \) are the solutions with \( u^{(i)'}(0) = e \). The particular way in which the linear combination of \( F \) and \( G \) in Eq.(20) has been written arises from the choice of the additive electromagnetic amplitudes \( (f, g)^{em} \), which contain the parts \( (f, g)^{ext} \) and \( (f, g)^{rel} \), and of the electromagnetic phase shifts \( \Sigma_{l \pm} \), which contain \( \Delta \sigma_{l \pm} = \sigma_{l \pm}^{ext} + \sigma_{l \pm}^{rel} \). This means that, in order to obtain the correct matrices \( t^{n}_{l \pm} \) as defined in Eq.(7), it is necessary to match to a linear combination of point charge Coulomb wavefunctions which correspond, not to the phase shift \( \sigma_t \), but to \( \sigma_t + \Delta \sigma_{l \pm} \). These are just the Coulomb wavefunctions which appear in braces in Eq.(20).

When the two linearly independent solutions \( u^{(i)}_{l \pm}(r) \) are correctly matched in the way prescribed in Eqs.(20) and (21), the result is two \( 2 \times 2 \) matrices \( a_{l \pm} \) and \( b_{l \pm} \), in terms of which the nuclear matrices \( t^{n}_{l \pm} \) are given by

\[
t^{n}_{l \pm} = b_{l \pm} a_{l \pm}^{-1}.
\]

(22)

Eq.(22) is the generalisation to two-channels of the one-channel result given in Sec.III of Ref.[1] (see the paragraph containing Eq.(20)). Further details of the two-channel procedure for calculating \( t^{n}_{l \pm} \) are given, without the complication of the presence of \( \Delta \sigma_{0 \pm} \), in Sec.3 of Ref.[6], where \( K \) is used there instead of \( t^{n} \).

The machinery for an interactive procedure exactly like that described in Ref.[1] for the \( \pi^+ p \) case has now been fully explained. The details of the analysis of the \( \pi^- p \) elastic scattering data are given in the next paper, which parallels the analysis of the \( \pi^+ p \) data described in Ref.[2]. The hadronic phase shifts \( \delta_{h}^3 \) were fixed throughout the \( \pi^- p \) analysis at the final values given in Ref.[2]. The starting point for the \( T = 1/2 \) hadronic phase shifts was the values from the analysis of Arndt et al. [7]. The parametric form of the \( T = 1/2 \) hadronic potentials was taken to be the same as that used for the hadronic potentials in Ref.[1]. At each stage of the iteration the components of the matrices \( t^{m}_{0 \pm}, t^{m}_{1 \pm} \) and \( t^{m}_{1 \pm} \) were calculated using Eqs.(22),(1) and (2). The conversion to the corrections in the form \( C_1, C_3 \) and \( \Delta \phi \) defined in Eq.(11) was done after the final values of these three matrices had been obtained.

We checked that the results of this iteration procedure are insensitive to the choice of the starting point.

IV. RESULTS FOR THE CORRECTIONS

In Tables 1, 2 and 3 we give the results for the electromagnetic corrections, in
the case of the $s$-, $p_{3/2}$- and $p_{1/2}$-waves respectively, in the form of the corrections $C_1$ and $C_3$ to the hadronic phase shifts and the correction $\Delta \phi$ to the isospin invariant mixing angle. They are given at 10 MeV intervals from $T_\pi = 10$ MeV to $T_\pi = 100$ MeV. Similar remarks to those made in Sec.IV of Ref.[1] about the accuracy of the corrections in the $\pi^+p$ case apply also to the results in Tables 1-3. In fact the only large correction is $C_3$ for the $p_{3/2}$-wave and here the uncertainty is well within the statistical error on the resonant phase shift given in Ref.[2]. For the rest of the corrections the uncertainty is negligible.

Inspection of the RSEs in Eq.(16) shows that the electromagnetic corrections may be separated into two parts, one due to the inclusion of $V^{em}$ as an addition to $V^h$ (Eq.(15)) and the other due to the difference between the masses of the particles in the two channels ($q_c \neq q_0$, $m_c \neq m_0$, $f_c \neq f_0$). The marked increase in $\Delta \phi$ for small $T_\pi$ is due to the latter effect. In Ref.[1] we have decomposed our results also into the contributions coming from the separate pieces of $V^{em}$ as given in Eq.(14). Giving this decomposition in the coupled channel case would involve us in complicated notation and would not convey any useful information: the relative importance of the single pieces varies with the partial wave, with the energy and with the particular correction ($C_1$, $C_3$ or $\Delta \phi$).

Comparison with the results of the NORDITA group [3] for the $s$- and $p_{3/2}$-waves is complicated by the different quantities ($\Delta_1$, $\Delta_3$, $\Delta_{13}$) that they use for the corrections. The relation with our corrections is

$$
\Delta_1 = -\frac{3}{2}C_1, \quad \Delta_3 = -3C_3, \quad \Delta_{13} = \frac{3}{\sqrt{2}} \Delta \phi \sin(\delta_1^0 - \delta_3^0),
$$

the last relation being valid for $\Delta \phi$ very small. The same comments apply as in Ref.[1]: the general trend of the corrections is the same for the two calculations, the difference almost certainly arising from the neglect in the NORDITA calculations of medium and short range effects due to $t$- and $u$-channel exchanges. For this reason we claim a higher degree of reliability for our results for the electromagnetic corrections as given in Tables 1-3 than for the results of the NORDITA calculation. For the $s$- and $p_{3/2}$-waves at $T_\pi = 90$ MeV and 100 MeV our results are in reasonably good agreement with those of Ref.[8]. It should be noted that our results in Tables 1-3 contain the influence of vacuum polarisation which is not always negligible, whereas those of Refs.[3,4,8] do not include it.

The corrections for the $p_{1/2}$-wave are given here for the first time. It turns out that for the analysis of the data they play a negligible role, due to the smallness of the hadronic phases in that partial wave.

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References


### TABLE 1: Values in degrees of the s-wave electromagnetic corrections $C_3$, $C_1$ and $\Delta \phi$ as functions of the pion lab kinetic energy $T_\pi$ (in MeV).

<table>
<thead>
<tr>
<th>$T_\pi$</th>
<th>$C_3$</th>
<th>$C_1$</th>
<th>$\Delta \phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>-0.195</td>
<td>0.221</td>
<td>0.179</td>
</tr>
<tr>
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<td>-0.013</td>
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</tr>
<tr>
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<td>0.021</td>
<td>-0.027</td>
</tr>
</tbody>
</table>

### TABLE 2: Values in degrees of the $p_{3/2}$-wave electromagnetic corrections $C_3$, $C_1$ and $\Delta \phi$ as functions of the pion lab kinetic energy $T_\pi$ (in MeV).

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<tr>
<th>$T_\pi$</th>
<th>$C_3$</th>
<th>$C_1$</th>
<th>$\Delta \phi$</th>
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<tbody>
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<td>$C_3$</td>
<td>$C_1$</td>
<td>$\Delta\phi$</td>
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**TABLE 3:** Values in degrees of the $p_{1/2}$-wave electromagnetic corrections $C_3$, $C_1$ and $\Delta\phi$ as functions of the pion lab kinetic energy $T_\pi$ (in MeV).