Cosmological gravitons and the expansion dynamics during the matter age

M. R. de Garcia Maia*, J. C. Carvalho†, and J. S. Alcaniz‡
Departamento de Física, Universidade Federal do Rio Grande do Norte, 59072-970 Natal RN Brazil

Extending previous results [Phys. Rev.D 56, 6351 (1997)], we estimate the back reaction of cosmological gravitons in the expansion dynamics, during the matter age. Tensor perturbations with scales larger than the Hubble length [1–8]. During noninflationary phases these perturbations become effective gravitational waves as they enter the Hubble radius, leading to a departure from the standard behaviour $a(t) \propto t^{3/2}$, with possible consequences to several cosmic processes, such as primordial nucleosynthesis. We examine the implications of this phenomenon during the matter dominated era, assuming an initial inflationary period. The dynamical equation obeyed by the scale factor is derived and numerically solved for different values of the relevant parameters involved.

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I. INTRODUCTION

In a recent paper [1], we have studied the back reaction of cosmological gravitons on the cosmic dynamics during the radiation epoch. The analysis was based on the following reasoning: the inequivalence of vacuum states at different moments of time leads to the production of tensor perturbations in scales larger than the Hubble length [1–8]. During noninflationary periods of expansion, these very long tensor perturbations (VLTP’s) become effective gravitational waves (EGW’s) as they enter the Hubble radius, thus adding new contributions to the energy density associated with the subhorizon waves $\rho_\gamma$ [1–5], [9]. Such a phenomenon can be studied as a process of creation of effective gravitons by using the macroscopic formalism to matter creation based on the thermodynamics of open systems [10,11]. In order to deal with the process, a creation pressure term is introduced in the continuity equation obeyed by $\rho_\gamma$. A dynamical equation for the scale factor $a(t)$ is then derived which takes into account the effective gravitons' back reaction. Full details of this approach can be found in Ref. [1] where the equation for $a(t)$ was numerically solved for a model in which the universe evolves from an arbitrary initial phase to a radiation dominated era. It was found that, if the barotropic index $\gamma$ of the equation of state in the first epoch is close to $2/3$, the back reaction of the effective gravitational waves makes $a(t)$ deviate appreciably from the standard behavior $a(t) \propto t^{3/2}$.

In the present paper we extend our analysis to the matter dominated period of the cosmic expansion. In particular, we aim to check a conjecture stated by Sahni [12], according to which the process mentioned above would ultimately lead the universe to expand linearly with time ($a(t) \propto t$), since the tensor perturbations would enter the Hubble radius during noninflationary periods of expansion (increasing $\rho_\gamma$) and leave the Hubble radius during inflationary periods (decreasing $\rho_\gamma$). In Ref. [1] we have shown that the scale factor would turn from $a(t) \propto t^{1/2}$ to $a(t) \propto t$ only if $\gamma \rightarrow 2/3$. Several models with the universe evolving linearly with time during its early phases have been proposed, most of them related to string motivated cosmologies [13,14]. An universe evolving linearly with time during most of its history would lead to important observational consequences, such as those related to the age of the universe, to the luminosity distance - redshift and the angular diameter distance - redshift relations, and to the galaxy number count as a function of the redshift [15,16] (see also [17]).

We should remark that, in order to the mechanism of graviton creation take place, no ‘exotic’ physics has to be assumed. It only requires the validity of quantum mechanics and of general relativity [18]. Therefore, the study of the possible influence of the cosmological gravitational waves in the expansion dynamics has an importance on its own, a fact that has not been fully appreciated in the literature.

The paper is organized as follows: In Section II we present the basic equations related to the spectrum of cosmological gravitational waves. In Section III we derive the dynamical equations that govern the expansion dynamics during the
matter age, taking into account the gravitons’ back reaction. In Section IV we show the numerical solutions of these equations and present our conclusions.

The system of units used is such that \( \hbar = c = k_B = 1 \).

II. BASIC EQUATIONS

We will consider a homogeneous and isotropic universe, for which the background line element takes the Friedmann-Robertson-Walker (FRW) form

\[ ds^2 = dt^2 - a^2(t)dl^2 = a^2(\eta)(d\eta^2 - dl^2) , \]

where \( t \) and \( \eta \) are, respectively, the cosmic and conformal times, related by

\[ dt = ad\eta . \]

We will restrict ourselves to the spatially flat case. If the pressure \( p \) and energy density \( \rho \) of the cosmic fluid are related by the equation of state

\[ p = (\gamma - 1)\rho , \]

then the scale factor is [3]

\[ a(t) = a_0 \left[ 1 + \frac{3\gamma H_0}{2} (t - t_0) \right]^{2/(3\gamma)} , \]

where \( H_0 \equiv H(t_0) \) and \( H \equiv \dot{a}/a.\)

We will further assume that the universe evolves from an initial arbitrary era (\( \gamma = \gamma \)) to a radiation dominated phase (\( \gamma = 4/3 \)) and then to a matter dominated period (\( \gamma = 1 \)).

As the inequivalence of vacua appear at different instants of time, we will focus on the time-dependent amplitude of the tensor perturbations \( \mu(k, \eta) \), which obeys [1,3,4]

\[ \mu''(k, \eta) + \left( k^2 - \frac{a''}{a} \right) \mu(k, \eta) = 0 . \]

In the above equation the primes indicate derivatives with respect to the conformal time and \( k \) is the comoving wave number, related to the physical wavelength \( \lambda \) and frequency \( \omega \) by

\[ k = \frac{2\pi a}{\lambda} = \omega a . \]

The properly normalized solutions of Eq. (5), corresponding to a scale factor given by Eq. (4), and representing adiabatic vacuum states [7,8], can be found in Refs. [3,4].

The spectrum of the EGW’s can be described by the quantity \( P_g(\omega) \), defined in such a way that \( P_g(\omega) \, d\omega \) represents the energy per unit volume between the frequencies \( \omega \) and \( \omega + d\omega \) [2–4]. Note that this quantity can be defined only for the EGW’s [2,9], i.e., for those perturbations such that

\[ \lambda \leq \lambda_H \equiv H^{-1} . \]

In the units used in this paper and assuming an initial vacuum state, \( P_g(\omega) \) is given by [2–4]

\[ P_g(\omega) = \frac{\omega^3}{\pi^2} \langle N(\omega) \rangle . \]

In the above equation \( \langle N(\omega) \rangle \) is the expectation number of the gravitons with frequency \( \omega \) which is found to be [1,3,4]

\[ \langle N(\omega) \rangle = |\beta(k)|^2 , \]

where \( \beta(k) \) and \( \alpha(k) \) (see below) are the Bogoliubov coefficients relating the creation and annihilation operators (which define the particle states) of different epochs. We will denote by \( \alpha_1, \beta_1 \) the coefficients associated with the first
transition from an arbitrary phase to the radiation dominated era, and by $\alpha_2, \beta_2$ those related with the transition from the radiation to the matter dominated epoch. These coefficients have been obtained in references [3,4] (see also [1]). The coefficient $\beta$ is then given by [2]

$$\beta = \beta_2(k)\alpha_1(k) + \alpha_2^*(k)\beta_1(k),$$

(10)

and the asterisk indicates the complex conjugate of a quantity.

The total energy density associated with the tensor perturbations is obtained by integrating $P_g(\omega)$,

$$\rho_g(t) = \int_{\omega_{\text{min}}(t)}^{\infty} P_g(\omega) \, d\omega,$$

(11)

where

$$\omega_{\text{min}}(t) = 2\pi H(t).$$

(12)

This infrared cutoff is obviously related to the condition (7) and, since it is time dependent, it is the origin of the process of creation of EGW’s. In a FRW scenario, it will be responsible for the fact that $\rho_g$ does not decay with $a^{-4}$, as one could expect for a massless particle such as the graviton.

The integral in Eq. (11) is usually written with a finite upper limit. This is done because the above method for obtaining the spectrum, based on the sudden transition approximation between cosmic eras, does not give accurate results for high frequency waves. In fact, it is reliable only for those waves whose periods are much greater than the transitions time scales [4,19]. However, due to the adiabatic theorem [8], the number of created particles should decrease exponentially for the high frequency modes. Hence, an ultraviolet cutoff is imposed and the error produced by doing so is supposed to be small. Our present results will not be affected by this approximation since we are obviously interested only on the very long modes.

### III. DYNAMICAL EQUATIONS DURING THE MATTER ERA

Following Ref. [1], we will suppose that, prior to the time $t_1$ the dominant material content of the Universe has an equation of state of the form (3). This stage can be either an inflationary or a noninflationary one. At $t_1$ a transition occurs, so that the new barotropic parameter is $4/3$. Due to the transition, tensor perturbations are created [1,3,4]. Some of these have wavelengths less than $H^{-1}(t_1)$ (EGW’s), but most are created with scales larger than $H^{-1}(t_1)$ (VLTP’s). A similar phenomenon will occur at a time $t_2$ when the universe becomes matter dominated ($\gamma = 1$). We will further assume that during this matter dominated period and until a time, say, $t_g$, the energy density $\rho_g$ associated with the EGW’s is negligible compared with the energy density of matter $\rho_m$. There is no restriction over the size of the interval $t_g - t_2$, that can be taken to be arbitrarily small. However, after $t_g$, the ongoing transformation of VLTP’s into EGW’s makes the continuous increase of $\rho_g$ start perturbing the dynamics. Thereafter, the Universe can be supposed to be filled by two fluids: dust and the one composed by the effective gravitons. The total energy density is then $\rho_m + \rho_g$. As gravitons require extremely high energies to interact with matter [4,20], the two fluids can be safely considered to be noninteracting. The effective gravitons behave as a perfect fluid with equation of state [21,22]

$$p_g = \frac{\rho_g}{3}.$$

(13)

Nevertheless, it is possible to associate a creation pressure term $\Pi_g$ to the creation process of these gravitons [1]. The field equations are then written as

$$8\pi G (\rho_m + \rho_g) = 3\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2},$$

(14)

$$8\pi G (p_m + p_g + \Pi_g) = -2\frac{\ddot{a}}{a} - 3\frac{\dot{a}^2}{a^2},$$

(15)

$$\dot{\rho}_g + 3H\rho_g = \Psi_g,$$

(16)
where the dots indicate derivatives with respect to the cosmic time, \( p_m = 0 \) is the pressure of the dust fluid and \( n_g \) and \( \Psi_g \) are the number density and creation rate of the effective gravitons, respectively. This last equation is the novel aspect introduced by the phenomenological formalism to particle (in this case, gravitons) creation [10,11]. The conservation of the energy-momentum tensor leads to

\[
\dot{\rho} + 3H (\rho + p + \Pi_g) = 0 ,
\]

where \( \rho \) is the total energy density (\( \rho = \rho_m + \rho_g \)) and \( p \) is the total pressure (\( p = p_m + p_g, p_m = 0 \)).

As the two fluids are noninteracting, this conservation equation can be split into

\[
\dot{\rho}_m + 3H \rho_m = 0
\]

and

\[
\dot{\rho}_g + 3H \left( \frac{4}{3} \rho_g + \Pi_g \right) = 0 .
\]

From Eqs. (14), (15), (18) and (19) we obtain [1]

\[
\frac{\ddot{a}}{a} + \frac{1}{2} \frac{\dot{a}^2}{a^2} = 4\pi G \left( \rho_g + \frac{\dot{\rho}_g}{3H} \right) ,
\]

\[
\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} (\rho_m + \rho_g) .
\]

It is important to mention that, in the method described in [1,3,4], the Bogoliubov coefficients that appear in the calculus of \( \langle N(\omega) \rangle \) are evaluated at the transition times \( t_1 \) and \( t_2 \). Hence, \( \rho_g(t) \) is univocally determined in terms of \( a(t) \) and \( \dot{a}(t) \), that is [1]

\[
\rho_g(t) = \rho_g(a(t), \dot{a}(t)) .
\]

Therefore, the system of coupled equations (18), (21) and (22) allows us to obtain \( a(t) \) taking into account the back reaction induced by the transformation of very long tensor perturbations into effective gravitational waves.

Let us define

\[
A(t) \equiv \frac{a(t)}{a_2} ,
\]

and

\[
a_2 \equiv a(t_2) ,
\]

\[
a_1 \equiv a(t_1) ,
\]

\[
H_1 \equiv H(t_1) ,
\]

\[
\sigma \equiv \frac{a_2}{a_1} ,
\]

Then Eqs. (8)–(11), (18), (21), and the expressions derived for the Bogoliubov coefficients in [3,4], lead to a lengthy set of equations that enable us to evaluate \( a(t) \) during the matter age. These equations take the form

\[
\dot{A}^2 = \frac{H_1^2}{\sigma^4} \frac{1}{A} \left[ 1 + \kappa \left( \frac{H_1}{m_{Pl}} \right)^2 \frac{1}{A} G(t) \right] ,
\]

where \( m_{Pl} \) is the Planck mass. The values of the constant \( \kappa \) and of the function \( G(t) \) depends whether \( \gamma = 0 \) (the initial era is a de Sitter one), or \( \gamma \neq 0 \) (we will restrict our analysis to inflationary models, so that \( \gamma < 2/3 \)).

For \( \gamma = 0 \):
\( \kappa = \frac{2}{3\pi} \),

\[
G(t) = \left( 1 + \frac{1}{16\sigma^2} \right) \ln \tau + \frac{b_1^2}{2048\pi^4} (\tau^4 - 1) + \frac{b_2^2}{32\pi^2} \ln b_2 + c_0, \quad (29)
\]

\[
\tau \equiv \frac{H_1}{b_2 \sigma^2 \dot{A}}, \quad (30)
\]

\[
c_0 \equiv 16\pi^4 \left( \frac{1}{b_1^4} - \frac{1}{b_2^4 \sigma^4} \right), \quad (31)
\]

\[
j(t) \equiv N_1(\sigma, \tau) \sin \theta_1 + N_2(\sigma, \tau) \cos \theta_1 + N_3(\sigma) \cos \theta_2 + N_4(\sigma) \sin \theta_2 + N_5(\sigma) J(t), \quad (32)
\]

\[
\theta_1(t) \equiv 4\pi\sigma(\sigma - 1) \frac{\dot{A}}{H_1}, \quad (33)
\]

\[
\theta_2 \equiv \frac{4\pi(\sigma - 1)}{b_2 \sigma}, \quad (34)
\]

\[
J(t) \equiv \int_{\theta_1(t)}^{\theta_2} \cos \theta d\theta, \quad (35)
\]

\[
N_1(\sigma, \tau) \equiv \frac{b_2}{4\pi} \left[ \frac{(3\sigma + 1)}{\sigma} \tau^3 - \frac{b_2(11\sigma^3 + 11\sigma^2 - 7\sigma + 1)}{\pi} \right], \quad (36)
\]

\[
N_2(\sigma, \tau) \equiv \frac{1}{2} \left[ \frac{(11\sigma^2 + 6\sigma - 1)}{\sigma^2} \tau^2 - \frac{b_2^2}{8\pi^2} \tau^4 \right], \quad (37)
\]

\[
N_3(\sigma) \equiv \frac{1}{2} \left[ \frac{b_2^2}{8\pi^2} - \frac{(11\sigma^2 + 6\sigma - 1)}{\sigma^2} \right], \quad (38)
\]

\[
N_4(\sigma) \equiv \frac{b_2}{4\pi} \left[ \frac{b_2(11\sigma^3 + 11\sigma^2 - 7\sigma + 1)}{\pi} - \frac{(3\sigma + 1)}{\sigma} \right], \quad (39)
\]

\[
N_5(\sigma) \equiv - \frac{b_2(\sigma - 1)}{\pi \sigma} (11\sigma^3 + 11\sigma^2 - 7\sigma + 1). \quad (40)
\]

On the above equations the free parameters \( b_1 \) and \( b_2 \) are related to the transitions time scales \( \Delta t_1 \) and \( \Delta t_2 \) that are assumed to take place at \( t_1 \) and \( t_2 \), respectively:

\[
\Delta t_1 = \frac{b_1}{H(t_1)} = \frac{b_1}{H_1}, \quad (41)
\]

\[
\Delta t_2 = \frac{b_2}{H(t_2)} = \frac{b_2}{H_2}. \quad (42)
\]
Thus, the larger $b_i$, the slower is the transition in comparison with the Hubble time at $t_i$, $H_i^{-1}$. It is usual to take $b_1 \sim b_2 \sim 1$, but, as for the first transition, there is no compelling reason to assume that it necessarily occurred in a time scale of the order of $H_i^{-1}$ in every model. For example, this transition may occur as fast as $10^{-4}H_i^{-1}$ in the ‘new’ inflationary scenarios [2].

When the first stage is not a de Sitter one, that is, for $\gamma < 2/3$, $\gamma \neq 0$, the constant $\kappa$ and the function $G(t)$ become

$$\kappa = \frac{4}{3},$$

$$G(t) = \frac{1}{\pi \sigma^4} G_1(t) + \frac{4}{|2m + 1|^4} C_0,$$  \hspace{1cm} (44)

where

$$m \equiv \frac{3 (2 - \gamma)}{2 (3 \gamma - 2)} < 0,$$  \hspace{1cm} (46)

$$C_0 \equiv \frac{1}{4} \left( m - \frac{1}{2} \right)^2 \left( m + \frac{1}{2} \right)^2 \int_{y_2}^{y_1} B_m(y) dy + D(y_1) - D(y_2),$$  \hspace{1cm} (47)

$$y_1 \equiv \frac{\pi |2m + 1|}{b_1},$$

$$y_2 \equiv \frac{\pi |2m + 1|}{b_2 \sigma},$$  \hspace{1cm} (49)

$$D(y) \equiv f_1(m, y) B_m(y) + f_2(m, y) B_{m+1}(y) + f_3(m, y) C_m(y) + \frac{y^4}{\pi},$$  \hspace{1cm} (50)

$$f_1(m, y) \equiv \frac{1}{4} \left[ y^5 + \frac{1}{2} \left( m + \frac{1}{2} \right) \left( m + \frac{3}{2} \right) y^3 - \frac{1}{2} \left( m - \frac{1}{2} \right)^2 \left( m + \frac{1}{2} \right) y \right],$$  \hspace{1cm} (51)

$$f_2(m, y) \equiv \frac{1}{4} \left[ y^5 + \frac{1}{2} \left( m + \frac{1}{2} \right) \left( m - \frac{1}{2} \right) y^3 \right],$$  \hspace{1cm} (52)

$$f_3(m, y) \equiv \frac{1}{2} \left( m + \frac{1}{2} \right) \left[ y^4 + \frac{1}{2} \left( m - \frac{1}{2} \right)^2 y^2 \right],$$  \hspace{1cm} (53)

$$B_m(y) \equiv J^2_m(y) + Y^2_m(y),$$

$$C_m(y) \equiv J_m(y) J_{m+1}(y) + Y_m(y) Y_{m+1}(y).$$  \hspace{1cm} (55)

$J_m$ and $Y_m$ are, respectively, the Bessel functions of the first and second kinds of order $m$, and

$$G_1(t) \equiv \frac{1}{32 \sigma} \ln \tau + P(m, \sigma, b_2) \left[ \frac{\pi^2}{b_2^2 (-2m - 3)} \left( \tau^{-2m-3} - 1 \right) + \frac{b_2}{512 \pi^2 (1 - 2m)} \left( \tau^{1 - 2m} - 1 \right) + \frac{1}{128 \pi} f(t) \right],$$  \hspace{1cm} (56)
\[ P(m, \sigma, b_2) \equiv \left( \frac{2b_2 \sigma}{\pi |2m+1|} \right)^{-2m} \Gamma^2(-m|2m+1), \]  

\[ f(t) \equiv P_1(m, \sigma) \tau^{-2m} \sin \theta_1 + \left[ \frac{4\pi}{b_2} P_2(m, \sigma) \tau^{-2m-1} - \frac{b_2}{4\pi(1-2m)} \tau^{1-2m} \right] \cos \theta_1 + P_3(m, \sigma, b_2) \cos \theta_2 - P_4(m, \sigma) I(t), \]  

\[ I(t) \equiv \int_{\theta(t)}^{\theta_2} \frac{\sin \theta}{\theta^{2m-1}} d\theta, \]  

\[ P_1(m, \sigma) \equiv \frac{3 \sigma - 2m(1+2\sigma)}{(-2m)(1-2m)\sigma}, \]  

\[ P_2(m, \sigma) \equiv \frac{(\sigma - 1)}{\sigma^2 |2m+1|} \left[ \frac{3 \sigma - 2m(1+2\sigma)}{(-2m)(1-2m) \sigma} + \frac{2\sigma(\sigma + 1)}{\sigma - 1} \right], \]  

\[ P_3(m, \sigma, b_2) \equiv \frac{b_2}{4\pi(1-2m)} - \frac{4\pi}{b_2} P_2(m, \sigma), \]  

\[ P_4(m, \sigma) \equiv -\frac{1}{\sigma |2m+1|} \left[ \frac{4(\sigma - 1)}{2m+1} \right]^{-2m} \left[ \frac{3 \sigma - 2m(1+2\sigma)}{(-2m)(1-2m) \sigma} + \frac{2\sigma(\sigma + 1)}{\sigma - 1} + \frac{2\sigma^2 |2m+1|}{(\sigma - 1)^2} \right], \]  

and \( \Gamma \) is the gamma function.

**IV. NUMERICAL RESULTS AND CONCLUSIONS**

We are interested in evaluating \( a(t) \) after \( t_2 \), hence the initial conditions are

\[ A(t_2) = 1 \]  

and

\[ \dot{A}(t_2) = \frac{H_1}{\sigma^2}. \]  

Note that, besides \( b_1 \) and \( b_2 \), there are three free parameters:

\[ h_1 \equiv H_1/m_{Pl}, \]  

\( \sigma \), which we relate to the duration of the radiation era \( t_2 - t_1 \) through

\[ \sigma^2 = 1 + 2H_1(t_2 - t_1), \]  

and \( m \) (or, equivalently, \( \gamma \)).

We have integrated Eq. (28) numerically for different values of the parameters \( b_1, b_2, h_1, \gamma \), and \( t_2 - t_1 \). We have taken as characteristic values \( b_1 = 1, b_2 = 1, h_1 = 10^{-50} \) and \( 0 \leq \gamma \leq 0.6 \). In Figure 1 we show the solutions for five values of \( t_2 - t_1 \), namely \( 3 \times 10^6 \) yr (curve a), \( 10^6 \) yr (curve b), \( 3 \times 10^5 \) yr (curve c), \( 10^5 \) yr (curve d), and \( 3 \times 10^4 \) yr (curve e). The time is measured in units of the time of the beginning of the matter era \( t_2 \). The important thing to notice is that the solutions are insensitive to all parameters, except \( t_2 - t_1 \). We see that different values of \( t_2 - t_1 \)
change the behaviour of $A(t)$ near $t_2$. All the curves go asymptotically as $t^{2/3}$, which is represented by the dotted curve. However, the net effect of decreasing the duration of the radiation era is to change the present value of $a(t)/a_2$.

We are led to conclude that the transformation of very long tensor perturbations into effective gravitational waves does not change the expansion dynamics during the matter epoch. This is in contrast to what happens during the radiation era, as it has been shown in Ref. [1]. Therefore, the Sahni’s conjecture [12] will hold only under very restrictive conditions.

We must remark, however, that, in the model analysed above, we have assumed a standard evolution from a radiation phase with $a(t) \propto t^{1/2}$ to a dust era with $a(t) \propto t^{2/3}$. We have not taken into account the modifications in the dynamics generated by the gravitons’ back reaction during the radiation dominated period [1]. A more realistic model would consider these modifications and then a transition to a matter dominated era. The numerical work becomes much more involved in this case. Moreover, primordial nucleosynthesis and the data related to the anisotropy of the cosmic microwave background will place severe constraints on the relevant parameters. These questions are presently under investigation.

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FIG. 1. A logarithmic plot of the quantity $A = a/a_2$ as a function of time [solution of Eq. (28)] for $b_1 = 1$, $b_2 = 1$, $b_3 = 10^{-50}$, $\gamma = 0.5$, and five values of $t_2 - t_1$: $3 \times 10^9$ yr (curve a), $10^9$ yr (curve b), $3 \times 10^8$ yr (curve c), $10^8$ yr (curve d), and $3 \times 10^7$ yr (curve e). The dotted curve corresponds to $a(t) \propto t^{2/3}$ and the log is to base 10. The solutions are insensitive to all parameters, except $t_2 - t_1$. 