The Universe With Bulk Viscosity

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ABSTRACT

Exact solutions for a model with variable $G$ and $\Lambda$ and bulk viscosity are obtained. The inflationary solution exists with a constant and non-constant energy density. For anisotropic universe the anisotropy energy is found to decay with time and remains constant for a static universe. For the inflationary solution with variable energy density the anisotropy energy decays exponentially with time. The gravitational constant is found to increase with time.

KEY WORDS: Variable $G$, $\Lambda$; Bianchi models; Bulk viscosity; Inflation

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1. INTRODUCTION

In a recent paper, Kalligas, Wessan and Everitt [1] have investigated a flat model with variable gravitational (G) and cosmological (Λ) “constants”. In the same line Beesham [4] has considered a viscous cosmological model with variable G and Λ. He considered a different energy conservation law from that of ref.[2]. Incidentally, he found similar solutions as in ref.[2]. Very recently, we have studied Bianchi type I model and we have shown that the universe isotropises during its course of expansion. In the same fashion, we wish to study the effect of anisotropy in the universe with the energy conservation advocated by Beesham. Kalligas et al. have found solutions for a static universe with zero total energy density while G and Λ are allowed to vary with time. However, their solution, does not seem to be physically sensible, since one does expect that G will change with time in an empty universe.

In the present case, we show that the static universe corresponds to a constant G, Λ and bulk viscosity. We have also obtained solutions with constant energy density that correspond to either static or inflationary universe. We remark that these solutions do not hold for our earlier work [2].

2. SOLUTIONS FOR THE ISOTROPIC UNIVERSE

For a flat Robertson Walker metric

\[ ds^2 = dt^2 - R^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \]  

with an imperfect-fluid energy momentum tensor

\[ T_{\mu\nu} = (\rho + p^*)u_\mu u_\nu - p^*g_{\mu\nu} \]  

and time-dependent G and Λ, Einstein’s field equations, give

\[ 3 \frac{\ddot{R}}{R} = -4\pi G(3p^* + \rho) + \Lambda, \]  

and

\[ 3 \frac{\dot{R}^2}{R^2} = 8\pi G\rho + \Lambda. \]  

The Bianchi identities yield

\[ 3(p^* + \rho)\dot{R} = -\left(\frac{\dot{G}}{G}\rho + \dot{\rho} + \frac{\dot{\Lambda}}{8\pi G}\right)R. \]  

where a dot denotes differentiation with respect to time t and \( p^* = p - 3\eta H \), η being
the coefficient of bulk viscosity, $H$ the Hubble constant. The bulk viscosity takes the form

$$
\eta = \eta_0 \rho^n, \quad \eta_0 \geq 0, \quad n = \text{const}.
$$

The equation of state relates the pressure ($p$) and the energy density ($\rho$) of the cosmic fluid by the equation

$$
p = (\gamma - 1) \rho
$$

where $\gamma = \text{constant}$. The energy conservation ($T_{\mu\nu} = 0$) splits eq.(5) into

$$
\dot{\rho} + 3(p + \rho) H = 9 \eta H^2,
$$

and

$$
8\pi \dot{G} \rho + \dot{\Lambda} = 0.
$$

### 2.1 Solution with constant energy density

One can satisfy eq.(7) with a constant energy density ($\rho = \text{const}$) with:

(i) $H = 0$, which implies a static universe.

(ii) $H = \frac{2}{3 \eta_0} = \text{const}$. The solution of this equation is of the form $R = \text{const.} \exp(\text{Ht})$.

We also notice that there is another possible inflationary solution but with a variable energy density. This corresponds to the case when $\eta = \eta_0 \rho$ and $H < \frac{2}{3 \eta_0}$ (see 3.1.c). We would like to remark that the classical inflation with an equation of state $p = -\rho$ is not permitted in this model.

### 3. SOLUTIONS FOR AN ANISOTROPIC UNIVERSE

For the Bianchi type I metric

$$
ds^2 = dt^2 - R_1^2 dx^2 - R_2^2 dy^2 - R_3^2 dz^2
$$

with an imperfect-fluid energy momentum tensor, Einstein’s field equations yield [3]

$$
\frac{\dot{R}_1 \dot{R}_2}{R_1 R_2} + \frac{\dot{R}_1 \dot{R}_3}{R_1 R_3} + \frac{\dot{R}_2 \dot{R}_3}{R_2 R_3} = -8\pi G \rho - \Lambda,
$$

$$
\frac{\dot{R}_1 \dot{R}_2}{R_1 R_2} + \frac{\dot{R}_3}{R_3} + \frac{\dot{R}_2}{R_2} = 8\pi G p^* - \Lambda,
$$

$$
\frac{\dot{R}_1 \dot{R}_3}{R_1 R_3} + \frac{\dot{R}_1}{R_1} + \frac{\dot{R}_3}{R_3} = 8\pi G p^* - \Lambda,
$$

$$
\frac{\dot{R}_2 \dot{R}_3}{R_2 R_3} + \frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3} = 8\pi G p^* - \Lambda,
$$

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and
\[ 8\pi \dot{G}\rho + \dot{\Lambda} + 8\pi G[\dot{\rho} + (\rho + p^*)(\frac{\dot{R}_1}{R_1} + \frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3})] = 0. \tag{14} \]
where a dot denotes differentiation with respect to time \( t \). From eq.(10)-(14) one obtains
\[ \frac{\ddot{R}_1}{R_1} + \frac{\ddot{R}_2}{R_2} + \frac{\ddot{R}_3}{R_3} = 4\pi G(\rho + 3p^*) - \Lambda. \tag{15} \]
where \( p^* = p - 3\eta H \). Applying the energy conservation law to eq.(14) one obtains
\[ \dot{\rho} + (\rho + p)(\frac{\dot{R}_1}{R_1} + \frac{\dot{R}_2}{R_2} + \frac{\dot{R}_3}{R_3}) = 9\eta H^2, \tag{16} \]
and
\[ \dot{\Lambda} + 8\pi \dot{G}\rho = 0, \tag{17} \]
where we have defined the average scale factor \( R \) by \( R = (R_1R_2R_3)^{1/3} \) and the equation of state \( p = (\gamma - 1)\rho, \gamma = \text{constant} \). Equation (16) then becomes
\[ \dot{\rho} + 3\gamma \rho H = 9\eta H^2. \tag{18} \]
Consider the ansatz \[2,5\]
\[ \Lambda = 3\beta H^2, \quad \beta \text{ const.}. \tag{19} \]
Let us now assume that the energy density is given by the power law
\[ \rho = At^m, \quad A, m \text{ const.}. \tag{20} \]
and the average scale factor
\[ R = Bt^\alpha, \quad \alpha, B \text{ const.}. \tag{21} \]
Substituting these equations in eqs.(17) and (18) using eq.(19), we get
\[ \rho = At^{-1/(1-n)}, \tag{22} \]
\[ \Lambda = 3\alpha \beta t^{-2}, \tag{23} \]
\[ G = \frac{3\beta \alpha^2(1-n)}{4\pi A(2n-1)} t^{(2n-1)/(1-n)}, \quad n \neq 1/2, \ n \neq 1. \tag{24} \]
and the condition \( m = \frac{1}{1-n} \). For a physical significance \( G > 0 \) so that \( n > 1/2 \). This implies that the gravitational constant is an ever-increasing function of time. We now consider the anisotropy energy \((\sigma)\) defined by

\[
8\pi G \sigma = \left( \frac{\dot{R}_1}{R_1} - \frac{\dot{R}_2}{R_2} \right)^2 + \left( \frac{\dot{R}_1}{R_1} - \frac{\dot{R}_3}{R_3} \right)^2 + \left( \frac{\dot{R}_2}{R_2} - \frac{\dot{R}_3}{R_3} \right)^2. \tag{25}
\]

Using eqs.(10) and (16) the above equation becomes

\[
8\pi G \sigma = 18\left( \frac{\dot{R}}{R} \right)^2 + 48\pi G \rho + 6\Lambda. \tag{26}
\]

We see that the anisotropy energy \((\sigma)\) becomes

\[
8\pi G \sigma = B t^{-2}, \quad B \quad \text{const.}, \tag{27}
\]

a result that has been obtained by [3].

3.1 Constant energy solution

Equation (18) can be written in the form

\[
\dot{\rho} + 3\gamma H \rho = 9\eta_0 \rho^n H^2. \tag{28}
\]

Now consider the following cases

(a) **Static universe** \((H = 0)\).

The above equation yields \( \rho = \text{const.} \), and eqs.(17) and (19) give \( \Lambda = 0 \) and \( G = \text{const.} \), respectively. It follows from eq.(26) that the anisotropy energy \( \sigma = 6\rho \). However, Kalligas et al. have shown that a static universe can only have \( \sigma = 0 \).

(b) **Inflationary universe with a constant energy density** \((H = \text{const.} \equiv H_0, n = 1)\)

Equation (28) gives \( H_0 = \frac{\gamma}{3\eta_0}, \rho = \text{const.} \). Equations (17) and (26) give \( \Lambda = \text{const.} \) and \( \sigma = \text{const.} \).

(c) **Inflationary solution with a variable energy density** \((H = H_0 = \text{const.})\)

Assume that the bulk viscosity to have the form \( \eta = \eta_0 \rho \). If \( H_0 < \frac{\gamma}{3\eta_0} \) then eqs.(28) and (17) give

\[
\rho = D \exp \left( -3\gamma H_0 - 9\eta_0 H_0^2 \right), \quad D = \text{const.} \quad \text{and} \quad G = \text{const.} \tag{29}
\]

It is evident, from eq.(26), that the anisotropy density decays exponentially with time in the late times.
4. CONCLUSION
In this paper we have studied both isotropic and anisotropic models with variable $G$, $\Lambda$ and bulk viscosity. We have found that a constant energy density would either lead to a static or inflationary universe. An inflationary solution with a variable energy density is also found. It is remarked by Kalligas et al. that a static universe can only have a zero anisotropy energy. In the present model the anisotropy energy is constant for the static universe and that the classical inflation of an equation of state $p = -\rho$ is not possible. The inflationary solutions are influenced by the effect of bulk viscosity. The anisotropy energy is found to decay exponentially with time during the de Sitter expansion. And the gravitational constant is an ever-increasing function of time.

REFERENCES