Cosmology and Heterotic M–Theory in Five–Dimensions

André Lukas\textsuperscript{1}, Burt A. Ovrut\textsuperscript{2} \footnote{†Lectures presented at the Advanced School on Cosmology and Particle Physics, June 1998, Peniscola, Spain.} \footnote{‡Supported in part by a Senior Alexander von Humboldt Award.} and Daniel Waldram\textsuperscript{3}

\textsuperscript{1}Department of Physics, Oxford University
1 Keble Road, Oxford OX1 3NP, United Kingdom

\textsuperscript{2}Department of Physics, University of Pennsylvania
Philadelphia, PA 19104–6396, USA

\textsuperscript{3}Department of Physics, Joseph Henry Laboratories,
Princeton University, Princeton, NJ 08544, USA

Abstract

In these lectures, we present cosmological vacuum solutions of Hořava-Witten theory and discuss their physical properties. We begin by deriving the five–dimensional effective action of strongly coupled heterotic string theory by performing a reduction, on a Calabi–Yau three–fold, of M–theory on $S^1/Z_2$. The effective theory is shown to be a gauged version of five–dimensional $N = 1$ supergravity coupled, for simplicity, to the universal hypermultiplet and four–dimensional boundary theories with gauge and universal gauge matter fields. The static vacuum of the theory is a pair of BPS three–brane domain walls. We show that this five–dimensional theory, together with the domain wall vacuum solution, provides the correct starting point for early universe cosmology in Hořava-Witten theory. Relevant cosmological solutions are those associated with the BPS domain wall vacuum. Such solutions must be inhomogeneous, depending on the orbifold coordinate as well as on time. We present two examples of this new type of cosmological solution, obtained by separation of variables. The first example represents the analog of a rolling radii solution with the radii specifying the geometry of the domain wall pair. This is generalized in the second example to include a nontrivial Ramond-Ramond scalar.
Introduction:

It has been known for a long time that there are five consistent superstring theories, all defined in ten-dimensional spacetime. These are the two $N = 2$ supersymmetric string theories called Type IIA and IIB and three $N = 1$ superstrings called Type I and heterotic $SO(32)$ and heterotic $E_8 \times E_8$. A few years ago, it was demonstrated that these apparently different theories are, in fact, all related to each other through an intricate web of so-called duality transformations. Furthermore, it became clear that these five superstring theories are simply part of a much larger moduli space associated with a more fundamental and unique theory, termed $M$-theory. Interestingly, it was shown that there is yet another region of the moduli space of $M$-theory that corresponds to the theory of eleven-dimensional supergravitation. It follows that $M$-theory cannot be a superstring theory, but, to date, the exact nature of $M$-theory remains unknown. Be that as it may, $M$-theory does emphasize the fundamental importance of eleven-dimensional supergravity as a starting point for exploring aspects of cosmology and particle physics at energies near or below the eleven-dimensional Planck scale. In these lectures, we will discuss what is presently known about the exact structure of $M$-theory at low energy and present the first results on the cosmological solutions and behaviour of this theory. We refer the reader to reference [1] for a detailed discussion of duality in superstring theory and to references therein.

The strongly coupled $E_8 \times E_8$ heterotic string has been identified as the eleven-dimensional limit of $M$-theory compactified on an $S^1/Z_2$ orbifold with a set of $E_8$ gauge supermultiplets on each ten-dimensional orbifold fixed plane [2, 3]. Witten has shown that there exists a consistent compactification of this $M$-theory limit on a deformed Calabi-Yau three-fold, leading to a supersymmetric $N = 1$ theory in four dimensions [4]. Matching at tree level to the phenomenological gravitational and grand-unified gauge couplings [4, 5], one finds that the orbifold must be larger than the Calabi-Yau radius, which is of the order of the inverse gauge coupling unification scale, about $10^{16}$ GeV. Typically, one finds that the orbifold radius can be chosen to be from a factor of four or five times the inverse unification scale to as large as inverse $10^{14}$ GeV, depending on circumstances. This suggests that there is a substantial regime where the universe appears five-dimensional. It is important, therefore, to determine the five-dimensional effective action describing heterotic $M$-theory in this regime. This theory constitutes a new setting for early universe cosmology, which has traditionally been studied in the framework of the four-dimensional effective action.

In previous papers [6, 7], we derived this five-dimensional effective theory by directly re-
ducing Hořava–Witten theory on a Calabi–Yau three–fold. We showed that a non–zero mode of the antisymmetric tensor field strength has to be included for a consistent reduction from eleven to five dimensions and that the correct five–dimensional effective theory of the strongly coupled heterotic string is given by a gauged version of five–dimensional supergravity. We explicitly included all \((1,1)\) moduli vector superfields and the universal hypermultiplet, as well as indicating how the remaining \((2,1)\) moduli could be incorporated. We also discussed the four–dimensional boundary gauge and matter supermultiplets and derived their effective action. In these lectures, we present the effective five-dimensional theory for the universal bulk fields only; that is, the gravity supermultiplet and the universal hypermultiplet. We show that the consistent reduction from eleven to five dimensions on a Calabi-Yau manifold requires the inclusion of non-zero values of the four-form field strength in the internal Calabi-Yau three–fold directions. This leads to a gauged five-dimensional supergravity action with a potential term. More precisely, given the universal hypermultiplet coset manifold \([8]\) \( \mathcal{M}_Q = SU(2,1)/SU(2) \times U(1) \), we find that a subgroup \( U(1) \subset SU(2) \times U(1) \) is gauged, with the vector field in the gravity supermultiplet as the corresponding gauge boson. Due to the potential, flat space is not a solution of this five-dimensional theory. However, the equations of motion do admit as a solution a pair of three-branes that preserves half of the remaining \(d = 5\) supersymmetries. This solution is supported by source terms on the fixed orbifold planes of the five-dimensional space. This pair of BPS three-branes constitutes the static “vacuum” of the five-dimensional theory and it is the appropriate background for a further reduction to four-dimensional \(N = 1\) supergravity theories. In such a reduction, four-dimensional space-time becomes identified with the three-brane worldvolume. We will show that the linearized version of this three-brane corresponds to Witten’s “deformed” Calabi-Yau solution, which was constructed only to first non-trivial order in powers of the eleven-dimensional Newton constant. Subsequent work discussing Hořava-Witten and related theories from the point of view of five–dimensional supergravity with boundaries can be found in \([9, 10]\).

It is clear from the above discussion that cosmology in effective Hořava-Witten theory should be studied in the five-dimensional context and will, generically, be inhomogeneous in the extra spatial dimension. What should realistic theories look like? In the ideal case, one would have a situation in which the internal six-dimensional Calabi-Yau space and the orbifold evolve for a period of time and then settle down to their “phenomenological” values while the three non–compact dimensions continue to expand. Then, for late time, when all physical scales are much larger than the orbifold size, the theory is effectively
four-dimensional and should, in the “static” limit, provide a realistic supergravity model of particle physics. As we have argued above, such realistic supergravity models originate from a reduction of the five-dimensional theory on its BPS domain wall vacuum state. Hence, in the “static” limit at late time, realistic cosmological solutions should reduce to the BPS domain wall vacuum or, perhaps, a modification thereof that incorporates spontaneous breaking of the remaining four-dimensional \( N = 1 \) supersymmetry. Consequently, one is forced to look for solutions which depend on the orbifold coordinate as well as on time. In a previous paper [11], we presented examples of such cosmological solutions in five-dimensional heterotic M-theory. In these lectures, we review these solutions and illustrate some of the characteristic cosmological features of the theory.

In earlier work [12, 13], we showed how a general class of cosmological solutions, that is, time-dependent solutions of the equations of motion that are homogeneous and isotropic in our physical \( d = 3 \) subspace, can be obtained in both superstring theories and M-theory defined in spacetimes without boundary. Loosely speaking, we showed that a cosmological solution could be obtained from any p-brane or D-brane by inverting the roles of the time and “radial” spatial coordinate. This method will clearly continue to work in Hořava-Witten theory as long as one exchanges time with a radial coordinate not aligned in the orbifold direction. An example of this in eleven-dimensions, based on the solution of [14], has been given in [15]. It can not, however, be applied to the fundamental domain wall since its radial direction coincides with the orbifold coordinate. This coordinate is bounded and cannot be turned into time. Instead, the domain wall itself should be made time dependent thereby leading to solutions that depend on both time and the orbifold coordinate. As a result, we have to deal with coupled partial differential equations, but, under certain constraints, these can by solved by separation of variables, though the equations remain non-linear. Essentially, we are allowing the moduli describing the geometry of the domain wall and the excitations of other five-dimensional fields, to become time-dependent. Technically, we will simply take the usual Ansätze for the five-dimensional fields, but now allow the functions to depend on both the time and radial coordinates. We will further demand that these functions each factor into a purely time dependent piece times a purely radial dependent piece. This is not, in general, sufficient to separate the equations of motion. However, we will show that, subject to certain constraints, separation of variables is achieved. We can solve these separated equations and find new, cosmologically relevant solutions. In these lectures, we will restrict our attention to two examples representing cosmological extensions of the pure BPS pair of three-branes.
The first example is simply the domain wall itself with two of its three moduli made time-dependent. We show that separation of variables occurs in this case. It turns out that these moduli behave like “rolling radii” [16], which constitute fundamental cosmological solutions in weakly coupled string theory. Unlike those rolling radii which represent scale factors of homogeneous, isotropic spaces, here they measure the separation of the two walls of the three-brane and its worldvolume size (which, at the same time, is the size of “our” three-dimensional universe). We have, therefore, a time-dependent domain wall pair with its shape staying rigid but its size and separation evolving like rolling radii.

For the second example, we consider a similar setting as for the first but, in addition, we allow a nonvanishing Ramond-Ramond scalar. This terminology is perhaps a little misleading, but relates to the fact that the scalar would be a Type II Ramond-Ramond field in the case where the orbifold was replaced by a circle. This makes connection with Type II cosmologies with non-trivial Ramond-Ramond fields discussed in [12, 13]. Separation of variables occurs for a specific time-independent form of this scalar. The orbifold-dependent part then coincides with the domain wall with, however, the addition of the Ramond-Ramond scalar. This non–vanishing value of the scalar breaks supersymmetry even in the static limit. We find that the time-dependent part of the equations fits into the general scheme of M-theory cosmological solutions with form fields as presented in ref. [12, 13]. Applying the results of these papers, the domain wall moduli are found to behave like rolling radii asymptotically for early and late times. The evolution rates in these asymptotic regions are, however, different and the transitions between them can be attributed to the nontrivial Ramond-Ramond scalar.

Let us summarize our conventions. We will consider eleven-dimensional spacetime compactified on a Calabi-Yau space $X$, with the subsequent reduction down to four dimensions effectively provided by a double-domain-wall background, corresponding to an $S^1/Z_2$ orbifold. We use coordinates $x^I$ with indices $I, J, K, \ldots = 0, \ldots, 9, 11$ to parameterize the full eleven-dimensional space $M_{11}$. Throughout these lectures, when we refer to orbifolds, we will work in the “upstairs” picture with the orbifold $S^1/Z_2$ in the $x^{11}$-direction. We choose the range $x^{11} \in [-\pi \rho, \pi \rho]$ with the endpoints being identified. The $Z_2$ orbifold symmetry acts as $x^{11} \rightarrow -x^{11}$. Then there exist two ten-dimensional hyperplanes fixed under the $Z_2$ symmetry which we denote by $M^{(i)}_{10}$, $i = 1, 2$. Locally, they are specified by the conditions $x^{11} = 0, \pi \rho$. Barred indices $\bar{I}, \bar{J}, \bar{K}, \ldots = 0, \ldots, 9$ are used for the ten-dimensional space orthogonal to the orbifold. Upon reduction on the Calabi-Yau space we have a five-dimensional spacetime $M_5$ labeled by indices $\alpha, \beta, \gamma, \ldots = 0, \ldots, 3, 11$. The orbifold fixed planes become
four-dimensional with indices $\mu, \nu, \rho, \ldots = 0, \ldots, 3$. We use indices $A, B, C, \ldots = 4, \ldots, 9$ for the Calabi-Yau space. The eleven-dimensional Dirac-matrices $\Gamma^I$ with $\{\Gamma^I, \Gamma^J\} = 2g^{IJ}$ are decomposed as $\Gamma^I = \{\gamma^a \otimes \lambda, 1 \otimes \lambda^A\}$ where $\gamma^a$ and $\lambda^A$ are the five- and six-dimensional Dirac matrices, respectively. Here, $\lambda$ is the chiral projection matrix in six dimensions with $\lambda^2 = 1$.

Spinors in eleven dimensions will be Majorana spinors with 32 real components throughout the paper. In five dimensions we use symplectic-real spinors $\psi^i$ where $i = 1, 2$ is an $SU(2)$ index, corresponding to the automorphism group of the $N = 1$ supersymmetry algebra in five dimensions. We will follow the conventions given in [18]. Fields will be required to have a definite behaviour under the $Z_2$ orbifold symmetry in $D = 11$. We demand a bosonic field $\Phi$ to be even or odd; that is, $\Phi(x^{11}) = \pm \Phi(-x^{11})$. For a spinor $\Psi$ the condition is $\Gamma_{11}\Psi(-x^{11}) = \pm \Psi(x^{11})$ so that the projection to one of the orbifold planes leads to a ten-dimensional Majorana-Weyl spinor with definite chirality. Similarly, in five dimensions, bosonic fields will be either even or odd. We can choose a basis for the $SU(2)$ automorphism group such that symplectic-real spinors $\psi^i$ satisfy the constraint $\gamma_{11}\psi^i(-x^{11}) = (\tau_3)_{ij}\psi^j(x^{11})$ where $\tau_a$ are the Pauli spin matrices, so $\tau_3 = \text{diag}(1, -1)$.

Lecture 1:

In this first lecture, we will discuss the Hořava–Witten theory of eleven-dimensional supergravity compactified on an $S^1/Z_2$ orbifold. We then, using the standard embedding of the spin connection into the gauge connection, discuss the compactification of this theory to four–dimensions so that a single $N = 1$ supersymmetry is preserved. The constraint that $N = 1$ supersymmetry be preserved causes a “deformation” of the background spacetime. We show, in detail, that mathematical simplification and physical clarity can be achieved by first compactifying Hořava–Witten on a Calabi–Yau three-fold to five–dimensions. We construct the five–dimensional effective theory in detail and show that it is a new, gauged form of supergravity that does not admit flat spacetime as its static vacuum. We end the lecture by discussing the supersymmetry variations of the gravitino and hypermultiplet fermions.
The strongly coupled heterotic string and Calabi-Yau solutions

To set the scene for our later discussion, we will now briefly review the effective description of strongly coupled heterotic string theory as eleven-dimensional supergravity with boundaries given by Horava and Witten [2, 3]. In addition, we present, in a simple form, the solutions of this theory [4] appropriate for a reduction to $N = 1$ theories in four dimensions using the explicit form of these solutions given in ref. [19].

The bosonic part of the action is of the form

$$S = S_{SG} + S_{YM}$$

where $S_{SG}$ is the familiar eleven-dimensional supergravity

$$S_{SG} = -\frac{1}{2\kappa^2} \int_{M^{11}} \sqrt{-g} \left[ R + \frac{1}{24} G_{IJKL} G^{IJKL} + \frac{\sqrt{2}}{1728} \epsilon_{I_{1}...I_{11}} C_{I_{1}I_{2}I_{3}} G_{I_{4}...I_{7}} G_{I_{8}...I_{11}} \right]$$

and $S_{YM}$ are the two $E_8$ Yang-Mills theories on the orbifold planes explicitly given by

$$S_{YM} = -\frac{1}{8\pi\kappa^2} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_{M^{(1)}} \sqrt{-g} \left\{ \operatorname{tr}(F_{i}^{(1)})^2 - \frac{1}{2} \operatorname{tr} R^2 \right\} - \frac{1}{8\pi\kappa^2} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_{M^{(2)}} \sqrt{-g} \left\{ \operatorname{tr}(F_{i}^{(2)})^2 - \frac{1}{2} \operatorname{tr} R^2 \right\}.$$  

Here $F_{i}^{(i)}$ are the two $E_8$ gauge field strengths and $C_{IJK}$ is the 3-form with field strength $G_{IJKL} = 24 \partial_{[I} C_{JKL]}$. In order for the above theory to be supersymmetric as well as anomaly free, the Bianchi identity for $G$ should receive a correction such that

$$(dG)_{11IJKL} = -\frac{1}{2\sqrt{2\pi}} \left( \frac{\kappa}{4\pi} \right)^{2/3} \left\{ J^{(1)} \delta(x^{11}) + J^{(2)} \delta(x^{11} - \pi \rho) \right\}_{IJKL}$$

where the sources are given by

$$J^{(i)} = \operatorname{tr} F^{(i)} \wedge F^{(i)} - \frac{1}{2} \operatorname{tr} R \wedge R.$$  

We note that there is a debate in the literature about the precise value of the Yang-Mills coupling constant in terms of $\kappa$. While we quote the original value [3, 20], the value found in ref. [21] is smaller. In the second case, the coefficients in the Yang-Mills action (3) and the Bianchi identity (4) should both be multiplied by $2^{-1/3}$. This potential factor will not be essential in the following discussion as it will simply lead to a redefinition of the five-dimensional coupling constants. We will comment on this point later on.
Under the $Z_2$ orbifold symmetry, the field components $g_{IJ}$, $g_{11,11}$, $C_{IJK}$ are even, while $g_{111}$, $C_{111}$ are odd. We note that the above boundary actions contain, in addition to the Yang-Mills terms, $\text{tr} R^2$ terms which were not part of the original theory derived in [3]. It was argued in ref. [19] that these terms are required by supersymmetry, since they pair with the $R^2$ terms in the Bianchi identity (4) in analogy to the weakly coupled case. The existence of these terms will be of some importance in the following.

One way to view this theory is to draw an analogy between the orbifold planes and D-branes in Type II theories. A collection of D$p$-branes is described by a $U(N)$ gauge theory. The D$p$-brane charge is measured by $\text{tr} \mathbf{1} = N$, while exciting a D$(p-2)$-brane charge corresponds to having a non-trivial $\text{tr} F$, and a D$(p-4)$-brane charge corresponds to non-trivial $\text{tr} F \wedge F$ and so on [22]. Similarly, if the original D-branes are on a curved manifold then there is also an induced charge for lower-dimensional branes given by $\text{tr} R \wedge R$ and higher even powers [23]. Applying this picture to our situation, the rôle of the $U(N)$ gauge field on the D-brane worldvolume is here played by the $E_8$ gauge fields on the orbifold planes. The correction to the Bianchi identity then has the interpretation of exciting an M5-brane charge in the orbifold plane. In ref. [14] this picture has been made explicit by constructing a gauge five-brane in this theory.

We would now like to discuss solutions of the above theory which preserve four of the 32 supercharges leading, upon compactification, to four dimensional $N = 1$ supergravities. This task is significantly complicated by the fact that the sources in the Bianchi identity (4) are located on the orbifold planes with the gravitational part distributed equally between the two planes. While the standard embedding of the spin connection into the gauge connection

$$\text{tr} F^{(1)} \wedge F^{(1)} = \text{tr} R \wedge R \tag{6}$$

leads to vanishing source terms in the weakly coupled heterotic string Bianchi identity (which, in turn, allows one to set the antisymmetric tensor gauge field to zero), in the present case, one is left with non-zero sources $\pm \frac{1}{2} \text{tr} R \wedge R$ on the two hyperplanes. As a result, the antisymmetric tensor field $G$ and, hence, the second term in the gravitino supersymmetry variation

$$\delta \Psi_I = D_I \eta + \frac{\sqrt{2}}{288} \left( \Gamma_{IJKLM} - 8g_{IJ} \Gamma_{KLM} \right) G^{JKLM} \eta + \cdots \tag{7}$$

do not vanish. Thus, straightforwardly compactifying on a Calabi-Yau manifold no longer provides a solution to the Killing spinor equation $\delta \Psi_I = 0$. The problem can, however, be treated perturbatively in powers of the eleven-dimensional Newton constant $\kappa$. To lowest
order, one can start with a manifold $M_4 \times S^1/Z_2 \times X$ where $X$ is a Calabi-Yau three-fold and $M_4$ is four-dimensional Minkowski space. This manifold has an $x^{11}$-independent (and hence chiral) Killing spinor $\eta$ which corresponds to four preserved supercharges. Then, one can determine the first order corrections to this background and the spinor $\eta$ so that the gravitino variation vanishes to order $\kappa^{2/3}$.

The existence of such a “deformed” background solution to order $\kappa^{2/3}$ has been demonstrated in ref. [4]. To see its explicit form, let us start with the zeroth order metric

$$ds^2_{11} = \eta_{\mu\nu} dx^\mu dx^\nu + R_0^2(dx^{11})^2 + V_0^{1/3} \Omega_{AB} dx^A dx^B,$$

where $\Omega_{AB}$ is a Calabi-Yau metric with Kähler form $\omega_{a\bar{b}} = i \Omega_{a\bar{b}}$. (Here $a$ and $\bar{b}$ are holomorphic and anti-holomorphic indices.) To keep track of the scaling properties of the solution, we have introduced moduli $V_0$ and $R_0$ for the Calabi-Yau volume and the orbifold radius, respectively. It was shown in [4] that, to order $\kappa^{2/3}$, the metric can be written in the form

$$ds^2_{11} = (1 + \hat{b}) \eta_{\mu\nu} dx^\mu dx^\nu + R_0^2 (1 + \hat{\gamma})(dx^{11})^2 + V_0^{1/3} (\Omega_{AB} + h_{AB}) dx^A dx^B$$

where the functions $\hat{b}$, $\hat{\gamma}$ and $h_{AB}$ depend on $x^{11}$ and the Calabi-Yau coordinates. Furthermore, as we have discussed, $G_{ABCD}$ and $G_{ABC11}$ receive a contribution of order $\kappa^{2/3}$ from the Bianchi identity source terms. To get the general explicit form of the corrections, one has to solve the relations given in ref. [4]. This can be done by dualizing the antisymmetric tensor field and using a harmonic expansion on the Calabi-Yau space [19].

Here, we quote those results simplified in two essential ways. First, we drop all terms corresponding to non-zero eigenvalue harmonics on the Calabi-Yau space. These terms will be of no relevance to the low energy theory, since they correspond to heavy Calabi-Yau modes which decouple at this order. Second, we write only the one massless term that is related to the Calabi-Yau breathing mode. This will be sufficient for all applications dealing only with the universal moduli. Given these simplifications, the corrections are explicitly

$$\hat{b} = -\frac{\sqrt{2} R_0}{3 V_0^{2/3}} \alpha_0 (|x^{11}| - \pi \rho / 2)$$

$$\hat{\gamma} = \frac{2 \sqrt{2} R_0}{3 V_0^{2/3}} \alpha_0 (|x^{11}| - \pi \rho / 2)$$

$$h_{AB} = \frac{\sqrt{2} R_0}{3 V_0^{2/3}} \alpha_0 (|x^{11}| - \pi \rho / 2) \Omega_{AB}$$

$$G_{ABCD} = \frac{1}{6} \alpha_0 \epsilon_{ABCD}^{\ EF} \omega_{EF} \epsilon(x^{11})$$

$$G_{ABC11} = 0$$
with
\[
\alpha_0 = -\frac{1}{8\sqrt{2}\pi v} \left( \frac{\kappa}{4\pi} \right)^{2/3} \int_X \omega \wedge \text{tr} R^{(I)} \wedge R^{(I)},
\]
\[
v = \int_X \sqrt{\Omega}.
\]

(15)

Here \(\epsilon(x^{11})\) is the step function which is +1 (−1) for \(x^{11}\) positive (negative). Note that, by dropping the massive modes, these expressions take a very simple form representing a linear increase of the corrections along the orbifold. Even more significantly, and unlike the exact solution including the heavy modes, the above approximation leads to a corrected metric \(\Omega_{AB} + h_{AB}\) that is still of Calabi-Yau type at each point on the \(S^1/Z_2\) orbifold. The Calabi-Yau volume (and, if all moduli are included, also its shape), however, is continuously changing across the orbifold. More generally, one can think of the internal part of the corrected metric as a curve in the Calabi-Yau moduli space.

Returning to the D-brane perspective, one can view the above configuration as the linearized solution for a collection of five-branes embedded in the orbifold planes. The relation (6) fixes equal amounts of five-brane charge, \(\frac{1}{2}\text{tr} R \wedge R\), on each orbifold fixed plane, where the five-branes are confined to live. Since \(\text{tr} R \wedge R \in H^{2,2}(X)\), we can associate a different five-brane charge for each independent element of \(H^{2,2}(X)\). The five-branes themselves are associated with Poincaré dual cycles. Thus they span the non-compact four-dimensional space together with a two-cycle in the Calabi-Yau space. In particular, from the five-dimensional point of view, they are three-branes localized on the orbifold planes. Witten’s construction ensures that this configuration of branes preserves one-eighth of the supersymmetry. Finally, restricting to just the Calabi-Yau breathing modes corresponds to keeping only the five-brane which spans the holomorphic two-cycle in the Calabi-Yau three-fold defined by the Kähler form.

The five-dimensional effective action

Phenomenologically, there is a regime where the universe appears five-dimensional. We would, therefore, like to derive an effective theory in the space consisting of the usual four space-time dimensions and the orbifold, based on the background solution discussed in the previous section. As we have already mentioned, we will consider the universal zero modes only; that is, the five-dimensional graviton supermultiplet and the breathing mode of the Calabi-Yau space, along with its superpartners. These form a hypermultiplet in five dimensions. Furthermore, to keep the discussion as simple as possible, we will not consider
boundary gauge matter fields. This simple framework suffices to illustrate our main ideas. The general case was presented in [7].

Naively, one might attempt to perform the actual reduction directly on the background given in eqs. (9) and (10)–(14). This would, however, lead to a complicated five-dimensional theory with explicit $x^{11}$-dependence in the action. Moreover, this background preserves only four supercharges whereas the minimal supergravity in five dimensions ($N = 1$) is invariant under twice this amount of supersymmetry.

A useful observation here is that, since we retain the dependence on the orbifold coordinate, we can actually absorb the metric deformations in (9) and (10)–(14) into the five-dimensional metric moduli. That is, the $x^{11}$-dependent scale factors $\hat{b}$ and $\hat{\gamma}$ of the four-dimensional space and of the orbifold can be absorbed into the five-dimensional (Einstein frame) metric $g_{\alpha\beta}$ while, analogously, the variation of the Calabi-Yau volume along the orbifold encoded in $h_{AB}$ can be absorbed into a modulus $V$. More precisely, we can perform the Kaluza-Klein reduction on the metric

$$ds^2_{11} = V^{-2/3}g_{\alpha\beta}dx^\alpha dx^\beta + V^{1/3}\Omega_{AB}dx^A dx^B. \tag{16}$$

This rewriting suggests a change of perspective: rather than reducing on the Witten vacuum, we can try to find an effective five-dimensional theory where we recover the Witten vacuum as a particular solution.

We see that, since we have absorbed the deformation into the moduli, the background corresponding to the metric (16) preserves eight supercharges, the appropriate number for a reduction down to five dimensions. It might appear that we are simply performing a standard reduction of eleven-dimensional supergravity on a Calabi-Yau space to five dimensions; for example, in the way described in ref. [24]. If this were the case, then it would be hard to understand how the resulting five-dimensional theory could encode any information about the deformed Calabi-Yau background. There are, however, two important ingredients that we have not yet included. One is obviously the existence of the boundary theories. We will return to this point shortly. First, however, let us explain a somewhat unconventional addition to the bulk theory that must be included.

Although we could absorb all metric corrections into the five-dimensional metric moduli, the same is not true for the 4-form field. Specifically, for the nonvanishing component

$^{2}$Note that we could not apply a similar method for a reduction down to four dimensions, as all moduli fields would then be $x^{11}$ independent. In this case, one should work with the background in the form (9), (10)–(14) as done in ref. [19].
$G_{ABCD}$ in eq. (13) there is no corresponding zero mode field. Therefore, in the reduction, we should take this part of $G$ explicitly into account. In the terminology of ref. [25], such an antisymmetric tensor field configuration is called a “non-zero mode”. More generally, a non-zero mode is a background antisymmetric tensor field that solves the equations of motion but, unlike antisymmetric tensor field moduli, has nonvanishing field strength. Such configurations, for a $p$-form field strength, can be identified with the cohomology group $H^p(M)$ of the manifold $M$ and, in particular, exist if this cohomology group is nontrivial.

In the case under consideration, the relevant cohomology group is $H^4(X)$ which is nontrivial for a Calabi-Yau manifold $X$ since $h^{2,2} = h^{1,1} \geq 1$. Again, the form of $G_{ABCD}$ in eq. (13) is somewhat special, reflecting the fact that we are concentrating here on the universal moduli. In the general case, $G_{ABCD}$ would be a linear combination of all harmonic $(2,2)$-forms.

The complete configuration for the antisymmetric tensor field that we use in the reduction is given by

$$C_{\alpha\beta\gamma}, \quad G_{\alpha\beta\gamma\delta} = 24 \partial_{[\alpha} C_{\beta\gamma\delta]}$$

$$C_{\alpha AB} = \frac{1}{6} A_\alpha \omega_{AB}, \quad G_{\alpha\beta AB} = \mathcal{F}_{\alpha\beta} \omega_{AB} = 2 \partial_{[\alpha} A_{\beta]} \omega_{AB}, \quad C_{ABC} = \frac{1}{6} \xi \omega_{ABC}, \quad G_{\alpha ABC} = \partial_\alpha \xi \omega_{ABC} \quad (17)$$

and the non-zero mode is

$$G_{ABCD} = \frac{\alpha_0}{6} \epsilon_{ABCD}^{\ EF} \omega_{EF} \epsilon(x^{11}) \quad (18)$$

where $\alpha_0$ was defined in eq. (15). Here, $\omega_{ABC}$ is the harmonic $(3,0)$ form on the Calabi-Yau space and $\xi$ is the corresponding (complex) scalar zero mode. In addition, we have a five-dimensional vector field $A_\alpha$ and 3-form $C_{\alpha\beta\gamma}$, which can be dualized to a scalar $\sigma$. The total bulk field content of the five-dimensional theory is then given by the gravity multiplet $(g_{\mu\nu}, A_\mu, \psi^i_\alpha, \bar{\psi}^i_\alpha)$ together with the universal hypermultiplet $(V, \sigma, \xi, \bar{\xi}, \zeta^i, \bar{\zeta}^i)$ where $\psi^i_\alpha$ and $\zeta^i$ are the gravitini and the hypermultiplet fermions respectively and $i = 1, 2$. From their relations to the eleven-dimensional fields, it is easy to see that $g_{\mu\nu}, g_{11,11}, A_{11}, \sigma$ must be even under the $Z_2$ action whereas $g_{a11}, A_\mu, \xi$ must be odd.

Examples of compactifications with non-zero modes in pure eleven-dimensional supergravity on various manifolds including Calabi-Yau three-folds have been studied in ref. [26].

3This can be seen from the mixed part of the Bianchi identity $\partial_\alpha G_{ABCD} = 0$ which shows that the constant $\alpha_0$ in eqs. (10)–(14) cannot be promoted as stands to a five-dimensional field. It is possible to dualize in five dimensions so the constant $\alpha_0$ is promoted to a five-form field, but we will not pursue this formulation here.
There is, however, one important way in which our non-zero mode differs from other non-zero modes in pure eleven-dimensional supergravity. Whereas the latter may be viewed as an optional feature of generalized Kaluza-Klein reduction, the non-zero mode in Hořava-Witten theory that we have identified cannot be turned off. This can be seen from the fact that the constant \( \alpha_0 \) in expression (18) cannot be set to zero. This is unlike the case in pure eleven-dimensional supergravity where it would be arbitrary, since it is fixed by eq. (15) in terms of Calabi-Yau data. This fact is, of course, intimately related to the existence of the boundary source terms, particularly in the Bianchi identity (4). As we will see, keeping the non-zero mode in the derivation of the five-dimensional action is crucial to finding a solution of this theory that corresponds to the deformed Calabi-Yau space discussed in the previous section.

Let us now turn to a discussion of the boundary theories. In the five-dimensional space \( M_5 \) of the reduced theory, the orbifold fixed planes constitute four-dimensional hypersurfaces which we denote by \( M_4^{(i)} \), \( i = 1, 2 \). Clearly, since we have used the standard embedding, there will be an \( E_6 \) gauge field \( A_\mu^{(1)} \) accompanied by gauginos and gauge matter fields on the orbifold plane \( M_4^{(1)} \). For simplicity, we will set these gauge matter fields to zero in the following. The field content of the orbifold plane \( M_4^{(2)} \) consists of an \( E_8 \) gauge field \( A_\mu^{(2)} \) and the corresponding gauginos. In addition, there is another important boundary effect which results from the non-zero internal gauge field and gravity curvatures. More precisely, note that

\[
\int_X \sqrt{\Omega} \operatorname{tr} F_{AB}^{(1)} F^{(1)AB} = \int_X \sqrt{\Omega} \operatorname{tr} R_{AB} R^{AB} = -16\sqrt{2}\pi v \left( \frac{4\pi}{\kappa} \right)^{2/3} \alpha_0 ,
\]

(19)

In view of the boundary actions (3), it follows that we will retain cosmological type terms with opposite signs on the two boundaries. Note that the size of those terms is set by the same constant \( \alpha_0 \), given by eq. (15), which determines the magnitude of the non-zero mode. The boundary cosmological terms are another important ingredient in reproducing the eleven-dimensional background as a solution of the five-dimensional theory.

We can now compute the five-dimensional effective action of Hořava-Witten theory. Using the field configuration (16)–(19) we find from the action (1)–(3) that

\[
S_5 = S_{\text{grav}} + S_{\text{hyper}} + S_{\text{bound}}
\]

(20)
where
\[ S_{\text{grav}} = - \frac{1}{2\kappa_5^2} \int_{M_5} \sqrt{-g} \left[ R + \frac{3}{2} \mathcal{F}_{\alpha\beta} \mathcal{F}^{\alpha\beta} + \frac{1}{\sqrt{2}} \epsilon_{\alpha\beta\gamma\delta} A_{\alpha} \mathcal{F}_{\beta\gamma} \mathcal{F}_{\delta} \right] \]  \hspace{1cm} (21)
\[ S_{\text{hyper}} = - \frac{1}{2\kappa_5^2} \int_{M_5} \sqrt{-g} \left[ \frac{1}{2V^2} \partial_\alpha V \partial^\alpha V + \frac{2}{V} \partial_\alpha \xi \partial^\alpha \bar{\xi} + \frac{V^2}{24} G_{\alpha\beta\gamma\delta} G^{\alpha\beta\gamma\delta} + \frac{\sqrt{2}}{24} \epsilon_{\alpha\beta\gamma\delta} \epsilon \mathcal{F}^{\alpha\beta\gamma\delta} \right] \]  \hspace{1cm} (22)
\[ S_{\text{bound}} = \frac{\sqrt{2}}{\kappa_5^2} \int_{M_4^{(i)}} \sqrt{-g} V^{-1} \alpha_0 - \frac{\sqrt{2}}{\kappa_5^2} \int_{M_4^{(i)}} \sqrt{-g} V^{-1} \alpha_0 \] 
\[ \quad - \frac{1}{16\pi\alpha_{\text{GUT}}} \sum_{i=1}^{2} \int_{M_4^{(i)}} \sqrt{-g} V \text{tr} F^{(i)\mu\nu} F_{\mu\nu}^{(i)} \]  \hspace{1cm} (23)

In this expression, we have now dropped higher-derivative terms. The four-form field strength \( G_{\alpha\beta\gamma\delta} \) is subject to the Bianchi identity
\[ (dG)_{11\mu\nu\rho\sigma} = - \frac{\kappa_5^2}{4\sqrt{2}\pi\alpha_{\text{GUT}}} \left\{ J^{(1)}(x^{11}) + J^{(2)}(x^{11} - \pi\rho) \right\}_{\mu\nu\rho\sigma} \]  \hspace{1cm} (24)
which follows directly from the eleven-dimensional Bianchi identity (4). The currents \( J^{(i)} \) have been defined in eq. (5). The five-dimensional Newton constant \( \kappa_5 \) and the Yang-Mills coupling \( \alpha_{\text{GUT}} \) are expressed in terms of eleven-dimensional quantities as \(^4\)
\[ \kappa_5^2 = \frac{\kappa^2}{v}, \quad \alpha_{\text{GUT}} = \frac{\kappa^2}{2v} \left( \frac{4\pi}{\kappa} \right)^{2/3}. \]  \hspace{1cm} (25)

We have checked the consistency of the truncation which leads to the above action by an explicit reduction of the eleven-dimensional equations of motion to five dimensions. Note that the potential terms in the bulk and on the boundaries arise precisely from the inclusion of the non-zero mode and the gauge and gravity field strengths, respectively. Since we have compactified on a Calabi-Yau space, we expect the bulk part of the above action to have eight preserved supercharges and, therefore, to correspond to minimal \( N = 1 \) supergravity in five dimensions. Accordingly, let us compare the result (21)–(23) to the known \( N = 1 \) supergravity-matter theories in five dimensions [18, 27, 28].

In these theories, the scalar fields in the universal hypermultiplet parameterize a quaternionic manifold with coset structure \( \mathcal{M}_Q = SU(2,1)/SU(2) \times U(1) \). Hence, to compare our

\(^4\)The following relations are given for the normalization of the eleven-dimensional action as in eq. (1). If instead the normalization of [21] is used, the expression for \( \alpha_{\text{GUT}} \) gets rescaled to \( \alpha_{\text{GUT}} = 2^{1/3} (\kappa^2/2v) \left( 4\pi/\kappa \right)^{2/3} \). Otherwise the action and Bianchi identities are unchanged, except that in the expression (19) for \( \alpha \) the RHS is multiplied by \( 2^{1/3} \).
action to these we should dualize the three-form $C_{\alpha\beta\gamma}$ to a scalar field $\sigma$ by setting (in the bulk)

$$G_{\alpha\beta\gamma\delta} = \frac{1}{\sqrt{2V^2}} \epsilon_{\alpha\beta\gamma\delta\epsilon} \left( \partial^\epsilon \sigma - i(\xi \partial^\epsilon \bar{\xi} - \bar{\xi} \partial^\epsilon \xi) - 2\alpha_0 \epsilon(x^{11}) \mathcal{A} \right). \quad (26)$$

Then the hypermultiplet part of the action (22) can be written as

$$S_{\text{hyper}} = -\frac{v}{2k^2} \int_{M_5} \sqrt{-g} \left[ h_{uv} \nabla_\alpha q^u \nabla^\alpha q^v + \frac{1}{3} V^{-2} \alpha_0^2 \right] \quad (27)$$

where $q^u = (V, \sigma, \xi, \bar{\xi})$. The covariant derivative $\nabla_\alpha$ is defined as $\nabla_\alpha q^u = \partial_\alpha q^u + \alpha_0 \epsilon(x^{11}) \mathcal{A}_\alpha k^u$ with $k^u = (0, -2, 0, 0)$. The sigma model metric $h_{uv} = \partial_u \partial_v K_Q$ can be computed from the Kähler potential

$$K_Q = -\ln(S + \bar{S} - 2C\bar{C}) , \quad S = V + \xi \bar{\xi} + i\sigma , \quad C = \xi . \quad (28)$$

Consequently, the hypermultiplet scalars $q^u$ parameterize a Kähler manifold with metric $h_{uv}$. It can be demonstrated that $k^u$ is a Killing vector on this manifold. Using the expressions given in ref. [29], one can show that this manifold is quaternionic with coset structure $M_Q$. Hence, the terms in eq. (27) that are independent of $\alpha_0$ describe the known form of the universal hypermultiplet action. How do we interpret the extra terms in the hypermultiplet action depending on $\alpha_0$? A hint is provided by the fact that one of these $\alpha_0$-dependent terms modifies the flat derivative in the kinetic energy to a generalized derivative $\nabla_\alpha$. This is exactly the combination that we would need if one wanted to gauge the $U(1)$ symmetry on $M_Q$ corresponding to the Killing vector $k^u$, using the gauge field $\mathcal{A}_\alpha$ in the gravity supermultiplet. In fact, investigation of the other terms in the action, including the fermions, shows that the resulting five-dimensional theory is precisely a gauged form of supergravity. Not only is a $U(1)$ isometry of $M_Q$ gauged, but at the same time a $U(1)$ subgroup of the $SU(2)$ automorphism group is also gauged.

What about the remaining $\alpha_0$-dependent potential term in the hypermultiplet action? From $d = 4$, $N = 2$ theories, we are used to the idea that gauging a symmetry of the quaternionic manifold describing hypermultiplets generically introduces potential terms into the action when supersymmetry is preserved (see for instance [30]). Such potential terms can be thought of as the generalization of pure Fayet-Iliopoulos (FI) terms. This is precisely what happens in our theory as well, with the gauging of the $U(1)$ subgroup inducing the $\alpha_0$-dependent potential term in (27). The general gauged action was discussed in detail in [7]. Certain pure FI terms were previously considered in [27], but, to our knowledge, such a theory with general gauging has not been constructed previously in five dimensions.
The phenomenon that the inclusion of non-zero modes leads to gauged supergravity theories has already been observed in Type II Calabi-Yau compactifications [31, 32], while the observation that the vacua of gauged theories correspond to dimensional reduction with non-trivial form-fields has a long history. Recent results relating to intersecting branes are described in [33]. From the form of the Killing vector, we see that it is only the scalar field $\sigma$, dual to the four-form $G_{\alpha\beta\gamma\delta}$, which is charged under the $U(1)$ symmetry. Its charge is fixed by $\alpha_0$. We note that this charge is quantized since, suitably normalized, $\text{tr}R \wedge R$ is an element of $H^{2,2}(X, \mathbb{Z})$. In the brane description of the theory, this is a reflection of the fact that the five-brane charge is quantized.

To analyze the supersymmetry properties of the solutions shortly to be discussed, we need the supersymmetry variations of the fermions associated with the theory (20). They can be obtained either by a reduction of the eleven-dimensional gravitino variation (7) or by generalizing the known five-dimensional transformations [18, 28] by matching onto gauged four-dimensional $N = 2$ theories. It is sufficient for our purposes to keep the bosonic terms only. Both approaches lead to

\[
\delta \psi^i = D_\alpha \epsilon^i + \frac{\sqrt{2}i}{8} (\gamma^a \partial_\alpha - 4 \delta^a_\alpha \gamma^\gamma) \mathcal{F}_{\beta\gamma} \epsilon^i - \frac{1}{2} V^{-1/2} \left( \partial_\alpha (\tau_1 - i \tau_2) \partial^i_j - \partial_\alpha \bar{\xi} (\tau_1 + i \tau_2) \partial^i_j \right) \epsilon^j \\
- \frac{\sqrt{2}i}{96} V \epsilon_{\alpha\beta\gamma\delta} G_{\beta\gamma\delta\epsilon} (\tau_3)_j^i \epsilon^j - \frac{\sqrt{2}}{12} \alpha_0 V^{-1} \epsilon (x^{11}) \gamma^a (\tau_3)_j^i \epsilon^j \\
\delta \bar{\zeta}^i = \frac{\sqrt{2}}{48} V \epsilon_{\alpha\beta\gamma\delta} G_{\alpha\beta\gamma\delta} \gamma^i \epsilon^i - \frac{i}{2} V^{-1/2} \gamma^a \left( \partial_\alpha (\tau_1 - i \tau_2) \partial^i_j + \partial_\alpha \bar{\xi} (\tau_1 + i \tau_2) \partial^i_j \right) \epsilon^j \\
+ \frac{i}{2} V^{-1} \gamma_\beta \partial^\beta V \epsilon^i - \frac{i}{\sqrt{2}} \alpha_0 V^{-1} \epsilon (x^{11}) (\tau_3)_j^i \epsilon^j
\]

(29)

where $\tau_i$ are the Pauli spin matrices.

In summary, we see that the relevant five-dimensional effective theory for the reduction of Hořava-Witten theory is a gauged $N = 1$ supergravity theory with bulk and boundary potentials. While we have calculated the theory only to order $\kappa^{2/3}$, one would expect that M-theory corrections can be described in the same type of theory. For this reason, it is very desirable to construct the most general gauged five-dimensional $N = 1$ supergravity theory coupled to general $N = 1$ four-dimensional boundary theories with vector and chiral multiplets. Part of this program was carried out in ref. [7]. In the context of global supersymmetry, such boundary theories in five dimensions have been studied in ref. [34].
Lecture 2:

In this second lecture, we discuss the static vacuum solution of the effective five-dimensional Hořava-Witten theory. We show that it is given by a pair of BPS domain walls, each located at one orbifold boundary plane. At leading non-trivial order, this domain wall solution is identical to the “deformations” of the eleven-dimensional background spacetime found by Witten. The reduction of the five-dimensional theory on this BPS domain wall then produces, at low energy, the four-dimensional $N = 1$ supersymmetric effective heterotic $M$-theory. We then present, in detail, the simplest time-dependent cosmological vacuum solution associated with this domain wall background. We discuss various aspects of its cosmology and show that it has at least one branch that corresponds to a radiation-like expanding three-dimensional universe and a contracting orbifold.

The domain-wall solution

Let us recapitulate what we have done so far. To arrive at a simple form for the five dimensional effective action, we have absorbed the deformation of the Calabi-Yau background metric into the five-dimensional moduli. Effectively, we could then carry out the reduction on a Calabi-Yau space but had to explicitly keep the antisymmetric tensor part of the background as a non-zero mode in the reduction. As a consequence, although Witten’s original background preserved only four supercharges, the effective bulk theory has twice that number of preserved supercharges, corresponding to minimal $N = 1$ supergravity in five dimensions. For consistency, we should now be able to find the deformations of the Calabi-Yau background as solutions of the effective five-dimensional theory. These solutions should break half the supersymmetry of the five-dimensional bulk theory and preserve Poincaré invariance in four dimensions. Hence, we expect there to be a three-brane domain wall in five dimensions with a worldvolume lying in the four uncompactified directions. This domain wall can be viewed as the “vacuum” of the five-dimensional theory, in the sense that it provides the appropriate background for a reduction to the $d = 4$, $N = 1$ effective theory.

This expectation is made stronger if we recall the brane picture of Witten’s background. We argued that this could be described by five-branes with equal amounts of five-brane charge living on the orbifold planes. From the five-dimensional perspective, the five-branes appear as three-branes living on the orbifold fixed planes. Thus, in five dimensions, Witten’s background must correspond to a pair of parallel three-branes.
We notice that the theory (20) has all of the prerequisites necessary for such a three-brane solution to exist. Generally, in order to have a \((D-2)\)-brane in a \(D\)-dimensional theory, one needs to have a \((D-1)\)-form field or, equivalently, a cosmological constant. This is familiar from the eight-brane [35] in the massive type IIA supergravity in ten dimensions [36], and has been systematically studied for theories in arbitrary dimension obtained by generalized (Scherk-Schwarz) dimensional reduction [37]. In our case, this cosmological term is provided by the bulk potential term in the action (20). From the viewpoint of the bulk theory, we could have multi three-brane solutions with an arbitrary number of parallel branes located at various places in the \(x^{11}\) direction. As is well known, however, elementary brane solutions have singularities at the location of the branes, needing to be supported by source terms. The natural candidates for those source terms, in our case, are the boundary actions. Given the anomaly-cancelation requirements, this restricts the possible solutions to those representing a pair of parallel three-branes corresponding to the orbifold planes.

From the above discussion, it is clear that in order to find a three-brane solution, we should start with the Ansatz

\[
\begin{align*}
  ds_5^2 &= a(y)^2dx^\mu dx^\nu \eta_{\mu\nu} + b(y)^2dy^2 \\
  V &= V(y)
\end{align*}
\]

where \(a\) and \(b\) are functions of \(y = x^{11}\) and all other fields vanish. The general solution for this Ansatz, satisfying the equations of motion derived from action (20), is given by

\[
\begin{align*}
  a &= a_0 H^{1/2} \\
  b &= b_0 H^2 \\
  V &= b_0 H^3
\end{align*}
\]

where \(a_0\), \(b_0\) and \(c_0\) are constants. We note that the boundary source terms have fixed the form of the harmonic function \(H\) in the above solution. Without specific information about the sources, the function \(H\) would generically be glued together from an arbitrary number of linear pieces with slopes \(\pm \frac{\sqrt{2}}{3} \alpha_0\). The edges of each piece would then indicate the location of the source terms. The necessity of matching the boundary sources at \(y = 0\) and \(\pi \rho\), however, has forced us to consider only two such linear pieces, namely \(y \in [0, \pi \rho]\) and \(y \in [-\pi \rho, 0]\). These pieces are glued together at \(y = 0\) and \(\pi \rho\) (recall here that we have identified \(\pi \rho\) and \(-\pi \rho\)). To see this explicitly, let us consider one of the equations of motion; specifically, the
equation derived from the variation of $g_{\mu\nu}$. For the Ansatz in (31), this is given by

$$a'' + \frac{a'^2}{a} - \frac{a'b'}{a} + \frac{1}{12} \frac{V'^2}{V^2} + \frac{a_0^2 b^2}{18 V^2} = \frac{\sqrt{2}a_0}{3} \frac{b}{V} (\delta(y) - \delta(y - \pi \rho))$$

(32)

where the prime denotes differentiation with respect to $y$. The term involving the delta functions arises from the stress energy on the boundary planes. Inserting the solution (32) in this equation, we have

$$\partial_y^2 H = \frac{2\sqrt{2}}{3} \alpha_0 (\delta(y) - \delta(y - \pi \rho))$$

(33)

which shows that the solution represents two parallel three-branes located at the orbifold planes.

We stress that this solution solves the five-dimensional theory (20) exactly, whereas the original deformed Calabi-Yau solution was only an approximation to order $\kappa^{2/3}$. It is straightforward to show that the linearized version of (31), that is, the expansion to first order in $\alpha_0 = O(\kappa^{2/3})$, coincides with Witten’s solution (9), (10)–(14) upon appropriate matching of the integration constants. Hence, we have found an exact generalization, good to all orders in $\kappa$, of the linearized eleven-dimensional solution.

Of course, we still have to check that our solution preserves half of the supersymmetries. When $g_{\alpha\beta}$ and $V$ are the only non-zero fields, the supersymmetry transformations (29) simplify to

$$\delta \psi^i_\alpha = D_{\alpha} \epsilon^i - \frac{\sqrt{2}}{12} \alpha_0 \epsilon(y) V^{-1} \gamma_3 (\tau_3)^{ij} \epsilon^j$$

(34)

$$\delta \zeta^i = \frac{i}{2} V^{-1} \gamma_3 \partial^3 V \epsilon^i - \frac{i}{\sqrt{2}} \alpha_0 \epsilon(y) V^{-1} (\tau_3)^{ij} \epsilon^j.$$ 

The Killing spinor equations $\delta \psi^i_\alpha = 0$, $\delta \zeta^i = 0$ are satisfied for the solution (31) if we require that the spinor $\epsilon^i$ is given by

$$\epsilon^i = H^{1/4} \epsilon^i_0, \quad \gamma_{11} \epsilon^i_0 = (\tau_3)^{ij} \epsilon^j_0$$

(35)

where $\epsilon^i_0$ is a constant symplectic Majorana spinor. This shows that we have indeed found a BPS solution preserving four of the eight bulk supercharges.

Let us discuss the meaning of this solution in some detail. First, we notice that it fits into the general scheme of domain wall solutions in various dimensions \footnote{In the notation of ref. [37], it corresponds to choosing $D = 5$, $\Delta = 4/3$ and $a(5) = 2$.}. It is, however, a new solution to the gauged supergravity action (20) in five dimensions which has not
been constructed previously. In addition, its source terms are naturally provided by the boundary actions resulting from Hořava-Witten theory. Most importantly, it constitutes the fundamental vacuum solution of a phenomenologically relevant theory. The two parallel three-branes of the solution, separated by the bulk, are oriented in the four uncompactified space-time dimensions, and carry the physical low-energy gauge and matter fields. Therefore, from the low-energy point of view where the orbifold is not resolved the three-brane worldvolume is identified with four-dimensional space-time. In this sense, the Universe lives on the worldvolume of a three-brane.

Although we have found an exact solution to the (lowest order) low energy theory, thereby improving previous results, it is not clear whether the solution will be exact in the full theory. Strominger [29] has argued that the all-loop corrections (corresponding to corrections to the effective action proportional to powers of $\kappa^{4/3}/V$, in our notation) to the quaternionic metric of the universal hypermultiplet can be actually absorbed into a shift of $V$, so that the metric is unchanged. This implies that our solution would be unaffected by such corrections. On the other hand, we have no general argument why the solution should be protected against corrections from higher derivative terms.

In any case, we believe that pursuing the construction of five-dimensional gauged supergravities with boundaries, and the analysis of their soliton structure, in the way indicated in this paper will provide important insights into early universe cosmology as well as low energy particle phenomenology.

It is the purpose of the remainder of these lectures to put this picture into the context of cosmology. Consequently, we are looking for time dependent solutions based on the static domain wall which we have just presented.

The domain-wall cosmological solution

In this section, we will present a cosmological solution related to the static domain wall vacuum of the previous section. As discussed in ref. [12, 13], a convenient way to find such a solution is to use Ansatz (31) where the $y = \tau^{11}$ coordinate in the functions $a, b$ and $V$ is replaced by the time coordinate $\tau$. However, in Hořava-Witten theory the boundary planes preclude this from being a solution of the equations of motion, since it does not admit homogeneous solutions. To see this explicitly, let us consider the $g_{00}$ equation of motion,
where we replace \( a(y) \rightarrow \alpha(\tau), b(y) \rightarrow \beta(\tau) \) and \( V \rightarrow \gamma(\tau) \). We find that

\[
\dot{\alpha}^2 + \frac{\dot{\alpha} \dot{\beta}}{\alpha \beta} - \frac{1}{12} \dot{\gamma}^2 - \frac{\alpha_0^2}{18} \frac{1}{\gamma^2} = -\frac{\sqrt{2} \alpha_0}{3} \frac{1}{\beta \gamma} (\delta(y) - \delta(y - \pi \rho)) ,
\]

(36)

where the dot denotes differentiation with respect to \( \tau \). Again, the term containing the delta functions arises from the boundary planes. It is clear that, because of the \( y \)-dependence introduced by the delta functions, this equation has no globally defined solution. The structure of equation (36) suggests that a solution might be found if one were to let functions \( a, b \) and \( V \) depend on both \( \tau \) and \( y \) coordinates. This would be acceptable from the point of view of cosmology, since any such solution would be homogeneous and isotropic in the spatial coordinates \( x^m \) where \( m, n, r, \ldots = 1, 2, 3 \). In fact, the previous Ansatz was too homogeneous, being independent of the \( y \) coordinate as well. Instead, we are interested in solutions where the inhomogeneous vacuum domain wall evolves in time.

We now construct a cosmological solution where all functions depend on both \( \tau \) and \( y \). We start with the Ansatz

\[
\begin{align*}
\text{ds}_5^2 &= -N(\tau,y)^2d\tau^2 + a(\tau,y)^2dx^m dx^n \eta_{mn} + b(\tau,y)^2dy^2 \\
V &= V(\tau,y)
\end{align*}
\]

(37)

Note that we have introduced a separate function \( N \) into the purely temporal part of the metric. This Ansatz leads to equations of motion that mix the \( \tau \) and \( y \) variables in a complicated non-linear way. In order to solve this system of equations, we will try to separate the two variables. That is, we let

\[
\begin{align*}
N(\tau,y) &= n(\tau)a(y) \\
a(\tau,y) &= \alpha(\tau)a(y) \\
b(\tau,y) &= \beta(\tau)b(y) \\
V(\tau,y) &= \gamma(\tau)V(y)
\end{align*}
\]

(38)

There are two properties of this Ansatz that we wish to point out. The first is that for \( n = \alpha = \beta = \gamma = 1 \) it becomes identical to (31). Secondly, we note that \( n \) can be chosen to be any function by performing a redefinition of the \( \tau \) variable. That is, we can think of \( n \) as being subject to a gauge transformation. There is no a priori reason to believe that separation of variables will lead to a solution of the equations of motion derived from the action (20). However, as we now show, there is indeed such a solution. It is instructive to
present one of the equations of motion. With the above Ansatz, the $g_{00}$ equation of motion is given by

$$\frac{a^2}{b^2} \left( \frac{a''}{a} + \frac{a'^2}{a^2} - \frac{a'}{ab} + \frac{1}{12} \frac{V''}{V^2} + \frac{\alpha_0^2 b^2 \beta^2}{18 V^2 \gamma^2} - \frac{\sqrt{2}}{3} \alpha_0 \frac{b}{V} (\delta(y) - \delta(y - \pi \rho)) \frac{\beta}{\gamma} \right) = \frac{\beta^2}{n^2} \left( \frac{\dot{\alpha}^2}{\alpha^2} + \frac{\dot{\beta}}{\alpha \beta} - \frac{1}{12} \frac{\dot{\gamma}^2}{\gamma^2} \right)$$

(39)

Note that if we set $n = \alpha = \beta = \gamma = 1$ this equation becomes identical to (32). Similarly, if we set $a = b = V = 1$ and take the gauge $n = 1$ this equation becomes the same as (36). As is, the above equation does not separate. However, the obstruction to a separation of variables is the two terms proportional to $\alpha_0$. Note that both of these terms would be strictly functions of $y$ only if we demanded that $\beta \propto \gamma$. Without loss of generality, one can take

$$\beta = \gamma$$

(40)

We will, henceforth, assume this is the case. Note that this result is already indicated by the structure of integration constants (moduli) in the static domain wall solution (32). With this condition, the left hand side of equation (39) is purely $y$ dependent, whereas the right hand side is purely $\tau$ dependent. Both sides must now equal the same constant which, for simplicity, we take to be zero. The equation obtained by setting the left hand side to zero is identical to the pure $y$ equation (32). The equation for the pure $\tau$ dependent functions is

$$\frac{\dot{\alpha}^2}{\alpha^2} + \frac{\dot{\beta}}{\alpha \beta} - \frac{1}{12} \frac{\dot{\gamma}^2}{\gamma^2} = 0$$

(41)

Hence, separation of variables can be achieved for the $g_{00}$ equation by demanding that (40) is true. What is more remarkable is that, subject to the constraint that $\beta = \gamma$, all the equations of motion separate. The pure $y$ equations are identical to those of the previous section and, hence, the domain wall solution (32) remains valid as the $y$-dependent part of the solution.

The full set of $\tau$ equations is found to be

$$\frac{\dot{\alpha}^2}{\alpha^2} + \frac{\dot{\beta}}{\alpha \beta} - \frac{1}{12} \frac{\dot{\gamma}^2}{\gamma^2} = 0$$

(42)

$$2 \frac{\ddot{\alpha}}{\alpha} - 2 \frac{\dot{\alpha}}{\alpha n} + \frac{\ddot{\beta}}{\beta n} + \frac{\dot{\beta}}{\beta n} + \frac{\dot{\alpha}^2}{\alpha^2} + 2 \frac{\dot{\alpha}}{\alpha \beta} + \frac{1}{4} \frac{\dot{\gamma}^2}{\gamma^2} = 0$$

(43)

$^6$From now on, we denote by $a, b, V$ the $y$-dependent part of the Ansatz (38).
\[
\frac{\ddot{\alpha}}{\alpha} - \frac{\dot{\alpha} \dot{n}}{\alpha n} + \frac{\dot{\alpha}^2}{\alpha^2} + \frac{1}{12} \frac{\dot{\gamma}^2}{\gamma^2} = 0
\] (44)

\[
\frac{\ddot{\gamma}}{\gamma} + 3 \frac{\dot{\alpha} \dot{\gamma}}{\alpha \gamma} + \frac{\dot{\beta} \dot{\gamma}}{\beta \gamma} - \frac{\dot{\gamma}^2}{\gamma^2} - \frac{\dot{n} \dot{\gamma}}{n \gamma} = 0
\] (45)

In these equations we have displayed \(\beta\) and \(\gamma\) independently, for reasons to become apparent shortly. Of course, one must solve these equations subject to the condition that \(\beta = \gamma\). As a first attempt to solve these equations, it is most convenient to choose a gauge for which

\[n = \text{const}\] (46)

so that \(\tau\) becomes proportional to the comoving time \(t\), since \(dt = n(\tau)d\tau\). In such a gauge, the equations simplify considerably and we obtain the solution

\[
\alpha = A|t - t_0|^p
\]

\[
\beta = \gamma = B|t - t_0|^q
\] (47)

where

\[p = \frac{3}{11} \left(1 \pm \frac{4}{3\sqrt{3}}\right), \quad q = \frac{2}{11} \left(1 \pm 2\sqrt{3}\right)\] (48)

and \(A, B\) and \(t_0\) are arbitrary constants. We have therefore found a cosmological solution, based on the separation Ansatz (38), with the \(y\)-dependent part being identical to the domain wall solution (32) and the scale factors \(\alpha, \beta, \gamma\) evolving according to the power laws (47). This means that the shape of the domain wall pair stays rigid while its size and the separation between the walls evolve in time. Specifically, \(\alpha\) measures the size of the spatial domain wall worldvolume (the size of the three-dimensional universe), while \(\beta\) specifies the separation of the two walls (the size of the orbifold). Due to the separation constraint \(\gamma = \beta\), the time evolution of the Calabi-Yau volume, specified by \(\gamma\), is always tracking the orbifold. From this point of view, we are allowing two of the three moduli in (32), namely \(a_0\) and \(b_0\), to become time-dependent. Since these moduli multiply the harmonic function \(H\), it is then easy to see why a solution by separation of variables was appropriate.

To understand the structure of the above solution, it is useful to rewrite its time dependent part in a more systematic way using the formalism developed in ref. [12, 13]. First, let us define new functions \(\hat{\alpha}, \hat{\beta}\) and \(\hat{\gamma}\) by

\[
\alpha = e^{\hat{\alpha}}, \quad \beta = e^{\hat{\beta}}, \quad \gamma = e^{6\hat{\gamma}}
\] (49)
and introduce the vector notation
\[
\vec{\alpha} = (\alpha^i) = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \end{pmatrix}, \quad \vec{d} = (d_i) = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.
\] (50)

Note that the vector \( \vec{d} \) specifies the dimensions of the various subspaces, where the entry \( d_1 = 3 \) is the spatial worldvolume dimensions, \( d_2 = 1 \) is the orbifold dimension and we insert 0 for the dilaton. On the “moduli space” spanned by \( \vec{\alpha} \) we introduce the metric
\[
G_{ij} = 2(d_i \delta_{ij} - d_i d_j)
\]
\[
G_{mn} = G_{nm} = 0
\]
\[
G_{nn} = 36,
\] (51)

which in our case explicitly reads
\[
G = -12 \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}.
\] (52)

Furthermore, we define \( E \) by
\[
E = e^{d \vec{\alpha}} n = e^{3\hat{\alpha} + \hat{\beta}} n.
\] (53)

The equations of motion (42)-(45) can then be rewritten as
\[
\frac{1}{2} E \dot{\vec{\alpha}}^T G \dot{\vec{\alpha}} = 0, \quad \frac{d}{d\tau} \left( E G \dot{\vec{\alpha}} \right) = 0.
\] (54)

It is straightforward to show that if we choose a gauge \( n = \text{const.} \), these two equations exactly reproduce the solution given in (47) and (48). The importance of this reformulation of the equations of motion lies, however, in the fact that we now get solutions more easily by exploiting the gauge choice for \( n \). For example, let us now choose the gauge
\[
n = e^{d \vec{\alpha}}.
\] (55)

Note that in this gauge \( E = 1 \). The reader can verify that this gauge choice greatly simplifies solving the equations. The result is that
\[
\dot{\hat{\alpha}} = 6\hat{\gamma} = C\tau + k_1
\]
\[
\dot{\hat{\beta}} = (6 \pm 4\sqrt{3})C\tau + k_2
\] (56)
where \( C, k_1 \) and \( k_2 \) are arbitrary constants. Of course this solution is completely equivalent to the previous one, eq. (47), but written in a different gauge. We will exploit this gauge freedom to effect in the last lecture.

To discuss cosmological properties we define the Hubble parameters

\[
\vec{H} = \frac{d}{dt} \vec{\alpha}
\]

where \( t \) is the comoving time. From (47) and (49) we easily find

\[
\vec{H} = \frac{\vec{p}}{t - t_0}, \quad \vec{p} = \begin{pmatrix} p \\ q \\ \frac{1}{6}q \end{pmatrix}.
\]

Note that the powers \( \vec{p} \) satisfy the constraints

\[
\vec{p}^T G \vec{p} = 0 \quad \vec{d} \cdot \vec{p} = 1.
\]

These relations are characteristic for rolling radii solutions [16] which are fundamental cosmological solutions of weakly coupled heterotic string theory. Comparison of the equations of motion (54) indeed shows that the scale factors \( \vec{\alpha} \) behave like rolling radii. The original rolling radii solutions describe freely evolving scale factors of a product of homogeneous, isotropic spaces. In our case, the scale factors also evolve freely (since the time-dependent part of the equations of motion, obtained after separating variables, does not contain a potential) but they describe the time evolution of the domain wall. This also proves our earlier claim that the potential terms in the five-dimensional action (20) do not directly influence the time-dependence but are canceled by the static domain wall part of the solution.

Let us now be more specific about the cosmological properties of our solution. First note from eq. (58) that there exist two different types of time ranges, namely \( t < t_0 \) and \( t > t_0 \). In the first case, which we call the \((-)\) branch, the evolution starts at \( t \to -\infty \) and runs into a future curvature singularity [12, 13] at \( t = t_0 \). In the second case, called the \((+)\) branch, we start out in a past curvature singularity at \( t = t_0 \) and evolve toward \( t \to \infty \). In summary, we therefore have the branches

\[
t \in \begin{cases} [\infty, t_0] & (-) \text{ branch} \\ [t_0, \infty] & (+) \text{ branch} \end{cases}
\]

For both of these branches we have two options for the powers \( \vec{p} \), defined in eq. (58), corre-
sponding to the two different signs in eq. (48). Numerically, we find

\[ \vec{p}_1 \simeq \begin{pmatrix} +.06 \\ +.81 \\ +.14 \end{pmatrix}, \quad \vec{p}_1 \simeq \begin{pmatrix} +.48 \\ -.45 \\ -.08 \end{pmatrix} \] (61)

for the upper and lower sign in (48) respectively. We recall that the three entries in these vectors specify the evolution powers for the spatial worldvolume of the three-brane, the domain wall separation and the Calabi-Yau volume. The expansion of the domain wall worldvolume has so far been measured in terms of the five-dimensional Einstein frame metric \( g_{\mu\nu}^{(5)} \). This is also what the above numbers \( p_1 \) reflect. Alternatively, one could measure this expansion with the four-dimensional Einstein frame metric \( g_{\mu\nu}^{(4)} \) so that the curvature scalar on the worldvolume is canonically normalized. From the relation

\[ g_{\mu\nu}^{(4)} = (g_{11,11})^{1/2} g_{\mu\nu}^{(5)} \] (62)

we find that this modifies \( p_1 \) to

\[ \tilde{p}_1 = p_1 + \frac{p_2}{2}. \] (63)

In the following, we will discuss both frames. We recall that the separation condition \( \beta = \gamma \) implies that the internal Calabi-Yau space always tracks the orbifold. In the discussion we can, therefore, concentrate on the spatial worldvolume and the orbifold, corresponding to the first and second entries in (61). Let us first consider the \((-)\) branch. In this branch \( t \in [-\infty, t_0) \) and, hence, \( t - t_0 \) is always negative. It follows from eq. (58) that a subspace will expand if its \( \vec{p} \) component is negative and contract if it is positive. For the first set of powers \( \vec{p}_1 \) in eq. (61) both the worldvolume and the orbifold contract in the five-dimensional Einstein frame. The same conclusion holds in the four-dimensional Einstein frame. For the second set, \( \vec{p}_1 \), in both frames the worldvolume contracts while the orbifold expands. Furthermore, since the Hubble parameter of the orbifold increases in time the orbifold undergoes superinflation.

Now we turn to the \((+)\) branch. In this branch \( t \in [t_0, \infty] \) and, hence, \( t - t_0 \) is always positive. Consequently, a subspace expands for a positive component of \( \vec{p} \) and contracts otherwise. In addition, since the absolute values of all powers \( \vec{p} \) are smaller than one, an expansion is always subluminal. For the vector \( \vec{p}_1 \) the worldvolume and the orbifold expand in both frames. On the other hand, the vector \( \vec{p}_1 \) describes an expanding worldvolume and a contracting orbifold in both frames. This last solution perhaps corresponds most closely to our notion of the early universe.

25
Let us briefly summarize the basic geometry and physics of the solutions discussed in this section, emphasizing the (+) branch, \( \vec{p}_1 \) solution for concreteness. We find that at the GUT scale of around \( 10^{16} \) GeV, Hořava-Witten theory compactifies on a Calabi-Yau three-fold to become an effective five-dimensional theory with two four-dimensional boundaries. Anomaly cancellation constrains this effective theory to be a specific gauged form of \( N = 1 \) supergravity coupled to hyper and vector supermultiplets in the bulk with associated \( N = 1 \) gauge theories on each boundary. The size of the fifth-dimension can be anywhere between four or five times the inverse GUT scale to much larger, perhaps inverse \( 10^{14} \) Gev. This theory does not admit flat space as a solution. Rather, the static vacuum is found to be a pair of BPS three-branes each located on one of the orbifold fixed planes. There are natural cosmological solutions associated with this vacuum state. These solutions expand or contract in the four-dimensional directions of the low energy Universe as well as expanding or contracting in the fifth, orbifold direction. Among a plethora of possibilities mentioned above, one solution, the (+) branch with \( \vec{p}_1 \), corresponds to subluminal expansion of real four-dimensional spacetime while the associated fifth-dimension contracts. One assumes that other, perhaps non-perturbative, physics will stabilize the moduli of the fifth-direction. If so, this solution would be a candidate for a Robertson-Walker expanding phase of the early Universe.

**Lecture 3:**

In this lecture, we explore cosmological solutions with non-vanishing Ramond-Ramond form backgrounds. Solution of these cosmological scenarios is mathematically intricate and requires the use of methods, related to Toda theory, introduced in our previous work. We present one complete example with a single Ramond-Ramond scalar turned on and discuss its physical interpretation in detail.

**Cosmological solutions with Ramond forms**

Thus far, we have looked for both static and cosmological solutions where the form fields \( \xi, A_\alpha \) and \( \sigma \) have been set to zero. As discussed in previous papers [12, 13], turning on one or several such fields can drastically alter the solutions and their cosmological properties. Hence, we would like to explore cosmological solutions with such non-trivial fields. For clarity, in this lecture we will restrict the discussion to turning on the Ramond-Ramond
scalar $\xi$ only.

The Ansatz we will use is the following. For the metric and dilaton field, we choose

$$
\begin{align*}
    ds_5^2 &= -N(\tau, y)^2 d\tau^2 + a(\tau, y)^2 dx^m dx^n \eta_{mn} + b(\tau, y)^2 dy^2 \\
    V &= V(\tau, y) .
\end{align*}
$$

(64)

For the $\xi$ field, we assume that $\xi = \xi(\tau, y)$ and, hence, the field strength $F_\alpha = \partial_\alpha \xi$ is given by

$$
F_0 = Y(\tau, y) , \quad F_5 = X(\tau, y) .
$$

(65)

All other components of $F_\alpha$ vanish. Note that since $\xi$ is complex, both $X$ and $Y$ are complex. Once again, we will solve the equations of motion by separation of variables. That is, we let

$$
\begin{align*}
    N(\tau, y) &= n(\tau) N(y) \\
    a(\tau, y) &= \alpha(\tau) a(y) \\
    b(\tau, y) &= \beta(\tau) b(y) \\
    V(\tau, y) &= \gamma(\tau) V(y)
\end{align*}
$$

(66)

and

$$
\begin{align*}
    X(\tau, y) &= \chi(\tau) X(y) \\
    Y(\tau, y) &= \phi(\tau) Y(y)
\end{align*}
$$

(67)

Note that, in addition to the $\xi$ field, we have also allowed for the possibility that $N(y) \neq a(y)$. Again, there is no a priori reason to believe that a solution can be found by separation of variables. However, as above, there is indeed such a solution, although the constraints required to separate variables are more subtle. It is instructive to present one of the equations of motion. With the above Ansatz, the $g_{00}$ equation of motion becomes

$$
\begin{align*}
    \frac{N^2}{b^2} \left( \frac{a''}{a} + \frac{a'^2}{a^2} - \frac{a' b'}{a b} + \frac{1}{12 \gamma^2} + \frac{\alpha_0^2}{18 V^2 \gamma^2} - \sqrt{2} \frac{a_0 b}{V} (\delta(y) - \delta(y - \pi \rho)) \frac{\beta}{\gamma} \right) &= \\
    \frac{\beta^2}{n^2} \left( \frac{\alpha^2}{\alpha^2} + \frac{1}{12 \gamma^2} - \frac{\beta^2}{n^2} \frac{Y^2}{V} \frac{\phi^2}{\gamma} - \frac{N^2}{3 b^2} \frac{|X|^2}{V} \frac{|x|^2}{\gamma} - \frac{\beta^2}{3 n^2} \frac{|Y|^2}{V} \frac{\phi^2}{\gamma}
\end{align*}
$$

(69)

Note that if we set $X = Y = 0$ and $N = a$ this equation becomes identical to (39). We now see that there are two different types of obstructions to the separation of variables. The first

---

In the following, $N, a, b, V$ denote the $y$-dependent part of the Ansatz (66).
type, which we encountered in the previous section, is in the two terms proportional to $\alpha_0$. Clearly, we can separate variables only if we demand that

$$\beta = \gamma$$

(70)
as we did previously. However, for non-vanishing $X$ and $Y$ this is not sufficient. The problem, of course, comes from the last two terms in (69). There are a number of options one could try in order to separate variables in these terms. It is important to note that $X$ and $Y$ are not completely independent, but are related to each other by the integrability condition $\partial_\tau X(\tau, y) = \partial_y Y(\tau, y)$. We find that, because of this condition, it is impossible to obtain a solution by separation of variables that has both $X(\tau, y)$ and $Y(\tau, y)$ non-vanishing. Now $X(\tau, y)$, but not $Y(\tau, y)$, can be made to vanish by taking $\xi = \xi(\tau)$; that is, $\xi$ is a function of $\tau$ only. However we can find no solution by separation of variables under this circumstance. Thus, we are finally led to the choice $\xi = \xi(y)$. In this case $Y(\tau, y) = 0$ and we can, without lose of generality, choose

$$\chi = 1.$$  

(71)

At this point, the only obstruction to separation of variables in equation (69) is the next to last term, $N^2|X|^2/3b^2V\gamma$. Setting $\gamma = \text{const}$ is too restrictive, so we must demand that

$$X = \frac{bV^{1/2}}{N} c_0 e^{i\theta(y)}$$

(72)

where $c_0$ is a non-zero but otherwise arbitrary real constant and $\theta(y)$ is an, as yet, undetermined phase. Putting this condition into the $\xi$ equation of motion

$$\partial_y \left( \frac{a^2 N}{bV} X \right) = 0$$

(73)

we find that $\theta$ is a constant $\theta_0$ and $a \propto V^{\frac{1}{2}}$ with arbitrary coefficient. Note that the last condition is consistent with the static vacuum solution (32). Inserting this result into the $g_{05}$ equation of motion

$$\frac{\dot{\alpha}}{\alpha} \left( \frac{a'}{a} - \frac{N''}{N} \right) = \frac{\dot{\beta}}{\beta} \left( \frac{a'}{a} - \frac{1}{6} \frac{V''}{V} \right)$$

(74)

we learn that $N \propto a$ with arbitrary coefficient. Henceforth, we choose $N = a$ which is consistent with the static vacuum solution (32). Inserting all of these results, the $g_{00}$ equation
of motion now becomes
\[
\frac{a^2}{b^2} \left( \frac{a''}{a} + \frac{a'^2}{a^2} - \frac{a'b'}{ab} + \frac{1}{12} \frac{V''}{V^2} + \frac{\alpha_0^2 b^2}{18 V^2} - \frac{\sqrt{2}}{3} \alpha_0 \frac{b}{\sqrt{V}} (\delta(y) - \delta(y - \pi \rho)) \right) = \\
\frac{\beta^2}{n^2} \left( \frac{\dot{\alpha}^2}{\alpha^2} + \frac{\dot{\alpha}}{\alpha} - \frac{\dot{\beta}}{\beta} - \frac{1}{12} \frac{\dot{\gamma}^2}{\gamma^2} \right) - \frac{c_0^2}{3} \frac{n^2}{\gamma} 
\]

Note that the left hand side is of the same form as the static vacuum equation (32). The effect of turning on the $\xi$ background is to add a purely $\tau$ dependent piece to the right hand side. Putting these results into the remaining four equations of motion, we find that they too separate, with the left hand side being purely $y$ dependent and the right hand side purely $\tau$ dependent. Again, we find that in these equations the left hand sides are identical to those in the static vacuum equations and the effect of turning on $\xi$ is to add extra $\tau$ dependent terms to the right hand sides. In each equation, both sides must now equal the same constant which, for simplicity, we take to be zero. The $y$ equations for $a$, $b$ and $V$ thus obtained by setting the left hand side to zero are identical to the static vacuum equations. Hence, we have shown that

\[
N = a = a_0 H^{1/2} \\
b = b_0 H^2 \\
V = b_0 H^3 \\
X = x_0 H^3
\]

where $x_0 = c_0 e^{i \theta_0} a_0^{-3/2}$ is an arbitrary constant.

The $\tau$ equations obtained by setting the right hand side to zero are the following.

\[
\frac{\dot{\alpha}^2}{\alpha^2} + \frac{\dot{\alpha}}{\alpha} - \frac{\dot{\beta}}{\beta} - \frac{\dot{\gamma}^2}{2 \gamma^2} = 0 
\]

\[
2 \frac{\ddot{\alpha}}{\alpha} - 2 \frac{\dot{\alpha}}{\alpha} + \frac{\ddot{\beta}}{\beta} - \frac{\dot{\beta}}{\beta} + \frac{\ddot{\gamma}}{\gamma} - \frac{\dot{\gamma}}{\gamma} - \frac{\dot{\alpha}^2}{\alpha^2} + 2 \frac{\dot{\alpha}}{\alpha} + 1 \frac{\dot{\gamma}^2}{\gamma^2} - c_0^2 \frac{n^2}{\beta^2 \gamma} = 0 
\]

\[
\frac{\ddot{\alpha}}{\alpha} - \frac{\dot{\alpha}}{\alpha} - \frac{\ddot{\gamma}}{\gamma} + \frac{\ddot{\gamma}}{\gamma} + \frac{\ddot{\gamma}}{\gamma} - \frac{\dot{\gamma}}{\gamma} - \frac{\dot{\alpha}^2}{\alpha^2} + 1 \frac{\dot{\gamma}^2}{\gamma^2} - c_0^2 \frac{n^2}{\beta^2 \gamma} = 0 
\]

\[
\frac{\ddot{\gamma}}{\gamma} + 3 \frac{\dot{\alpha}}{\alpha} + \frac{\ddot{\alpha}}{\alpha} - \frac{\dot{\beta} \dot{\gamma}}{\beta \gamma} - \frac{\dot{\gamma}}{\gamma} - \frac{\dot{\gamma}}{\gamma} - \frac{\dot{\alpha}^2}{\alpha^2} + 2 \frac{\dot{\alpha}}{\alpha} = 0 
\]
In these equations we have, once again, displayed $\beta$ and $\gamma$ independently, although they should be solved subject to the condition $\beta = \gamma$. Note that the above equations are similar to the $\tau$ equations in the previous section, but each now has an additional term proportional to $c_0^2$. These extra terms considerably complicate finding a solution of the $\tau$ equations. Here, however, is where the formalism introduced in the previous section becomes important. Defining $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ as in (49), and $\vec{\alpha}, E$ and $G$ as in (50), (53) and (52) respectively, the equations (77)-(80) can be written in the form

$$\frac{1}{2}E\dot{\vec{\alpha}}^T G \dot{\vec{\alpha}} + E^{-1}U = 0, \quad \frac{d}{d\tau} (EG\dot{\vec{\alpha}}) + E^{-1} \frac{\partial U}{\partial \vec{\alpha}} = 0$$

where the potential $U$ is defined as

$$U = 2c_0^2 e^{\vec{q} \cdot \vec{\alpha}}$$

with

$$\vec{q} = \begin{pmatrix} 6 \\ 0 \\ -6 \end{pmatrix}.$$  \hspace{1cm} (82)

We can now exploit the gauge freedom of $n$ to simplify these equations. Choose the gauge

$$n = e^{(\vec{d} - \vec{q} \cdot \hat{\vec{\alpha}})}$$

where $\vec{d}$ is defined in (50). Then $E$ becomes proportional to the potential $U$ so that the potential terms in (81) turn into constants. Thanks to this simplification, the equations of motion can be integrated which leads to the general solution [12, 13]

$$\vec{\alpha} = \vec{c} \ln |\tau_1 - \tau| + \vec{w} \ln \left( \frac{s\tau}{\tau_1 - \tau} \right) + \vec{k}$$

where $\tau_1$ is an arbitrary parameter which we take, without loss of generality, to be positive and

$$\vec{c} = 2 \frac{G^{-1}\vec{q}}{<\vec{q}, \vec{q}>}, \quad s = \text{sign}(<\vec{q}, \vec{q}>).$$

The scalar product is defined as $<\vec{q}, \vec{q}> = \vec{q}^T G^{-1} \vec{q}$. The vectors $\vec{w}$ and $\vec{k}$ are integration constants subject to the constraints

$$\begin{align*}
\vec{q} \cdot \vec{w} &= 1 \\
\vec{w}^T G \vec{w} &= 0 \\
\vec{q} \cdot \vec{k} &= \ln (c_0^2 |<\vec{q}, \vec{q}>|)
\end{align*}$$

30
This solution is quite general in that it describes an arbitrary number of scale factors with equations of motion given by (81). Let us now specify to our example. For \( G \) and \( \vec{q} \) as given in eq. (52) and (83) we find that

\[
< \vec{q}, \vec{q} > = 1
\]  

hence \( s = 1 \), and

\[
\vec{c} = \begin{pmatrix}
0 \\
-2 \\
-\frac{1}{3}
\end{pmatrix}.
\]  

(89)

Recall that we must, in addition, demand that \( \beta = \gamma \). Note that the last two components of \( \vec{c} \) are consistent with this equality. We can also solve the constraints (87) subject to the condition \( \beta = \gamma \). The result is

\[
\vec{w} = \begin{pmatrix}
w_3 + \frac{1}{6} \\
6w_3 \\
w_3
\end{pmatrix}, \quad \vec{k} = \begin{pmatrix}
k_3 + \frac{1}{6} \ln c_0^2 \\
6k_3 \\
k_3
\end{pmatrix}
\]  

(90)

where

\[
w_3 = -\frac{1}{6} \pm \sqrt{\frac{3}{12}}
\]  

(91)

and \( k_3 \) is arbitrary. We conclude that in the gauge specified by (84), the solution is given by

\[
\dot{\alpha} = (w_3 + \frac{1}{6}) \ln \left( \frac{\tau}{\tau_1 - \tau} \right) + k_3 + \frac{1}{6} \ln c_0^2 \\
\dot{\beta} = -2 \ln |\tau_1 - \tau| + 6w_3 \ln \left( \frac{\tau}{\tau_1 - \tau} \right) + 6k_3 \\
\dot{\gamma} = -\frac{1}{3} \ln |\tau_1 - \tau| + w_3 \ln \left( \frac{\tau}{\tau_1 - \tau} \right) + k_3
\]  

(92)

with \( w_3 \) as above. As a consequence of \( s = 1 \), the range for \( \tau \) is restricted to

\[
0 < \tau < \tau_1
\]  

(93)

in this solution. Let us now summarize our result. We have found a cosmological solution with a nontrivial Ramond-Ramond scalar \( \xi \) starting with the separation Ansatz (66). To achieve separation of variables we had to demand that \( \beta = \gamma \), as previously, and that the Ramond-Ramond scalar depends on the orbifold coordinate but not on time. Then the
orbifold dependent part of the solution is given by eq. (76) and is identical to the static domain wall solution with the addition of the Ramond-Ramond scalar. The time dependent part, in the gauge (84), is specified by eq. (92). Furthermore, we have found that the time-dependent part of the equations of motion can be cast in a form familiar from cosmological solutions studied previously [12, 13]. Those solutions describe the evolution for scale factors of homogeneous, isotropic subspaces in the presence of antisymmetric tensor fields and are, therefore, natural generalizations of the rolling radii solutions. Each antisymmetric tensor field introduces an exponential type potential similar to the one in eq. (82). For the case with only one nontrivial form field, the general solution could be found and is given by eq. (85). We have, therefore, constructed a strong coupling version of these generalized rolling radii solutions with a one-form field strength, where the radii now specify the domain wall geometry rather than the size of maximally symmetric subspaces. We stress that the potential $U$ in the time-dependent equations of motion does not originate from the potentials in the action (20) but from the nontrivial Ramond-Ramond scalar. The potentials in the action are canceled by the static domain wall part of the solution, as in the previous example.

From the similarity to the known generalized rolling radii solutions, we can also directly infer some of the basic cosmological properties of our solution, using the results of ref. [12, 13]. We expect the integration constants to split into two disjunct sets which lead to solutions in the (−) branch, comoving time range $t \in [-\infty, t_0]$, and the (+) branch, comoving time range $t \in [t_0, \infty]$, respectively. The (−) branch ends in a future curvature singularity and the (+) branch starts in a past curvature singularity. In both branches the solutions behave like rolling radii solutions asymptotically; that is, at $t \to -\infty, t_0$ in the (−) branch and at $t \to t_0, \infty$ in the (+) branch. The two asymptotic regions in both branches have different expansion properties in general and the transition between them can be attributed to the nontrivial form field.

Let us now analyze this in more detail for our solution, following the method presented in ref. [12, 13]. First we should express our solution in terms of the comoving time $t$ by integrating $dt = n(\tau)d\tau$. The gauge parameter $n(\tau)$ is explicitly given by

$$n = e^{(\vec{d} - \vec{q}) \cdot \vec{w}} |\tau_1 - \tau|^{-x+\Delta-1} |\tau|^{x-1}$$

where

$$x = \vec{d} \cdot \vec{w}, \quad \Delta = \vec{d} \cdot \vec{c}.$$  

Given this expression, the integration cannot easily be performed in general except in the asymptotic regions $\tau \to 0, \tau_1$. These regions will turn out to be precisely the asymptotic
rolling-radii limits. Therefore, for our purpose, it suffices to concentrate on those regions. Eq. (94) shows that the resulting range for the comoving time depends on the magnitude of \( \Delta \) and \( x \) (note that \( \Delta \) is a fixed number, for a given model, whereas \( x \) depends on the integration constants). It turns out that for all values of the integration constants we have either \( x < \Delta \) or \( x > 0 > \Delta \). This splits the space of integration constant into two disjunct sets corresponding to the (−) and the (+) branch as explained before. More precisely, we have the mapping

\[
\tau \rightarrow t \in \begin{cases} 
[-\infty, t_0] & \text{for } x < \Delta < 0 , \text{ (−) branch} \\
[t_0, +\infty] & \text{for } x > 0 > \Delta , \text{ (+) branch} 
\end{cases}
\]  

(96)

where \( t_0 \) is a finite arbitrary time (which can be different for the two branches). We recall that the range of \( \tau \) is \( 0 < \tau < \tau_1 \). The above result can be easily read off from the expression (94) for the gauge parameter. Performing the integration in the asymptotic region we can express \( \tau \) in terms of the comoving time and find the Hubble parameters, defined by eq. (57), and the powers \( \vec{p} \). Generally, we have

\[
\vec{p} = \begin{cases} 
\frac{w}{x} & \text{at } \tau \simeq 0 \\
\frac{\dot{w} - \dot{c}}{x-\Delta} & \text{at } \tau \simeq \tau_1 
\end{cases}
\]  

(97)

Note that, from the mapping (96), the expression at \( \tau \simeq 0 \) describes the evolution powers at \( t \rightarrow -\infty \) in the (−) branch and at \( t \simeq t_0 \) in the (+) branch; that is, the evolution powers in the early asymptotic region. Correspondingly, the expression for \( \tau \simeq \tau_1 \) applies to the late asymptotic regions; that is, to \( t \simeq t_0 \) in the (−) branch and to \( t \rightarrow \infty \) in the (+) branch. As before, these powers satisfy the rolling radii constraints (59).

Let us now insert the explicit expression for \( \vec{d}, \vec{w} \) and \( \vec{c} \), eqs. (50),(90) and (89), that specify our example into those formulae. First, from eq. (95) we find that

\[
x = -1 \pm 3 \sqrt{\frac{3}{4}} , \quad \Delta = -2 .
\]  

(98)

Note that the space of integration constants just consists of two points in our case, represented by the two signs in the expression for \( x \) above. Clearly, from the criterion (96) the upper sign leads to a solution in the (+) branch and the lower sign to a solution in the (−) branch. In each branch we therefore have a uniquely determined solution. Using eq. (97) we can calculate the asymptotic evolution powers in the (−) branch

\[
\vec{p}_{-,-\infty} = \begin{pmatrix} +.06 \\
+.81 \\
+.13 
\end{pmatrix} , \quad \vec{p}_{-,-t_0} = \begin{pmatrix} +.48 \\
-.45 \\
-.08 \end{pmatrix} .
\]  

(99)
Correspondingly, for the (+) branch we have

\[
\vec{p}_{+, t\to t_0} = \begin{pmatrix} +.48 \\ -.45 \\ -.08 \end{pmatrix}, \quad \vec{p}_{+, t\to \infty} = \begin{pmatrix} +.06 \\ +.81 \\ +.13 \end{pmatrix}.
\] (100)

Note that these vectors are in fact the same as in the (−) branch, with the time order being reversed. This happens because they are three conditions on the powers \(\vec{p}\) that hold in both branches, namely the two rolling radii constraints (59) and the separation constraint \(\beta = \gamma\), eq. (70), which implies that \(p_3 = 6p_2\). Since two of these conditions are linear and one is quadratic, we expect at most two different solutions for \(\vec{p}\). As in the previous solution, the time variation of the Calabi-Yau volume (third entry) is tracking the orbifold variation (second entry) as a consequence of the separation condition and, hence, needs not to be discussed separately. The first entry gives the evolution power for the spatial worldvolume in the five-dimensional Einstein frame. For a conversion to the four-dimensional Einstein frame one should again apply eq. (63). It is clear from the above numbers, however, that this conversion does not change the qualitative behaviour of the worldvolume evolution in any of the cases. Having said this, let us first discuss the (−) branch. At \(t \to -\infty\) the powers are positive and, hence, the worldvolume and the orbifold are contracting. The solution then undergoes the transition induced by the Ramond-Ramond scalar. Then at \(t \approx t_0\) the worldvolume is still contracting while the orbifold has turned into superinflating expansion. In the (+) branch we start out with a subluminally expanding worldvolume and a contracting orbifold at \(t \approx t_0\). After the transition both subspaces have turned into subluminal expansion.

To conclude, Ramond forms have a major effect on the behaviour of cosmological solutions in five–dimensional Hořava-Witten theory. Generically, they change the coefficients governing the asymptotic behaviour of these solutions in a given branch at very early and very late times. This allows the orbifold to, say, start out contracting and then, at some intermediate time to halt, turn around and superluminally expand. For the moment, it is not clear how such solutions fit into a theory of the real Universe. However, they open the door to new and novel mechanisms in early Universe cosmology which may someday be necessary to explain observable phenomenon.
Conclusion

In these lectures we have presented examples of cosmological solutions in five-dimensional Hořava-Witten theory. They are physically relevant in that they are related to the exact BPS three–brane pair in five dimensions, whose \( d = 4 \) worldvolume theory exhibits \( N = 1 \) supersymmetry. A wider class of such cosmological solutions can be obtained and will be presented elsewhere. We expect solutions of this type to provide the fundamental scaffolding for theories of the early universe derived from Hořava-Witten theory, but they are clearly not sufficient as they stand. The most notable deficiency is the fact that they are vacuum solutions, devoid of any matter, radiation or potential stress-energy. Inclusion of such stress-energy is essential to understand the behaviour of early universe cosmology. A study of its effect on the cosmology of Hořava-Witten theory is presently underway [38].

Acknowledgments

A. L. is supported in part by the European Community under contract No. FMRXCT 960090. B. A. O. is supported in part by DOE under contract No. DE-AC02-76-ER-03071. D. W. is supported in part by DOE under contract No. DE-FG02-91ER40671.

References


37